An Introduction to Description Logic III

Knowledge Bases and Reasoning Tasks

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Knowledge Bases

General Remarks

- A Knowledge Base (KB) is the place where the information relative to an **Ontology** is stored.
- So, it is the place where the **background knowledge**, necessary to perform some kind of semantic reasoning, is stored.
- A KB provides basically **three kinds of information**:
  - terminological knowledge, true for every element of the domain, in form of **definitions** of concepts in term of other concepts, or **subsumptions** between concepts;
  - assertional knowledge, relative to particular individuals or pair of individuals, in form of **assertions** relating, e.g. that a particular individual is an instance of a concept;
  - relational knowledge, true for every pair of elements of the domain, in form of **definitions** of roles in term of other roles, or **subsumptions** between roles;

Terminological Knowledge

- The terminological knowledge is **true for every** individual in a given interpretation domain.
- A **Terminology**, Terminological Box or **TBox** is the place where this terminological knowledge about concepts is stored.
- Terminological information appears in form of **inclusions** or **subsumptions** between (complex) concepts.
- Inclusions **between concepts** are allowed in the more basic languages and are object of research since the 90's.

Concept Inclusions

- A **concept inclusion axiom** is an expression of the form:

$$C \sqsubseteq D$$

where C, D are concepts.

- Given an interpretation $\mathcal{I}$, the inclusion axiom $C \sqsubseteq D$ is **true** in $\mathcal{I}$ iff:

$$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}.$$

- The presence of $C \sqsubseteq D$ in a knowledge base $\mathcal{KB}$ **constraints** all the interpretations $\mathcal{I}$ that are models of $\mathcal{KB}$ to satisfy the inclusion $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

Examples

$$\texttt{Mother} \sqsubseteq \texttt{Female} \sqcap \texttt{Person}$$

"every mother is a female person"

$$\texttt{Parent} \equiv \texttt{Father} \sqcup \texttt{Mother}$$

"a parent is either a father or a mother (and nothing else)"

$$\texttt{Human} \equiv \texttt{Mammal} \sqcap \texttt{Biped}$$

"a human is a biped mammal (and nothing else is)"

$$\texttt{Region} \sqsubseteq \exists \texttt{hasPart}^{-}.\texttt{Country}$$

"a region is a part of a country"

Assertional Knowledge

- The assertional knowledge is **true for a particular** individual or pair of individuals in a given interpretation domain.
- An **Assertional Box** or **ABox** is the place where this assertional knowledge about concepts, roles and individuals is stored.
- Assertional information appears in form of concept or role **assertions**.
- A **concept assertion** is a statement asserting that an individual a is an instance of a concept C.
- A **role assertion** is a statement asserting that a pair of individuals a, b is an instance of a role R.

Concept Assertions

- A **concept assertion axiom** is an expression of the form:

$$C(a)$$

where C is a concept and a is an individual.

- Given an interpretation $\mathcal{I}$, the assertion axiom $C(a)$ is **true** in $\mathcal{I}$ iff:

$$a^{\mathcal{I}} \in C^{\mathcal{I}}.$$

- The presence of $C(a)$ in a knowledge base $\mathcal{KB}$ **constraints** all the interpretations $\mathcal{I}$ that are models of $\mathcal{KB}$ to satisfy the relation $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

Examples

$$\texttt{Mammal} \sqcap \texttt{Biped}(\texttt{Marco})$$

"Marco is a biped mammal"

$$\texttt{Region}(\texttt{Moravia})$$

"Moravia is a region"

$$\texttt{Town} \sqcap \exists \texttt{hasCapital}^{-}.\texttt{Country}(\texttt{Prague})$$

"Prague is a town which is capital city of a country"

$$\texttt{Country} \sqcap \exists \texttt{hasBorderWith}\{\texttt{Germany},\texttt{Austria}\}(\texttt{CzechRepublic})$$

"Czech Republic is a country which has borders with Germany and Austria"

Role Assertions

- A **role assertion axiom** is an expression of the form:

$$R(a, b)$$

  where R is a role and a, b are individuals.

- Given an interpretation $\mathcal{I}$, the role assertion axiom $R(a, b)$ is **true** in $\mathcal{I}$ iff:

$$\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}.$$

- The presence of $R(a, b)$ in a knowledge base $\mathcal{KB}$ **constraints** all the interpretations $\mathcal{I}$ that are models of $\mathcal{KB}$ to satisfy the relation $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}$.

Examples

hasRegion(CzechRepublic,Moravia)

"Czech Republic has Moravia as a region"

$\mathtt{hasRegion}^{-}\mathtt{(Moravia,CzechRepublic)}$

"Moravia is a region of Czech Republic"

hasCapital(CzechRepublic,Prague)

"Czech Republic has capital city Prague"

hasBorderWith(Germany,CzechRepublic)

"Germany has border with Czech Republic"

Relational Knowledge

- The relational knowledge is **true for every** pair of individuals in a given interpretation domain.
- A **Role Box** or **RBox** is the place where this terminological knowledge about roles is stored.
- Relational information appears in form of **inclusions** or **subsumptions** between (complex) roles.
- Inclusions **between roles** are allowed in the more complex languages (those with $\mathcal{H}$ or $\mathcal{R}$) and are object of research in the last 10 years.

Role Inclusions

- A **role inclusion axiom** is an expression of the form:

$$R \sqsubseteq P$$

where R, P are roles.

- Given an interpretation $\mathcal{I}$, the role inclusion axiom $R \sqsubseteq P$ is **true** in $\mathcal{I}$ iff:

$$R^{\mathcal{I}} \subseteq P^{\mathcal{I}}.$$

- The presence of $R \sqsubseteq P$ in a knowledge base $\mathcal{KB}$ **constraints** all the interpretations $\mathcal{I}$ that are models of $\mathcal{KB}$ to satisfy the inclusion $R^{\mathcal{I}} \subseteq P^{\mathcal{I}}$.

Examples

$$\mathtt{hasMother} \sqsubseteq \mathtt{hasParent}$$

"if a has b as mother, then a has b as parent"

$$\mathtt{isPartOf} \sqsubseteq \mathtt{hasPart}^{-}$$

"if a is part of b, then b has a as a part"

$$\mathtt{hasAncestor} \circ \mathtt{hasAncestor} \sqsubseteq \mathtt{hasAncestor}$$

"the ancestor of an ancestor is an ancestor"

$$\mathtt{hasPart} \sqcap \mathtt{hasBorderWith} \sqsubseteq \bot$$

"it is false that both a has b as a part and a has borders with b"

Acyclic TBoxes

- The presence of a TBox usually produces a great **growth in the complexity** of the calculus.
- For this reason, so-called **Acyclic TBoxes** are often considered.
- An Acyclic TBox is a **definitional** TBox **without cycles**, that is:
  - a TBox is said to be **definitional** when there appear at most one inclusion axiom of the form:

    $$A \equiv C$$

    for each atomic concept A;
  - a TBox is said to be **cyclic** or a set of **General Concept Inclusions** (GCIs), when there is a sequence of inclusion axioms $C_1 \sqsubseteq D_1, \ldots, C_n \sqsubseteq D_n$ and a set of concepts $A_1, \ldots, A_{n-1}$, such that, for every $1 < m < n$, A_m appears both in D_{m-1} and in C_m and A_1 appears both in D_n and in C_1.

Knowledge Bases

- A **terminological box** (TBox) is a finite set of concept inclusion axioms.
- An **assertional box** (ABox) is a finite set of assertion axioms.
- A **relational box** (RBox) is a finite set of role inclusion axioms.
- An **Knowledge Base** (KB) is a triple

$$\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{R})$$

where $\mathcal{T}$ is a TBox, $\mathcal{A}$ is an ABox and $\mathcal{R}$ is an RBox (each one possibly empty).

Reasoning

Reasoning tasks

- A **Reasoning task** provided by a DL system is a property of concepts or knowledge bases that the system is supposed to compute or verify.
- A reasoning task can be considered either **with respect to**:
  - an empty knowledge base,
  - a non-empty knowledge base but with empty TBox and RBox,
  - a non-empty knowledge base with acyclic TBox and empty RBox,
  - a non-empty knowledge base with empty RBox,
  - a non-empty knowledge base with acyclic RBox.

Knowledge base consistency

- A knowledge base $\mathcal{K}$ is said to be **consistent** when there is an interpretation $\mathcal{I}$ that satisfies every axiom in $\mathcal{K}$.
- In symbols $\mathcal{I} \models \mathcal{K}$.
- The **empty Knowledge Base** $\mathcal{KB} = \emptyset$ is assumed to be satisfied by every interpretation $\mathcal{I}$.
- The notion of KB consistency, as we will see, is a **central notion** among the reasoning tasks for DL.

Example

The KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where:

$$\mathcal{T} = \{ \quad \texttt{Female} \sqcap \texttt{Male} \sqsubseteq \ \bot \quad \}$$
$$\mathcal{A} = \{ \quad \forall \texttt{hasChild.Male(Marco)} \quad \}$$

is satisfiable in the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where:

- $\Delta^{\mathcal{I}} = \{v\}$,
- $\texttt{Marco}^{\mathcal{I}} = v$,
- $\texttt{Male}^{\mathcal{I}} = \{v\}$,
- $\texttt{Female}^{\mathcal{I}} = \emptyset$,
- $\texttt{hasChild}^{\mathcal{I}} = \{\langle v, v\rangle\}$.

Concept satisfiability

- A concept C is said to be **satisfiable** with respect to the knowledge base $\mathcal{K}$ when there exists an interpretation $\mathcal{I}$ satisfying $\mathcal{K}$, such that $C^{\mathcal{I}} \neq \emptyset$.

- This notion has been often considered with respect to the **empty KB**.

- Since **every interpretation** $\mathcal{I}$ is a model of the empty KB, then a concept is satisfiable when there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$.

Example

The concept:

$$C = \texttt{Female} \sqcap \forall \texttt{hasChild.Male}$$

is satisfiable with respect to the KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where:

$$\mathcal{T} = \{ \quad \texttt{Female} \sqcap \texttt{Male} \sqsubseteq \bot \quad \}$$
$$\mathcal{A} = \{ \quad \forall \texttt{hasChild.Male(Marco)} \quad \}$$

Since the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where:

- $\Delta^{\mathcal{I}} = \{v, w\}$,
- $\texttt{Marco}^{\mathcal{I}} = v$,
- $\texttt{Male}^{\mathcal{I}} = \{w\}$,
- $\texttt{Female}^{\mathcal{I}} = \{v\}$,
- $\texttt{hasChild}^{\mathcal{I}} = \{\langle v, w \rangle\}$.

is a model of $\mathcal{K}$ that satisfies C.

Concept Subsumption

- A concept D is said to **subsume** a concept C with respect to the knowledge base $\mathcal{K}$ when, in every interpretation $\mathcal{I}$ that satisfies $\mathcal{K}$, it holds that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- This notion has been often considered with respect to the **empty KB**.
- Since **every interpretation** $\mathcal{I}$ is a model of the empty KB, then a concept D subsumes a concept C when in every interpretation $\mathcal{I}$ it holds that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

Example

The concept Male is subsumed by the concept $\neg$Female with respect to the KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where:

$$\mathcal{T} = \{ \quad \texttt{Female} \sqcap \texttt{Male} \sqsubseteq \bot \quad \}$$
$$\mathcal{A} = \{ \quad \forall \texttt{hasChild.Male(Marco)} \quad \}$$

Suppose, on the contrary, that there is an interpretation $\mathcal{I}$ that satisfies every axiom in $\mathcal{K}$ but such that concept Male is not subsumed by the concept $\neg$Female. Hence $\texttt{Male}^{\mathcal{I}} \not\subseteq \texttt{Female}^{\mathcal{I}} \setminus \Delta^{\mathcal{I}}$. Then $\texttt{Male}^{\mathcal{I}} \cap \texttt{Female}^{\mathcal{I}} \neq \emptyset$. Hence $\mathcal{I}$ does not satisfies axiom $\texttt{Female} \sqcap \texttt{Male} \sqsubseteq \bot$ and, therefore, $\mathcal{I}$ is not a model of $\mathcal{K}$, a contradiction.

Entailment

- An axiom φ is said to be **entailed** by a knowledge base $\mathcal{K}$ when, in every model $\mathcal{I}$ of $\mathcal{K}$, it holds that $\varphi^{\mathcal{I}} = 1$.
- In symbols $\mathcal{K} \models \varphi$.
- If φ is a concept inclusion axiom, this notion coincides with **concept subsumption**.
- Since **every interpretation** $\mathcal{I}$ is a model of the empty KB, then an axiom φ is entailed by the **empty KB** when it is true in every interpretation $\mathcal{I}$.

Example

The axioms:

$$\neg\exists \texttt{hasChild}.\texttt{Female}(\texttt{Marco}) \qquad \text{and} \qquad \texttt{Female} \sqsubseteq \neg\texttt{Male}$$

are entailed by the KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where:

$$\mathcal{T} = \{ \quad \texttt{Female} \sqcap \texttt{Male} \sqsubseteq \bot \quad \}$$
$$\mathcal{A} = \{ \quad \forall \texttt{hasChild}.\texttt{Male}(\texttt{Marco}) \quad \}$$

We prove the first one. Suppose, on the contrary, that there is an interpretation $\mathcal{I}$ that satisfies every axiom in $\mathcal{K}$ but such that Marco is not an instance of the concept $\neg\exists \texttt{hasChild}.\texttt{Female}$. Hence $\texttt{Marco}^{\mathcal{I}} \notin (\exists \texttt{hasChild}.\texttt{Female})^{\mathcal{I}} \setminus \Delta^{\mathcal{I}}$. Then $\texttt{Marco}^{\mathcal{I}} \in (\exists \texttt{hasChild}.\texttt{Female})^{\mathcal{I}}$. By axiom $\texttt{Female} \sqcap \texttt{Male} \sqsubseteq \bot$ we have that $\texttt{Marco}^{\mathcal{I}} \in (\exists \texttt{hasChild}.\neg\texttt{Male})^{\mathcal{I}} = (\neg\forall \texttt{hasChild}.\texttt{Male})^{\mathcal{I}}$, contradicting axiom $\forall \texttt{hasChild}.\texttt{Male}(\texttt{Marco})$.

Reduction to knowledge base consistency

Each one of the above reasoning problems can be **reduced to knowledge base (in)consistency** in the following way:

- Concept C is **satisfiable with respect to the knowledge base** $\mathcal{K}$ if and only if the new knowledge base $\mathcal{K} \cup \{C(a)\}$ is consistent, where a is an individual name not occurring in $\mathcal{K}$.
- Concept D **subsumes concept** C **with respect to the knowledge base** $\mathcal{K}$ if and only if the new knowledge base $\mathcal{K} \cup \{(C \sqcap \neg D)(a)\}$ is inconsistent, where a is a new individual name.
- An axiom φ (either inclusion or assertion) is **entailed** by a knowledge base $\mathcal{K}$ if and only if the new knowledge base $\mathcal{K} \cup \{\neg\varphi\}$ is inconsistent. Here $\neg\varphi = \neg C(a)$, if $\varphi = C(a)$ and $\neg\varphi = C \sqcap \neg D(a)$, for a new individual name a, if $\varphi = C \sqsubseteq D$.