An Introduction to Description Logic VII

Structural subsumption algorithms

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Introduction

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- A **structural subsumption algorithm** is one of first kind of procedures for DL. The name is due to the fact that they look at the **syntactical structure** of concepts.
- It is suitable for solving **concept subsumption** with respect to **empty knowledge bases** in DL languages with **low expressivity**.
- We are mainly following the 1984 paper **The Tractability of Subsumption in Frame-Based Description Languages**, by R.J. Brachman and H.J. Levesque.
- The advantage structural subsumption algorithms is that they are relatively **fast** and **simple**.
- The disadvantage is that they are **incomplete for more expressive languages**.

The language $\mathcal{FL}^-$

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- The name $\mathcal{FL}$ stands for **frame language** because it has more or less the same expressive power of frame-based systems.
- Frame languages were studied in the 80's.
- Below we define the language $\mathcal{FL}^-$:

$$
\begin{array}{lcll}
C, D & \longrightarrow & A & \text{atomic concept} \\
 & & C \sqcap D & \text{conjunction} \\
 & & \forall R.C & \text{value restriction} \\
 & & \exists R.\top & \text{restricted existential quantif.}
\end{array}
$$

Consistency and satisfiability in $\mathcal{FL}^-$

- In $\mathcal{FL}^-$ concepts and axioms are **trivially satisfiable**.
- The reason for this is that in $\mathcal{FL}^-$ there is **no negation**.
- Hence a **trivial model** $\mathcal{I}_{\mathbf{D}} = (\Delta^{\mathcal{I}_{\mathbf{D}}}, \cdot^{\mathcal{I}_{\mathbf{D}}})$ for any concept or knowledge base on a given signature $\mathbf{D} = \langle N_I, N_C, N_R \rangle$ in the following way:
  - $\Delta^{\mathcal{I}_{\mathbf{D}}} = \{v\}$,
  - $a^{\mathcal{I}_{\mathbf{D}}} = v$, for every individual name $a \in N_I$,
  - $A^{\mathcal{I}_{\mathbf{D}}} = \Delta^{\mathcal{I}_{\mathbf{D}}}$, for every concept name $A \in N_C$,
  - $R^{\mathcal{I}_{\mathbf{D}}} = \Delta^{\mathcal{I}_{\mathbf{D}}} \times \Delta^{\mathcal{I}_{\mathbf{D}}}$, for every role name $R \in N_C$.

Concept Subsumption

- A concept D is said to **subsume** a concept C when, in every interpretation $\mathcal{I}$ it holds that

$$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}.$$

- We will consider this notion with respect to the **empty KB**.
- Differently from satisfiability, in $\mathcal{FL}^-$ it has **no trivial solution**, since the trivial model above is just one among all possible interpretations.

Example

For example, concept

$$\texttt{Person}$$

is **not subsumed** by concept

$$\texttt{Person} \sqcap \texttt{Male}.$$

Indeed, even though in the trivial model $\mathcal{I}_{\mathbf{D}}$ the inclusion $\texttt{Person}^{\mathcal{I}_{\mathbf{D}}} \sqsubseteq \texttt{Person} \sqcap \texttt{Male}^{\mathcal{I}_{\mathbf{D}}}$ holds, nevertheless, in the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where:

- $\Delta^{\mathcal{I}} = \{v, w\}$,
- $\texttt{Person}^{\mathcal{I}} = \{v\}$,
- $\texttt{Male}^{\mathcal{I}} = \{w\}$,

we have that $\texttt{Person}^{\mathcal{I}} = \{v\} \not\subseteq \{w\} = \texttt{Person}^{\mathcal{I}} \cap \texttt{Male}^{\mathcal{I}}$.

The structural subsumption algorithm for $\mathcal{FL}^-$

Structural subsumption algorithm $SUBS?[D, C]$ from [Brachman and Levesque, 1984]

1: Flatten both C and D by removing all nested $\sqcap$ operators.

2: Collect all arguments to an $\forall R.$ for a given role R.

3: Assuming that $C := C_1 \sqcap \ldots \sqcap C_n$ and $D := D_1 \sqcap \ldots \sqcap D_m$, then return **true** iff for each C_i:

(a) if D_i is an atom or a $\exists R.\top$, then one of C_j is D_i.

(b) if D_i is $\forall R.E$ then one of the C_j is $\forall R.F$, where $SUBS?[F, E]$.

Behavior

- From step 1 we have:

$$((C_1 \sqcap C_2) \sqcap C_3) \sqcap (C_4 \sqcap C_5) \quad \leadsto \quad C_1 \sqcap C_2 \sqcap C_3 \sqcap C_4 \sqcap C_5$$

  which means that the conjunctions are treated as **sets of concepts**.

- From step 2 we have:

$$\forall R.C_1 \sqcap \forall R.(C_2 \sqcap \forall R.C_3) \quad \leadsto \quad \forall R.(C_1 \sqcap C_2 \sqcap \forall R.C_3)$$

  which is possible since with classical semantics the following equivalence **always holds**:

$$\forall R.C_1 \sqcap \forall R.C_2 \equiv \forall R.(C_1 \sqcap C_2)$$

- After steps 1 and 2 we obtain **normalized concepts** with:
  - sets of atomic and quantified concepts...
  - which are eventually inside the scope of universal quantifiers...
  - that appear only once every role and nesting degree.

- From step 3 the algorithm **inductively checks** whether every concept in the consequent appears in the antecedent:

$$\underline{C_1} \sqcap C_2 \sqcap \forall R.(C_3 \sqcap \underline{C_4}) \quad \overset{\checkmark}{\sqsubseteq} \quad \underline{C_1} \sqcap \forall R.\underline{C_4}$$

$$\underline{C_1} \sqcap C_2 \sqcap \forall R.(C_3 \sqcap C_4) \quad \overset{!}{\not\sqsubseteq} \quad \underline{C_1} \sqcap C_4 \sqcap \forall R.C_2$$

Soundness

- By **induction** on the nesting degree of C and D.
- Suppose that $SUBS?[D, C]$ **returns "true"**.
- If the nesting degree of both concepts is 0 the result is straightforward.
- Let the nesting degree of some concept be $\geqslant 0$:
- then **either** every conjunct D_i appears in C,
- . . . **or** it is of the form $\forall R.E$.
- In the second case there is a conjunct C_i in C of the form $\forall R.E$ such that $SUBS?[F, E]$ **returns "true"**.
- By i.h. we have that for every interpretation $\mathcal{I}$ it holds $E^{\mathcal{I}} \subseteq F^{\mathcal{I}}$.
- Hence for every interpretation $\mathcal{I}$ it holds $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

Completeness

- In order to prove completeness, we assume that $SUBS?[D, C]$ **returns "false"**.

- We will consider three cases and, for each of them, we **define an interpretation** $\mathcal{I}$ that does not satisfy the subsumption.

- Let $C := C_1 \sqcap \ldots \sqcap C_n$ and $D := D_1 \sqcap \ldots \sqcap D_m$, then $SUBS?[D, C]$ **returns "false"** when:

    1. some **atomic** D_i does not appear in C,
    2. some D_i is an **existentially quantified concept** $\exists R.\top$ and does not appear in C,
    3. some D_i is a **universally quantified concept** $\forall R.F$ and for every concept $\forall R.E$ that appears in C, $SUBS?[F, E]$ **returns "false"**.

Case 1

- Suppose that some **atomic** D_i does not appear in C and consider the **interpretation** $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where:
  - $\Delta^{\mathcal{I}} = \{v, w\}$,
  - $R^{\mathcal{I}} = \{\langle v, w\rangle, \langle w, w\rangle\}$, for every role R that appears in C or D,
  - $A^{\mathcal{I}} = \{v, w\}$, for every atomic concept A different from D_i,
  - $D_i^{\mathcal{I}} = \{w\}$.
- Hence, for every role R we have:
  - $(\exists R.\top)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \colon R^{\mathcal{I}}(x, y) \text{ and } y \in \Delta^{\mathcal{I}}\} = \{v, w\}$,
  - $(\forall R.F)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \colon \text{ if } R^{\mathcal{I}}(x, y) \text{ then } y \in F^{\mathcal{I}}\} = \{v, w\}$.
- Therefore $C^{\mathcal{I}} = \bigcap_{1 \leqslant j \leqslant n} C_j = \{v, w\} \nsubseteq \{w\} = \bigcap_{1 \leqslant i \leqslant m} D_i = D^{\mathcal{I}}$.

Case 2

- Suppose that some D_i is an **existentially quantified concept** $\exists R.\top$ which does not appear in C and consider the **interpretation** $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where:
  - $\Delta^{\mathcal{I}} = \{v, w\}$,
  - $P^{\mathcal{I}} = \{\langle v, w\rangle, \langle w, w\rangle\}$, for every role P different from R,
  - $A^{\mathcal{I}} = \{v, w\}$, for every atomic concept A,
  - $R_i^{\mathcal{I}} = \{\langle w, w\rangle\}$.
- Hence, for every role P different from R and every role S including R we have:
  - $(\exists P.\top)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \colon P^{\mathcal{I}}(x, y) \text{ and } y \in \Delta^{\mathcal{I}}\} = \{v, w\}$,
  - $(\forall S.F)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \colon \text{ if } S^{\mathcal{I}}(x, y) \text{ then } y \in F^{\mathcal{I}}\} = \{v, w\}$.
- Therefore $C^{\mathcal{I}} = \bigcap_{1 \leqslant j \leqslant n} C_j = \{v, w\} \nsubseteq \{w\} = \bigcap_{1 \leqslant i \leqslant m} D_i = D^{\mathcal{I}}$.

Case 3

- Suppose that some D_i is a **universally quantified concept** $\forall R.F$ and for every concept $\forall R.E$ that appears in C, $SUBS?[F, E]$ **returns "false"** because of some concept G. Consider the **interpretation** $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where:
  - $\Delta^{\mathcal{I}} = \{v, w, z\}$,
  - $P^{\mathcal{I}} = \{\langle v, w \rangle, \langle w, w \rangle\}$, for every role P different from R,
  - $A^{\mathcal{I}} = \{v, w\}$, for every atomic concept A, except for G.
  - $R_i^{\mathcal{I}} = \{\langle w, w \rangle, \langle v, z \rangle\}$.
  - $G^{\mathcal{I}} = \{v, w, z\}$, for every atomic concept A,
- Hence, for every role S including R we have:
  - $(\exists S.\top)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : S^{\mathcal{I}}(x, y) \text{ and } y \in \Delta^{\mathcal{I}}\} = \{v, w\}$,

- for every role P different from R we have:
  - $(\forall P.F)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : \text{ if } P^{\mathcal{I}}(x,y) \text{ then } y \in F^{\mathcal{I}}\} = \{v, w, z\}$,
- for R we have:
  - $(\forall R.F)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : \text{ if } R^{\mathcal{I}}(x,y) \text{ then } y \in F^{\mathcal{I}}\} = \{v, w, z\}$.
  - $(\forall R.E)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : \text{ if } R^{\mathcal{I}}(x,y) \text{ then } y \in E^{\mathcal{I}}\} = \{w, z\}$.
- Therefore $C^{\mathcal{I}} = \bigcap_{1 \leqslant j \leqslant n} C_j = \{v, w\} \nsubseteq \{w\} = \bigcap_{1 \leqslant i \leqslant m} D_i = D^{\mathcal{I}}$.

Concluding, in all three cases $C^{\mathcal{I}}$ **is not a subset** of $D^{\mathcal{I}}$ when *SUBS?*$[D, C]$ **returns "false"**.

Computational complexity

In order to define the complexity of algorithm $SUBS?[D, C]$, let n be the **length of the longer argument**. Then:

- **Step 1** can be done in time linear in n (just erase parenthesis).
- **Step 2** may require that the entire concepts C and D are checked out a number of times **equal to their length**. Hence it can be done in $\mathcal{O}(n^2)$ time.
- **Step 3** may require that each of the concepts C and D is checked out a number of times **equal to the length of the other**. Hence it can be done in $\mathcal{O}(n^2)$ time.

Hence, algorithm $SUBS?[D, C]$ **operates in** $\mathcal{O}(n^2)$ time.