An Introduction to Description Logic V

Relations to Modal Logic

Marco Cerami

Palacký University in Olomouc
Department of Computer Science
Olomouc, Czech Republic

Olomouc, November 27^{th} 2014

european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

Preliminaries:

Modal Logic

Language and formulas

Language

- A countable set of propositional variables $Prop = \{p, q, \ldots\}$,
- the classical propositional constants $\top$ and $\bot$,
- the classical propositional connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$,
- two unary modal connectives $\Box$ and $\Diamond$.

Formulas

The set Φ of modal formulas is inductively built from *Prop* in the following way:

- Propositional variables and constants are formulas,
- if φ and ψ are formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\neg\varphi$ are formulas,
- if φ is a formula, then $\Box\varphi$ and $\Diamond\varphi$ are formulas.

Kripke models

A **Kripke frame** is a structure $\mathfrak{F} = \langle W, R \rangle$, where:

- W is a non-empty set of elements, often called **possible worlds**,
- $R \subseteq W \times W$ is a binary relation on W, called the **accessibility relation** of W.

A **Kripke model** is a structure $\mathfrak{M} = \langle W, R, V \rangle$, where:

- $\langle W, R \rangle$ is a Kripke frame,
- $\mathbf{V} : \mathbf{Prop} \longrightarrow \mathcal{P}(\mathbf{W})$ is a function that assigns a set of possible worlds to every propositional variable.

Satisfaction of a formula

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and $w \in W$, then:

$\mathfrak{M}, w \models p$	iff	$w \in V(p)$
$\mathfrak{M}, w \models \top$		always
$\mathfrak{M}, w \models \bot$		never
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	both $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \vee \psi$	iff	either $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \Box\varphi$	iff	for every $v \in W$ s.t. $R(w, v)$, it holds that $\mathfrak{M}, v \models \varphi$
$\mathfrak{M}, w \models \Diamond\varphi$	iff	there exists $v \in W$ s.t. $R(w, v)$ and $\mathfrak{M}, v \models \varphi$

Local and Global Satisfiability

- We say that a formula φ is **locally satisfiable**, if there exists a model $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$, such that

$$\mathfrak{M}, w \models \varphi$$

- We say that a formula φ is **globally satisfiable**, in a model $\mathfrak{M} = \langle W, R, V \rangle$, if φ is (locally) satisfiable in every point $w \in W$. In symbols:

$$\mathfrak{M} \models \varphi$$

Validity

- We say that a formula φ is **valid in a frame** $\mathfrak{F} = \langle W, R \rangle$, if for every model $\mathfrak{M} = \langle W, R, V \rangle$ and every $w \in W$, it holds that $\mathfrak{M}, w \models \varphi$. In symbols

$$\mathfrak{F} \models \varphi$$

- We say that a formula φ is **valid in a class of frames F** if it is valid in every frame $\mathfrak{F} \in \mathbf{F}$. In symbols:

$$\mathbf{F} \models \varphi$$

- We say that a formula φ is **valid**, if it is valid in every class of frames $\mathbf{F}$. In symbols:

$$\models \varphi$$

Semantic Consequence relations

Let $\Gamma \cup \varphi$ be a set of modal formulas and $\mathbf{M}$ a class of models, then:

- We say that a formula φ is a **local consequence** of Γ over $\mathbf{M}$, if for all models $\mathfrak{M} = \langle W, R, V\rangle \in \mathbf{M}$ and all points $w \in W$, it holds that
  - if $\mathfrak{M}, w \vDash \Gamma$, then $\mathfrak{M}, w \vDash \varphi$.

  In symbols: $\Gamma \vDash^{l}_{\mathbf{M}} \varphi$.

- We say that a formula φ is a **global consequence** of Γ over $\mathbf{M}$, if for all models $\mathfrak{M} = \langle W, R, V\rangle \in \mathbf{M}$ it holds that
  - if $\mathfrak{M} \vDash \Gamma$, then $\mathfrak{M} \vDash \varphi$.

  In symbols: $\Gamma \vDash^{g}_{\mathbf{M}} \varphi$.

Further notions

- A **universal modality** $\Box_U$ is a modality whose accessibility relation is the total relation $W \times W$.

- A **multi-modal** language, is a modal language with more than one couple of modal operators in the same language.

- For different couples $(\Box_1, \Diamond_1), \ldots, (\Box_m, \Diamond_m)$ of modal operators of a multi-modal languages, the **respective accessibility relations** $R_1, \ldots, R_m$ are supposed to be different relations on the domain.

- We are considering the framework of the **minimal multi-modal logic $\mathbf{K}_m$**.

Translating Description Logic into Multi-Modal Logic

Translation of the signature

Given a description signature $\mathbf{D} = \langle N_I, N_C, N_R \rangle$, we define the multi-modal language

$$\mathbf{L_D} := \mathbf{L} \cup \{\Box_R, : R \in N_R\} \cup \{\Diamond_R, : R \in N_R\}$$

where:

- $Prop_{\mathbf{D}} = \{p_A \colon A \in N_C\}$ is the set of propositional variables,
- $\mathbf{L}$ is the set of propositional connectives,
- $\{\Box_R, : R \in N_R\} \cup \{\Diamond_R, : R \in N_R\}$ is a set of unary modal operators.

We can define the **translation** $\tau : N_C \longrightarrow Prop_{\mathbf{D}}$ from the set of concept names into the set of propositional variables:

$$\tau(A) \quad := \quad p_A$$

Translation of complex concepts in $\mathcal{ALC}$

This translation can be inductively extended over the set of complex concept in $\mathcal{ALC}$ in the following way:

$$
\begin{aligned}
\tau(\neg C) &:= \neg\tau(C), \\
\tau(C \sqcap D) &:= \tau(C) \wedge \tau(D) \\
\tau(C \sqcup D) &:= \tau(C) \vee \tau(D) \\
\tau(\forall R.C) &:= \Box_R \tau(C) \\
\tau(\exists R.C) &:= \Diamond_R \tau(C)
\end{aligned}
$$

Translation of DL interpretations

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a DL interpretation, then we can **define the Kripke model** $\mathfrak{M}_{\mathcal{I}} = \langle W_{\mathcal{I}}, \{R_{\mathcal{I}} \colon R \in N_R\}, V_{\mathcal{I}} \rangle$, where:

- $W_{\mathcal{I}} = \Delta^{\mathcal{I}}$,
- for each role name $R \in N_R$, $R_{\mathcal{I}}$ is an **accessibility relation** on $W_{\mathcal{I}}$, i.e. a binary relation $R_{\mathcal{I}} \subseteq W_{\mathcal{I}} \times W_{\mathcal{I}}$, such that, for every $v, w \in W_{\mathcal{I}}$, it holds that

$$R_{\mathcal{I}}(v, w) \qquad \text{iff} \qquad R^{\mathcal{I}}(v, w),$$

- for each element $v \in W_{\mathcal{I}}$ and for every **propositional variable** $p_A \in Prop_{\mathcal{D}}$, it holds that

$$v \in V_{\mathcal{I}}(p_A) \qquad \text{iff} \qquad v \in A^{\mathcal{I}}.$$

Soundness of the translation

For every $\mathcal{ALC}$ concept C and every $v \in \Delta^{\mathcal{I}}$, it holds that

$$v \in V_{\mathcal{I}}(\tau(C)) \qquad \text{iff} \qquad v \in C^{\mathcal{I}}.$$

$$\begin{tikzcd}
\textit{DL concepts} \arrow[r, dashed, "\tau"] \arrow[d, leftrightarrow] & \textit{modal formulas} \arrow[d, leftrightarrow] \\
\textit{DL interpretations} \arrow[r, "\cdot^{\mathcal{I}}"] & \textit{Kripke models}
\end{tikzcd}$$

Translation of axioms

- A **concept inclusion axiom** $C \sqsubseteq D$ **can not be translated** into a modal formula without an **universal modality**:

$$\Box_U(\tau(C) \to \tau(D))$$

- Nevertheless, the **satisfiability of an axiom** in an interpretation $\mathcal{I}$ corresponds to the notion of **global satisfiability**:

$$\mathcal{I} \models C \sqsubseteq D \qquad \text{iff} \qquad \mathfrak{M}_{\mathcal{I}} \models_g \tau(C) \to \tau(D)$$

- An **assertion axiom** can not be translated into any modal formula.

Translation of the reasoning tasks

- Since **we have not a translation of assertions**:
  - we **can not obtain** a translation of every reasoning task from a translation of knowledge base consistency,
  - **subsumption** and **entailment** coincide.

- A **TBox** $\mathcal{T} = \{C_i \sqsubseteq D_i : 0 \leq i \leq n\}$ is satisfiable iff the formula

$$\bigwedge_{i=0}^{n} \tau(C_i) \rightarrow \tau(D_i)$$

  is globally satisfiable.

- A concept C is **satisfiable w. r. t. a TBox** $\mathcal{T}$ iff $\tau(C)$ is **locally satisfiable** in a model of $\mathcal{T}$.

- An inclusion axiom $C \sqsubseteq D$ is **entailed by a TBox** $\mathcal{T}$ iff the formula $\tau(C) \rightarrow \tau(D)$ is a **global consequence** of $\tau[\mathcal{T}]$.

Translating Modal Logic into Description Logic

Translation of the signature

Given a multi-modal language $\mathbf{L} = \{\wedge, \vee, \neg\} \cup \{\Box_i, : i \in I\} \cup \{\Diamond_i, : i \in I\}$ and a set of propositional variables $Prop = \{p_1, p_2, \ldots\}$, we define the description signature $\mathcal{D}_{\mathbf{L}} = \langle N_I^{\mathbf{L}}, N_C^{\mathbf{L}}, N_R^{\mathbf{L}} \rangle$, where:

- $N_I^{\mathbf{L}} := \emptyset$,
- $N_C^{\mathbf{L}} := \{A_p : p \in Prop\}$,
- $N_R^{\mathbf{L}} := \{R_i : \Box_i \in \mathbf{L}\}$.

We can define the **translation** $\rho : Prop \longrightarrow N_C^{\mathbf{L}}$ from the set of propositional variable into the set of concept names:

$$\rho(p) \quad := \quad A_p$$

Translation of multi-modal formulas

This translation can be inductively extended over the set of complex concepts in the following way:

$$
\begin{aligned}
\rho(\neg\varphi) &:= \neg\rho(\varphi) \\
\rho(\varphi \wedge \psi) &:= \rho(\varphi) \sqcap \rho(\psi) \\
\rho(\varphi \vee \psi) &:= \rho(\varphi) \sqcup \rho(\psi) \\
\rho(\Box_i\varphi) &:= \forall R_i.\rho(\varphi) \\
\rho(\Diamond_i\varphi) &:= \exists R_i.\rho(\varphi)
\end{aligned}
$$

Translation of Kripke models

Let $\mathfrak{M} = \langle W, \{R_1, \ldots, R_m\}, V\rangle$ be a Kripke model, then we can **define the DL interpretation** $\mathcal{I}_{\mathfrak{M}} = (\Delta^{\mathcal{I}_{\mathfrak{M}}}, \cdot^{\mathcal{I}_{\mathfrak{M}}})$, where:

- $\Delta^{\mathcal{I}_{\mathfrak{M}}} := W$,
- for each concept name $A_p \in N_C^{\mathbf{L}}$, the interpretation $A_p^{\mathcal{I}_{\mathfrak{M}}}$ is a **subset** of $\Delta^{\mathcal{I}_{\mathfrak{M}}}$, such that, for every $v \in \Delta^{\mathcal{I}_{\mathfrak{M}}}$, it holds that

$$v \in A_p^{\mathcal{I}_{\mathfrak{M}}} \qquad \text{iff} \qquad v \in V(p),$$

- for each role name $R_i \in N_R^{\mathbf{L}}$, $R_i^{\mathcal{I}_{\mathfrak{M}}}$ is a **binary relation** $R_i^{\mathcal{I}_{\mathfrak{M}}}$ in $\Delta^{\mathcal{I}_{\mathfrak{M}}}$, such that, for every $v, w \in \Delta^{\mathcal{I}_{\mathfrak{M}}}$, it holds that

$$R_i^{\mathcal{I}_{\mathfrak{M}}}(v, w) = R_i(v, w).$$

Soundness of the translation

For every $\mathbf{K}_m$ formula φ and every $v \in W$, it holds that

$$v \in (\rho(\varphi))^{\mathcal{I}_{\mathfrak{M}}} \qquad \text{iff} \qquad v \in V(\varphi).$$

$$\begin{tikzcd}
\textit{DL concepts} \arrow[d, leftrightarrow] & \textit{modal formulas} \arrow[l, dashed, "\rho"'] \arrow[d, leftrightarrow] \\
\textit{DL interpretations} & \textit{Kripke models} \arrow[l, "\cdot_{\mathfrak{M}}"]
\end{tikzcd}$$

Translation of the logic

- A formula φ is **locally satisfiable**, iff the concept $\rho(\varphi)$ is satisfiable w.r.t. the empty TBox.
- A formula φ is **globally satisfiable**, in a model $\mathfrak{M} = \langle W, R, V \rangle$, iff the inclusion axiom $\top \sqsubseteq \rho(\varphi)$ is satisfied in the interpretation $\mathcal{I}_{\mathfrak{M}}$.
- A formula φ is **valid** iff the concept $\rho(\varphi)$ is subsumed by concept $\top$.
- A formula φ is a **local consequence** of a set of formulas Γ iff the assertion axiom $\rho(\varphi)(a)$ is entailed by the ABox $\{\rho(\psi)(a) \colon \psi \in \Gamma\}$.
- A formula φ is a **global consequence** of Γ over **M**, iff the inclusion axiom $\top \sqsubseteq \rho(\varphi)$ is entailed by the TBox $\{\top \sqsubseteq \rho(\psi) \colon \psi \in \Gamma\}$

Bilateral relations

Translations between the syntaxes

For every DL Concept C and every modal formula φ it holds that:

- $\rho(\tau(C)) = C$,
- $\tau(\rho(\varphi)) = \varphi$.

$$\begin{tikzcd}
\textit{DL concepts} \arrow[r, bend left, "\tau"] & \textit{modal formulas} \arrow[l, bend left, "\rho"]
\end{tikzcd}$$

Translations between the semantics

For every DL interpretation $\mathcal{I}$ and every Kripke model $\mathfrak{M}$ it holds that:

- $\mathcal{I} = \mathcal{I}_{\mathfrak{M}_{\mathcal{I}}}$,
- $\mathfrak{M} = \mathfrak{M}_{\mathcal{I}_{\mathfrak{M}}}$.

$$
\begin{tikzcd}
\textit{DL interpretations} \arrow[r, bend left, "\cdot_{\mathcal{I}}"] & \textit{Kripke models} \arrow[l, bend left, "\cdot_{\mathfrak{M}}"]
\end{tikzcd}
$$

The respective loics do not coincide

- Some intermediate notions proper of Modal Logic, such as **validity w.r.t. a frame** or a **class of frames** are not expressible in DL, due to its lack of structurality.

- Even though an ABox is considered as a translation of local consequence, this notion is not a translation of the notion of general ABox reasoning, since it is related to a particular form of what in the literature is called **local ABox**.

- The notions of **global consequence** in ML and **entailment from a TBox** in DL seem to coincide.

- The notions of **local satisfiability** in ML and **satisfiability w.r.t. an empty TBox** in DL seem to coincide.