An Introduction to Description Logic VI

Relations to Formal Concept Analysis

Marco Cerami

Palacký University in Olomouc
Department of Computer Science
Olomouc, Czech Republic

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Introduction

General Remarks

- **Differently** from the case of modal and first order logics, Formal Concept Analysis is a formalism that appears to be **deeply dissimilar** from DL.
- The result of an account of the relations between FCA and DL can **depend on the point of view** or on the goals of this account.
- We are mainly following the ideas provided in the PhD thesis **Learning Description Logic Knowledge Bases from Data using methods from Formal Concept Analysis** by F. Distel.
- Nevertheless, **our purpose** is to see each formalism from the point of view of the other.
- This makes our exposition **quite different** from Distel's one.

Some dissimilarities

At first sight there are **deep differences** between both formalisms. Some of them are among the following:

- The **formal language** of FCA is quite limited if compared to the variety of concept constructors in DL.
- The basic information is usually **entirely determined** in FCA, while in DL is left open to interpretation.
- The **goals** of each formalism appear to be hardly performed by the other.

In the following we discuss these items with some more detail.

Language limitations

- FCA lacks concept constructors, like disjunction or negation, but above all the use of **roles**.
- The lack of roles could be overcome by considering the framework of **relational concept analysis**, but this goes beyond our scope.
- **Other concept constructors** are not expressible in FCA.
- We will consider the **fragment** of DL with only $\sqcap$ in the language. Following Distel's dissertation, we call this fragment $\mathcal{L}_{\sqcap}$.
- For reasons related to the particular nature of FCA, we will consider $\mathcal{L}_{\sqcap}$ with the constructor for nominals $\mathcal{L}_{\sqcap}\mathcal{O}$.

Closed vs open world assumption

- In FCA is usually accepted the **closed world assumption**. That is, if a relation xIy between an object end an attribute is not explicitly stated in a context, then it does not hold.

- In DL is usually accepted the **open world assumption**. That is, even though a relation $C(a)$ between an individual and a concept is not explicitly stated in a context, it does not mean that it does not hold.

- The open and closed world assumption are concerned also with the **existence** of objects or individuals not explicitly defined at the beginning, but, without roles and negation, there is no difference between DL and FCA under this point of view.

Interpretaions and tables

- This difference is related to the former one.
- Indeed, the **closed world assumption** is due to the fact that a **formal context** is a basic starting point for analysis in FCA.
- In a table all the **basic information is exhaustively stated**.
- In DL, the place where all the information is exhaustively stated are **interpretations**.
- But interpretations in DL are not a basic starting point, rather a **complementary tool**.
- The basic information contained in a **knowledge base** is open to be **realized, enriched and fixed** by interpretations.

Example

According to the KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where:

$$\mathcal{T} = \{ \quad \texttt{Female} \sqcap \texttt{Male} \sqsubseteq \bot \quad \}$$
$$\mathcal{A} = \{ \quad \forall \texttt{hasChild.Male(Marco)} \quad \}$$

individual Marco **can be interpreted** as an instance of Male or Female or neither, but not both.

According to the following formal context, that carries part of the information in $\mathcal{K}$:

I	Female	Male	hasChild.Male
Marco			×

object Marco **is definitely** neither Male nor Female.

Reasoning services

- In this sense, notions such **satisfiability of concepts** do not make sense in FCA, even though we can entirely translate concepts.
- On the other side, the **extensional determinacy** of attributes and classes is hardly accounted by DL syntax, and a constant appeal to **a particular interpretation** is always needed when translating concepts.
- This is due to the fact that in FCA there is no need to **range** over different contexts, while in DL this is the expected behavior.

Preliminaries:

Formal Concept Analysis

Formal contexts

A **formal context** is a triple $\langle X, Y, I \rangle$ where:

- X is a set of **objects**,
- Y is a set of **attributes**,
- $I \subseteq X \times Y$ is a **binary relation** between X and Y.

I	y_1	y_2	y_3	y_4
x_1	×	×	×	×
x_2	×		×	×
x_3		×	×	×
x_4		×	×	×
x_5	×			

Formal Concepts

- The operator $\cdot^{\uparrow} \colon 2^X \longrightarrow 2^Y$ is defined on every $A \subseteq X$ by:

$$A^{\uparrow} = \{y \in Y | \text{ for each } x \in A \colon \langle x, y \rangle \in I\}$$

- The operator $\cdot^{\downarrow} \colon 2^Y \longrightarrow 2^X$ is defined on every $B \subseteq Y$ by:

$$B^{\downarrow} = \{x \in X | \text{ for each } y \in B \colon \langle x, y \rangle \in I\}$$

- A **formal concept** is a pair $\langle A, B \rangle$, with $A \subseteq X$ and$B \subseteq Y$ such that:

$$A = B^{\downarrow} \qquad \text{and} \qquad B = A^{\uparrow}$$

- For two formal concepts $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$, we have that:

$$\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle \qquad \text{iff} \qquad A_1 \subseteq A_2 \qquad \text{iff} \qquad B_2 \subseteq B_1$$

- A **concept lattice** $\mathcal{B}(X, Y, I)$ is the collection of all formal concepts of a formal context $\langle X, Y, I \rangle$.

Attribute Implications

- An **attribute implication** is an expression of the form:

$$A \Rightarrow B$$

where $A, B \subseteq Y$ are sets of attributes.

- An attribute implication $A \Rightarrow B$ is **true in a set** $M \subseteq Y$ of attributes iff

$$A \subseteq M \quad \text{imples} \quad B \subseteq M$$

- An attribute implication $A \Rightarrow B$ is **true in a formal context** $\langle X, Y, I \rangle$ iff it is true in every set of the family:

$$\mathcal{M} = \{\{x\}^{\uparrow} \,|\, x \in X\}$$

Translating Description Logic into Formal Concept Analysis

Syntax

- Given a description signature $\mathbf{D} = \langle N_I, N_C \rangle$, we define a sets of objects an attributes:
  - $N_I \subseteq X$
  - $N_C \subseteq Y$
- We can define the **translation** $\tau : \mathbf{D} \longrightarrow X \cup Y$ from the signature into the sets of objects and attributes:

$$\begin{array}{rcl} \tau(a) & := & x \in X \\ \tau(A) & := & y \in Y \end{array}$$

and **extend** the translation to **complex concepts**:

$$\begin{array}{rcl} \tau(\{a_1, \ldots, a_n\}) & := & \{\tau(a_1), \ldots, \tau(a_n)\} \\ \tau(\{A_1 \sqcap \ldots \sqcap A_m\}) & := & \{\tau(A_1), \ldots, \tau(A_m)\} \end{array}$$

Semantics

An **interpretation** $\mathcal{I}$ is translated into a **formal context**

$$\langle X_{\mathcal{I}}, Y_{\mathcal{I}}, I_{\mathcal{I}} \rangle$$

where:

- $X_{\mathcal{I}} = \Delta^{\mathcal{I}}$,
- $Y_{\mathcal{I}} = \{A^{\mathcal{I}} \colon A \in N_C\}$,
- for every $v \in \Delta^{\mathcal{I}}$ and $A \in N_C$, it holds that

$$(v, A^{\mathcal{I}}) \in I_{\mathcal{I}} \quad \text{iff} \quad v \in A^{\mathcal{I}}.$$

Assertion axioms

A set $\mathcal{A}$ of **concept assertion axioms** or ABox can be viewed as a **partial context**.

$$\langle X_\mathcal{A}, Y_\mathcal{A}, I_\mathcal{A} \rangle$$

where:

- $X_\mathcal{A}$ are the individual names appearing in $\mathcal{A}$,
- $Y_\mathcal{A}$ are the atomic concept appearing in $\mathcal{A}$,
- for every $a \in X_\mathcal{A}$ and $A \in Y_\mathcal{A}$, it holds that

$$(\tau(a), \tau(A)) \in I_\mathcal{A} \qquad \text{iff} \qquad A(a) \in \mathcal{A}.$$

Hence, an ABox $\mathcal{A}$ is **satisfiable** if its translation $\langle X_\mathcal{A}, Y_\mathcal{A}, I_\mathcal{A} \rangle$ can be extended to a formal context. That is, it is always **trivially satisfiable**.

Inclusion axioms

A set $\mathcal{T}$ of **concept inclusion axioms** or TBox can be viewed as a set $T_{\mathcal{T}}$ of **attribute implications** or theory, where

$$\tau(C \sqsubseteq D) \quad = \quad \tau(C) \Rightarrow \tau(D).$$

Hence, a TBox $\mathcal{T}$ is **satisfiable** if there exists a formal context $\langle X, Y, I \rangle$ such that $T_{\mathcal{T}}$ is true in $\langle X, Y, I \rangle$.

Reasoning tasks

- A knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is **consistent** if there exists a formal context $\langle X, Y, I \rangle$ which extends $\langle X_{\mathcal{A}}, Y_{\mathcal{A}}, I_{\mathcal{A}} \rangle$ where $T_{\mathcal{T}}$ is true.
- A concept C is **satisfiable w.r.t. a knowledge base** $\mathcal{K}$ if there exists a formal context $\langle X, Y, I \rangle$ satisfying $\mathcal{K}$, where $\tau(C)^{\downarrow} \neq \emptyset$.
- A concept C is **subsumed** by concept D w.r.t. a knowledge base $\mathcal{K}$ if for every formal context $\langle X, Y, I \rangle$ satisfying $\mathcal{K}$, it holds that $\tau(C) \Rightarrow \tau(D)$.
- An axiom φ is **entailed** by a knowledge base $\mathcal{K}$ if every formal context $\langle X, Y, I \rangle$ satisfying $\mathcal{K}$ satisfies $\tau(\varphi)$.

Translating Formal Concept Analysis into Description Logic

Objects and attributes

Given a formal context $\mathbb{K} = \langle X, Y, I \rangle$, we define the description signature $\mathbf{D}_{\mathbb{K}} = \langle N_I^{\mathbb{K}}, N_C^{\mathbb{K}} \rangle$, where:

- $N_I^{\mathbb{K}} := X$,
- $N_C^{\mathbb{K}} := Y$,

We can define the **translation** $\rho : X \cup Y \longrightarrow \mathbf{D}_{\mathbb{K}}$ from the sets of objects and attributes into the signature:

$$
\begin{aligned}
\rho(x) &:= x \\
\rho(y) &:= A_y
\end{aligned}
$$

and **extend** the translation to **sets** of objects and attributes:

$$
\begin{aligned}
\rho(\{x_1, \ldots, x_n\}) &:= \{x_1, \ldots, x_n\} \\
\rho(\{y_1, \ldots, y_m\}) &:= \rho(y_1) \sqcap \ldots \sqcap \rho(y_m)
\end{aligned}
$$

The binary relation

Given a formal context $\mathbb{K} = \langle X, Y, I \rangle$, we define the interpretation $\mathcal{I}_{\mathbb{K}} = (\Delta^{\mathcal{I}_{\mathbb{K}}}, \cdot^{\mathcal{I}_{\mathbb{K}}})$ where:

- $\Delta^{\mathcal{I}_{\mathbb{K}}}$ is a non-empty set.
- $\cdot^{\mathcal{I}_{\mathbb{K}}}$, is a function with the signature $\mathbf{D}_{\mathbb{K}}$ as domain such that:
  - $x^{\mathcal{I}_{\mathbb{K}}} \in \Delta^{\mathcal{I}_{\mathbb{K}}}$, for every $x \in N_I^{\mathbb{K}}$,
  - $\rho(y)^{\mathcal{I}_{\mathbb{K}}}$ is a set in $\Delta^{\mathcal{I}_{\mathbb{K}}}$, for every $\rho(y) \in N_C^{\mathbb{K}}$.
  - $x^{\mathcal{I}_{\mathbb{K}}} \in \rho(y)^{\mathcal{I}_{\mathbb{K}}} \quad \text{iff} \quad (x, y) \in I$.

The operator $\cdot^{\uparrow}$

A translation of the operator $\cdot^{\uparrow}$ can be defined in the following way:

$$\rho(A^{\uparrow}) = \sqcap\{A_y \in N_C^{\mathbb{K}} \colon \rho(A)^{\mathcal{I}_{\mathbb{K}}} \subseteq A_y^{\mathcal{I}_{\mathbb{K}}}\}, \qquad \text{for every } A \subseteq X.$$

that is:

1. take a set of **objects** $A = \{x_1, \ldots, x_n\}$,
2. obtain the **nominal concept** $\rho(A) = \{\rho(x_1), \ldots, \rho(x_n)\}$,
3. obtain the **interpretation** $\rho(A)^{\mathcal{I}_{\mathbb{K}}}$,
4. consider all the **atomic concepts** $A_y \in N_C^{\mathbb{K}}$ such that $\rho(A)^{\mathcal{I}_{\mathbb{K}}} \subseteq A_y^{\mathcal{I}_{\mathbb{K}}}$,
5. the **conjunction** of those A_y's is exactly the concept $\rho(A^{\uparrow})$.

The operator $\cdot^{\downarrow}$

A translation of the operator $\cdot^{\downarrow}$ can be defined in the following way:

$$\rho(B^{\downarrow}) = \{x \in N_I^{\mathbb{K}} \colon x \in \rho(B)^{\mathcal{I}_{\mathbb{K}}}\}, \qquad \text{for every } B \subseteq Y.$$

that is:

1. take a set of **attributes** $B = \{y_1, \ldots, y_m\}$,
2. obtain the **concept conjunction** $\rho(B) = \rho(y_1) \sqcap \ldots \sqcap \rho(y_m)$,
3. obtain the **interpretation** $\rho(B)^{\mathcal{I}_{\mathbb{K}}}$,
4. consider all the **individual names** $x \in N_I^{\mathbb{K}}$ such that $x^{\mathcal{I}_{\mathbb{K}}} \in B^{\mathcal{I}_{\mathbb{K}}}$,
5. the **nominal concept** $\{x_1, \ldots, x_n\}$ built up from these x's is exactly the concept $\rho(B^{\downarrow})$.

Formal concepts

The translation $\rho(\langle A, B\rangle)$ of a **formal concept** is then a pair

$$\langle \rho(A), \rho(B)\rangle,$$

where:

- $\rho(A) = \{\rho(x_1), \ldots, \rho(x_n)\}$ is a **nominal concept**, built up from the translations of the elements of A,
- $\rho(B) = \rho(y_1) \sqcap \ldots \sqcap \rho(y_m)$ is a **conjunction** of atomic concepts, built up from the translations of the elements of A,
- $\rho(A)^{\mathcal{I}_{\mathbb{K}}} = \rho(B)^{\mathcal{I}_{\mathbb{K}}}$.

Attribute Implications

A set T of **attribute implications** or theory can be translated as a set $\mathcal{T}_T$ of **concept inclusion axioms** or TBox, where

$$\rho(A \Rightarrow B) \quad = \quad \rho(A) \sqsubseteq \rho(B).$$

Hence, a theory T is **true in a formal context** $\mathbb{K} = \langle X, Y, I \rangle$ if the interpretation $\mathcal{I}_{\mathbb{K}}$ satisfies every inclusion axiom in $\mathcal{T}_T$.

More expressive languages

Adding further constructors

- Even though **other concept constructors** are not expressible in FCA, we can consider complex concepts as basic attributes.
- The obvious shortcoming is that, even with a limited machinery, we can have **infinite complex concepts**:
  - $\exists \texttt{isMarriedTo}.\top$,
  - $\exists \texttt{isMarriedTo}.\exists \texttt{isMarriedTo}.\top$,
  - $\exists \texttt{isMarriedTo}.\exists \texttt{isMarriedTo}.\exists \texttt{isMarriedTo}.\top$,
  - ...
- So, there is the need of **limiting the number** of complex concepts in order to manage them by means of a finite set of attributes.

Effects of the open world assumption

- Consider the KB $\mathcal{K} = (\mathcal{A})$, where:

$$\mathcal{T} = \{ \quad \texttt{Female} \sqcap \texttt{Male} \sqsubseteq \bot \quad \}$$
$$\mathcal{A} = \{ \quad \texttt{Male} \sqcap \exists \texttt{isMarriedTo.Female(Marco)} \quad \}$$

- If we use **our former definition** for the operator $\cdot^{\downarrow}$ we obtain the undesired consequence that $\rho(\{\texttt{Female}\})^{\downarrow} = \emptyset$.

- For this reason Distel defines the operator $\cdot^{\downarrow}$ **directly on interpretations**:

$$C^{\downarrow} = C^{\mathcal{I}}, \qquad \text{for every concept } C.$$

- As a consequence, the representation of a **formal concept** in DL is no more a pair of DL concepts.

Model based most specific concept

- Let $\mathcal{L}$ be the set of **all possible concepts** from a given signature, $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an interpretation and $X \subseteq \Delta^{\mathcal{I}}$. The **model based most specific concept** of X is a concept Csuch that:
  - $X \subseteq C^{\mathcal{I}}$,
  - for every concept $D \in \mathcal{L}$ such that $X \subseteq D^{\mathcal{I}}$ it holds that $C \sqsubseteq D$.
- The model based most specific concept is the way to **represent the $\cdot^{\uparrow}$ operator** in DL.
- Now the representation of a **formal concept** in DL is a pair:

$$\langle X, C \rangle \in \Delta^{\mathcal{I}} \times \mathcal{L}$$

  where
  - $X = C^{\mathcal{I}}$,
  - C is the most specific concept of X.

Cyclic interpretations

Let's take an example from Distel's dissertation. Consider the signature $\mathbf{D} = (N_C, N_R)$, where:

- $N_C = \{\texttt{Male},\texttt{Female}\}$,
- $N_R = \{\texttt{isMarriedTo}\}$,

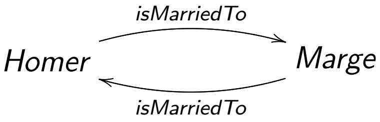
and the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, with

- $\Delta^{\mathcal{I}} = \{\texttt{Homer},\texttt{Marge}\}$,
- $\texttt{Male}^{\mathcal{I}} = \{\texttt{Homer}\}$,
- $\texttt{Female}^{\mathcal{I}} = \{\texttt{Marge}\}$,
- $\texttt{isMarriedTo}^{\mathcal{I}} = \{(\texttt{Homer},\texttt{Marge}),(\texttt{Marge},\texttt{Homer})\}$,

- Define the concept:

$$C_k = \exists \mathtt{isMarriedTo}. \overset{k \text{ times}}{\ldots} \exists \mathtt{isMarriedTo}. \top$$

- For every $k \in \mathbb{N}$ we have that $C_k^{\mathcal{I}} = \{\mathtt{Homer}, \mathtt{Marge}\}$:



- Moreover, $C_k \sqsubseteq Cj$ if and only if $k \geq j$, for every $k, j \in \mathbb{N}$.

- Now, suppose that D is the most specific concept of the set $\{\texttt{Homer},\texttt{Marge}\}$, that is, $D = \{\texttt{Homer},\texttt{Marge}\}^{\uparrow}$, then:
  - since $\{\texttt{Homer},\texttt{Marge}\} \subseteq D^{\mathcal{I}}$, then $D \neq \bot$,
  - since for every $k \in \mathbb{N}$ it holds that $\{\texttt{Homer},\texttt{Marge}\} \subseteq C^{\mathcal{I}}$, then $D \sqsubseteq C_k$, for every $k \in \mathbb{N}$.
- Hence $\{\texttt{Homer},\texttt{Marge}\}^{\uparrow}$ **can not exists**.
- This is true for **standard semantics**. In the dissertation Distel proves that a model based most specific concept always exists if we consider the so-called **greatest-fixpoint semantics**.
- Under greatest-fixpoint semantics it can be defined a general framework for **using FCA methods inside DL**.

Induced contexts

- A central notion of this general framework is that of **induced contexts**.
- The starting point are a **finite interpretation** $\mathcal{I}$ and a **finite set of complex concepts** Y.
- The **context induced by** $\mathcal{I}$ **and** Y is the formal context

$$\mathbb{K}_{\mathcal{I},Y} = \langle \Delta^{\mathcal{I}}, Y, I_{\mathcal{I},Y} \rangle$$

where

$$I_{\mathcal{I},Y} = \{(v, C) \colon C \in Y \text{ and } v \in C^{\mathcal{I}}\}.$$