From Classical Description Logic to *n*-graded Fuzzy Description Logic

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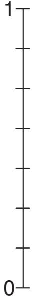
Finite-valued FDL

This work is a contribution towards the study of Fuzzy Description Logics valued on a finite chain.



- $C_n = \{0 = r_1 < r_2 < ... < r_{n-1} < r_n = 1\}$
- the semantics for conjunction is a *t*-norm $*$
- the semantics for implication is its residuum $\to_*$
  $x \to_* y = max\{z : z * x \leq y\}$

Finite divisible *t*-norms

$Ł_n$ $\quad x *_Ł y = \max\{0, x + y - 1\}$

$$x \to_Ł y = \min\{1, 1 - x + y\}$$

G_n $\quad x *_G y = \min\{x, y\}$

$$x \to_G y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$$

BL_n $\quad x *_{BL} y = \begin{cases} x * y, & \text{if } x, y \text{ belong to the same component} \\ \min\{x, y\}, & \text{otherwise} \end{cases}$

$$x \to_{BL} y = \begin{cases} x \to y, & \text{if } x, y \text{ belong to the same component} \\ x \to_G y, & \text{otherwise} \end{cases}$$

First Order Language

A Predicate Language is a pair $\Sigma = \langle \mathcal{C}, \mathcal{P} \rangle$, where:

- $\mathcal{C} = \{c, d, \ldots\}$ is a countable set of individual constants,
- $\mathcal{P} = \{P, Q, \ldots\}$ is a countable set of predicate symbols.

Logical symbols:

- a counatable set $Var = \{x, y, \ldots\}$ of individual variables,
- a propositional language $\mathbb{L} = \{\&, \rightarrow, \sim, \bot, \langle \overline{r} \rangle_{r \in C_n}\}$,
- quantifiers $\forall$ and $\exists$,

The Logics $\tilde{L}_n^*(C_n)\forall$

Given a divisible finite t-norm over a chain of n elements, the first order logic $\tilde{L}_n^*(C_n)\forall$ is presented in the Hilbert-style calculus defined as follows:
Axioms: the ones of the $*$-based propositional logic where now the formulas are read as predicate formulas, plus the following axioms on quantifiers:

$$
\begin{array}{ll}
(\forall 1) & (\forall x)\varphi(x) \to \varphi(t) \text{ (t substitutable for x in $\varphi(x)$)},\\
(\exists 1) & \varphi(t) \to (\exists x)\varphi(x) \text{ (t substitutable for x in $\varphi(x)$)},\\
(\forall 2) & (\forall x)(\varphi \to \psi) \to (\varphi \to (\forall x)\psi) \text{ (x not free in φ)},\\
(\exists 2) & (\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi) \text{ (x not free in ψ)},\\
(\forall 3) & (\forall x)(\varphi \vee \psi) \to (\forall x)\varphi \vee \psi \text{ (x not free in ψ)}.
\end{array}
$$

Deduction rules: Modus Ponens and Generalization.

Semantics: $*$-interpretations

Given a divisible finite t-norm $*$, an $*$-interpretation for $\Sigma = \langle \mathcal{C}, \mathcal{P} \rangle$ is a tuple

$$\mathbf{M} = \langle M, \{a^{\mathbf{M}} : a \in \mathcal{C}\}, \{P^{\mathbf{M}} : P \in \mathcal{P}\} \rangle$$

where:

- M is a non-empty set, called domain,
- each $a^{\mathbf{M}}$ is an element of M,
- $P^{\mathbf{M}} : M^k \rightarrow C_n$ is a k-ary n-graded relation on M, where k is the arity of P.

Semantics: evaluations

Given an $*$-interpretation M and an assignation $v : Var \rightarrow M$, an evaluation of a formula φ is inductively defined as follows:

$P^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v})$,	if $\varphi = P(t_1, \ldots, t_m)$
0,	if $\varphi = \bot$
1,	if $\varphi = \top$
r,	if $\varphi = \bar{r}$
r_{n-i+1},	if $\varphi = \sim \alpha$ and $\|\alpha\|^*_{\mathbf{M},v} = r_i$
$\|\alpha\|^*_{\mathbf{M},v} * \|\beta\|^*_{\mathbf{M},v}$,	if $\varphi = \alpha \& \beta$
$\|\alpha\|^*_{\mathbf{M},v} \rightarrow_* \|\beta\|^*_{\mathbf{M},v}$,	if $\varphi = \alpha \rightarrow \beta$
$\inf\{\|\alpha(a, b_1, \ldots, b_n)\|^*_{\mathbf{M},v} : a \in M\}$,	if $\varphi = (\forall x)\alpha(x, x_1, \ldots, x_m)$
$\sup\{\|\alpha(a, b_1, \ldots, b_n)\|^*_{\mathbf{M},v} : a \in M\}$,	if $\varphi = (\exists x)\alpha(x, x_1, \ldots, x_m)$

Properties of $\tilde{L}_n^*(C_n)\forall$

The logics $\tilde{L}_n^*(C_n)\forall$ have the following useful properties:

- Strong Completeness with respect to the corresponding semantics.
- Decidability, due to the presence of witnessed model property

Description Signature

A description signature is a tuple $\mathcal{D} = \langle N_I, N_T, N_A, N_R \rangle$, where:

- $N_I = \{a, b, \ldots\}$ is a countable set of individual names,
- $N_T = \{\bar{r}_1, \ldots, \bar{r}_n\}$ is a finite set of truth constants,
- $N_A = \{A, B, \ldots\}$ is a countable set of concept names,
- $N_R = \{R, S, \ldots\}$ is a countable set of role names,

Logical symbols:

- the set of concept constructors $\{\boxplus, \boxtimes, \sqsupset, \sim, \bot, \top\}$,
- quantifiers $\forall$ and $\exists$,

FDL Concepts

$\bot$	empty concept
$\top$	universal concept
A	atomic concept
$\sim A$	restricted strong complementary concept
$\sim C$	strong complementary concept
$\neg C$	weak complementary concept
$C \boxplus D$	concept strong union
$C \boxtimes D$	concept strong intersection
$C \sqcup D$	concept weak union
$C \sqcap D$	concept weak intersection
$C \sqsupset D$	concept implication
$\forall R.C$	universal quantification
$\exists R.C$	existential quantification
$\exists R.\top$	restricted existential quantification

From FDL to First Order Fuzzy Logic

Given a description signature

$$\mathcal{D} = \langle N_I, N_T, N_A, N_R \rangle$$

we define the first order signature

$$\Sigma_{\mathcal{D}} = \langle \mathcal{C}_{\mathcal{D}}, \mathcal{P}_{\mathcal{D}} \rangle$$

in the following way:

- $\mathcal{C}_{\mathcal{D}}$=$N_I$,
- $\mathcal{P}_{\mathcal{D}}$=$N_T \cup N_A \cup N_R$,

where

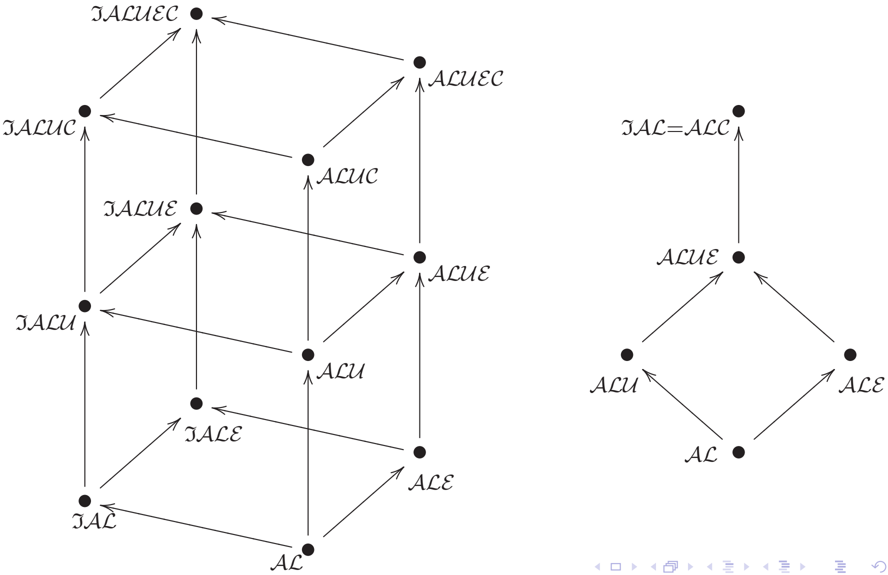
- each individual name in N_I is an individual constant,
- each truth constant in N_T is a nullary predicate symbol,
- each atomic concept in N_A is a unary predicate symbol,
- each atomic role in N_R is a binary predicate symbol.

Instance of a concept

The instance of a concept C is then a first order formula:

$A(t)$,	if C is an atomic concept A,
$\sim D(t)$,	if $C = \sim D$,
$\neg D(t)$,	if $C = \neg D$,
$D(t) \veebar E(t)$,	if $C = D \boxplus E$,
$D(t) \& E(t)$,	if $C = D \boxtimes E$,
$D(t) \vee E(t)$,	if $C = D \sqcup E$,
$D(t) \wedge E(t)$,	if $C = D \sqcap E$,
$D(t) \rightarrow E(t)$,	if $C = D \sqsupset E$,
$(\forall x)(R(t,x) \rightarrow C(x))$,	if $C = \forall R.C$,
$(\exists x)(R(t,x) \rightarrow C(x))$,	if $C = \exists R.C$.

A new Hierarchy of languages



Knowledge Bases

Let C, D be FDL-concepts without occurrences of any truth constant other than $\bot$ or $\top$, R be an atomic role and a, b be constant objects. Finally let $r \in C_n$.

fuzzy concept inclusions	$\langle C \sqsubseteq D, \succcurlyeq \bar{r} \rangle$	$\bar{r} \to (\forall x)(C(x) \to D(x))$
	$\langle C \sqsubseteq D, \preccurlyeq \bar{r} \rangle$	$(\forall x)(C(x) \to D(x)) \to \bar{r}$
	$\langle C \sqsubseteq D, \approx \bar{r} \rangle$	$\bar{r} \leftrightarrow (\forall x)(C(x) \to D(x))$
fuzzy concept assertions	$\langle C(a), \succcurlyeq \bar{r} \rangle$	$\bar{r} \to C(a)$
	$\langle C(a), \preccurlyeq \bar{r} \rangle$	$C(a) \to \bar{r}$
	$\langle C(a), \approx \bar{r} \rangle$	$\bar{r} \leftrightarrow C(a)$
fuzzy role assertions	$\langle R(a, b), \succcurlyeq \bar{r} \rangle$	$\bar{r} \to R(a, b)$

Reasoning tasks

- C is $*$-satisfiable to a degree greater (resp. lower) or equal than r iff the first order sentence $\bar{r} \rightarrow C(x)$ (resp. $C(x) \rightarrow \bar{r}$) is $*$-satisfiable.
- C is $*$-valid to a degree greater (resp. lower) or equal than r iff the first order sentence $\bar{r} \rightarrow (\forall x)C(x)$ (resp. $(\forall x)C(x) \rightarrow \bar{r}$) is $*$-valid.
- C is $*$-subsumed by D to a degree greater (resp. lower) or equal than r iff the first order sentence $\bar{r} \rightarrow (\forall x)(C(x) \rightarrow D(x))$ (resp. $(\forall x)(C(x) \rightarrow D(x)) \rightarrow \bar{r}$) is $*$-valid.

Reduction to satisfiability

- C is $*$-valid to a degree greater (resp. lower) or equal than r_m

  iff

  C is not $*$-satisfiable to a degree lower or equal than r_{m-1} (resp. r_{m+1}),

- C is $*$-subsumed by D to a degree greater (resp. lower) or equal than r_m

  iff

  the concept $C \sqsupset D$ is not $*$-satisfiable to a degree lower (resp. greater) or equal than r_{m-1} (resp. r_{m+1}).

Future work

- Study the decidability and complexity of satisfiability with respect to a Knowledge Base.
- Explore the relations with Many-valued Modal Logic.
- Explore the relations with Guarded fragments of First Order Fuzzy Logic.
- Obtain decidability results for more expressive FDL.

Thank you!!