On Finitely Valued Fuzzy Description Logics: The Łukasiewicz Case

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Finite Łukasiewicz chains

Given a positive integer $n \geq 2$, a finite Łukasiewicz chain is a structure:

$$\mathbf{Ł}_n := \langle L_n, \otimes, N, 0 \rangle$$

where

- $L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$,
- $a \otimes b := \max\{0, a + b - 1\}$, for each $a, b \in L_n$,
- $N(a) = 1 - a$, for each $a \in L_n$.

Definable operations

$a \Rightarrow b$	:=	$N(a \otimes N(b))$	$\min\{0, 1 - a + b\}$
$a \wedge b$	:=	$a \otimes (a \Rightarrow b)$	$\min\{a, b\}$
$a \vee b$	:=	$(a \Rightarrow b) \Rightarrow b$	$\max\{a, b\}$
$a \oplus b$	:=	$N(N(a) \otimes N(b))$	$\min\{1, a + b\}$
1	:=	$N(0)$	1

The n-valued Łukasiewicz propositional logic

Consider the set of formulas built from a countable set of propositional variables $\Phi = \{p_j : j \in J\}$ using the connectives:

- & strong conjunction
- $\sim$ involutive negation
- $\bar{0}$ falsity truth constant

A propositional evaluation is a map $e : \Phi \to L_n$ which is extended to all $\langle \&, \sim, \bar{0} \rangle$-formulas by setting:

- $e(\varphi \& \psi) = e(\varphi) \otimes e(\psi)$
- $e(\sim \varphi) = N(e(\varphi))$
- $e(\bar{0}) = 0.$

The n-valued Łukasiewicz logic, $\Lambda(\mathbf{Ł}_n)$

$$\Gamma \models_{\mathbf{Ł}_n} \varphi \iff \text{for every evaluation } e, \text{ if } e[\Gamma] \subseteq \{1\}, \text{ then } e(\varphi) = 1.$$

Adding truth constants

The logic $\Lambda(\mathbf{Ł}_n^c)$ is obtained by adding to the language one truth constant $\bar{r}$ for each $r \in L_n \setminus \{0\}$;

the semantics of the constant $\bar{r}$ is its canonical value r, i.e.

$$e(\bar{r}) = r$$

It is finitely axiomatizable from an axiomatization of $\Lambda(\mathbf{Ł}_n)$ by adding the so-called book-keeping axioms:

$$(bk_1) \qquad \bar{r} \& \bar{s} \leftrightarrow \overline{r \otimes s}$$
$$(bk_2) \qquad \sim \bar{r} \leftrightarrow \overline{N(r)}$$

Signature and terms

Let $\Sigma = \langle \mathcal{C}, \mathcal{P} \rangle$ be a first order signature (without functional symbols), where:

- $\mathcal{C}$ is a countable set of object constants
- $\mathcal{P}$ is a countable set of predicate symbols, each one with arity $k \geq 1$.

Given a countable set *Var* of individual variables, the set of Terms over a predicate signature is defined inductively as follows:

- every variable $x \in Var$ is a term,
- every constant $a \in \mathcal{C}$ is a term,

Formulas

The set of Formulas over a first order signature Σ is defined inductively as follows:

- $\bot$ and $\top$ are formulas,
- if $t_1, \ldots, t_n$ are terms and P is an n-ary predicate, then $P(t_1, \ldots, t_n)$ is a formula (called atomic formula),
- if $\varphi_1, \ldots, \varphi_n$ are formulas and $\circ$ is an n-ary logical operator, then $\circ(\varphi_1, \ldots, \varphi_n)$ is a formula,
- if $\varphi(x)$ is a formula, then $(\forall x)\varphi(x)$ and $(\exists x)\varphi(x)$ are formulas.

$Ł_n^c$-interpretations

$\mathbf{M} = \langle M, \{a^{\mathbf{M}} : a \in \mathcal{C}\}, \{P^{\mathbf{M}} : P \in \mathcal{P}\} \rangle$, where:

- M is a non-empty set;
- for each object constant $a \in \mathcal{C}$, $a^{\mathbf{M}}$ is an element of M;
- for each k-ary predicate symbol P, $P^{\mathbf{M}}$ is an n-graded k-ary relation defined on M, that is, a function $P^{\mathbf{M}} : M^k \to L_n$.

$v : Var \to M$ is called an assignation of the variables in $\mathbf{M}$.

The value of a term $\|t\|_{\mathbf{M},v}$:

- $v(x)$ when t is a variable x,
- $a^{\mathbf{M}}$ when t is a constant a.

The truth value of a formula $\varphi(x_1, \ldots, x_n)$ for the assignation v, denoted by $\|\varphi\|_{\mathbf{M},v}$ is a value in L_n defined inductively as follows:

$$
\begin{array}{ll}
P^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_k\|_{\mathbf{M},v}), & \text{if } \varphi = P(t_1, \ldots, t_k); \\
r, & \text{if } \varphi = \bar{r} \in \{\bar{0}, \bar{r}_1, \ldots, \bar{r}_{n-1}\}; \\
1 - \|\alpha\|_{\mathbf{M},v}, & \text{if } \varphi = \sim\alpha; \\
\|\alpha\|_{\mathbf{M},v} \otimes \|\beta\|_{\mathbf{M},v}, & \text{if } \varphi = \alpha \& \beta; \\
\inf\{\|\alpha(a, b_1, \ldots, b_n)\|_{\mathbf{M}} : a \in M\}, & \text{if } \varphi = (\forall x)\alpha(x, x_1, \ldots, x_n).
\end{array}
$$

A $\mathbf{Ł}_n^c$-interpretation $\mathbf{M}$ is a model, of a set of formulas Γ if, for each $\varphi \in \Gamma$, and each assignation v,

$$\|\varphi\|_{\mathbf{M},v} = 1$$

The logic $\Lambda(\mathbf{Ł}_n^c)\forall$ is defined by a finite set of axioms.

The logic $\Lambda(\mathbf{Ł}_n^c)\forall$ is strongly complete with respect to interpretations over $\mathbf{Ł}_n^c$.

Description signature

A description signature is a tuple $\mathcal{D} = \langle N_I, N_A, N_R \rangle$, where

- N_I, a set of individual names
- N_A a set of concept names (the atomic concepts)
- N_R a set of role names (the atomic roles)

Descriptions

An $\langle \mathcal{ALC}_{\text{Ł}_n^c}, \mathcal{D} \rangle$-description:

$$
\begin{array}{rcrl}
C, C_1, C_2 & \rightsquigarrow & A \mid & \text{(atomic concept)} \\
 & & \bot \mid & \text{(empty description)} \\
 & & \top \mid & \text{(universal description)} \\
 & & \mathfrak{r} \mid & \text{(constant description)} \\
 & & \neg C \mid & \text{(strong complementary concept)} \\
 & & C_1 \boxplus C_2 \mid & \text{(concept strong union)} \\
 & & C_1 \boxtimes C_2 \mid & \text{(concept strong intersection)} \\
 & & \forall R.C \mid & \text{(universal quantification)} \\
 & & \exists R.C \mid & \text{(existential quantification)} \\
 & & R \mid & \text{(atomic role)}
\end{array}
$$

Further constructors are defined as follows:

$$
\begin{array}{ll}
C_1 \sqsupset C_2 := \neg(C_1 \boxtimes \neg C_2) & \text{(residuated implication)} \\
C_1 \sqcap C_2 := C_1 \boxtimes (C_1 \sqsupset C_2) & \text{(weak intersection)} \\
C_1 \sqcup C_2 := \neg(\neg C_1 \sqcap \neg C_2) & \text{(weak union)}
\end{array}
$$

$$\mathcal{D} = \langle N_I, N_A, N_R \rangle \quad \Longrightarrow \quad \Sigma_{\mathcal{D}} = \langle \mathcal{C}_{\mathcal{D}}, \mathcal{P}_{\mathcal{D}} \rangle$$

where:

- $\mathcal{C}_{\mathcal{D}} = N_I$,
- $\mathcal{P}_{\mathcal{D}} = N_A \cup N_R$.

An instance of a concept D:

$$\begin{cases}
\bar{0}, & \text{if } D = \bot, \\
\bar{1}, & \text{if } D = \top, \\
\bar{r}, & \text{if } D = \mathfrak{r}, \\
A(t) & \text{if } D \text{ is an atomic concept } A, \\
\sim C(t) & \text{if } D = \neg C, \\
C_1(t) \veebar C_2(t) & \text{if } D = C_1 \boxplus C_2, \\
C_1(t) \& C_2(t) & \text{if } D = C_1 \boxtimes C_2, \\
(\forall y)(R(t,y) \to C(y)) & \text{if } D = \forall R.C, \\
(\exists y)(R(t,y) \& C(y)) & \text{if } D = \exists R.C.
\end{cases}$$

Modal finite-valued Łukasiewicz logics

For every natural number m, the language of multi-modal finite-valued Łukasiewicz logics with truth constants, denoted by μ_m, is obtained by expanding the language of $\Lambda(\mathbf{Ł}_n^c)$ with m unary connectives $\Box_1, \ldots, \Box_m$.

An n-valued Kripke m-frame is a tuple $\mathfrak{F} = \langle W, R_1, \ldots, R_m \rangle$, where:

- W is a non-empty set (the set of worlds)
- $R_1, \ldots, R_m$ are binary relations (the accessibility relations) valued in L_n

A Kripke $\langle Ł_n^c, m\rangle$-model is a pair $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$, where:

- $\mathfrak{F}$ is an n-valued Kripke m-frame
- V is a valuation $V\colon \Phi \times W \to L_n$ such that:
  - $V(\varphi \& \psi, w) = V(\varphi, w) \otimes V(\psi, w)$,
  - $V({\sim}\varphi, w) = 1 - V(\varphi, w)$,
  - $V(\bar{0}, w) = 0$;
  - $V(\bar{r}, w) = r$;
  - $V(\Box_k \varphi, w) = \inf\{R_k(w, w') \Rightarrow V(\varphi, w') : w' \in W\}$.

From modal to first order logics

Language translation:

- $p_j \quad \Longrightarrow \quad P_j$
- $\Box_k \quad \Longrightarrow \quad R_k$

We define the standard translation τ_x from μ_m-formulas to $\mathcal{L}_{\mu_m}(\Phi)$-formulas as follows:

$$
\begin{aligned}
\tau_x(p_j) &= P_j(x), \\
\tau_x(\varphi \& \psi) &= \tau_x(\varphi) \,\&\, \tau_x(\psi), \\
\tau_x(\sim \varphi) &= \sim \tau_x(\varphi), \\
\tau_x(\Box_k \varphi) &= (\forall y)(R_k(x, y) \to \tau_y(\varphi)), \\
\tau_x(\bar{0}) &= \bar{0}, \\
\tau_x(\bar{r}) &= \bar{r}.
\end{aligned}
$$

Model translation

$$\mathfrak{M} = \langle W, R_1, \ldots, R_m, V \rangle \Longrightarrow \mathbf{M}_{\mathfrak{M}} = \langle W, (P_j^{\mathbf{M}_{\mathfrak{M}}})_{j \in J}, R_1^{\mathbf{M}_{\mathfrak{M}}}, \ldots, R_m^{\mathbf{M}_{\mathfrak{M}}} \rangle$$

where:

- $P_j^{\mathbf{M}_{\mathfrak{M}}} : W \to L_n$ such that $P_j^{\mathbf{M}_{\mathfrak{M}}}(w) = V(p_j, w)$
- $R_k^{\mathbf{M}_{\mathfrak{M}}} = R_k$

Then:

- $\|\tau_x(\varphi)(w)\|_{\mathbf{M}_{\mathfrak{M}}} = V(\varphi, w)$.
- $\mathfrak{M}, w \models^1 \varphi \iff \|\tau_x(\varphi)(w)\|_{\mathbf{M}_{\mathfrak{M}}} = 1$.
- $\mathfrak{M} \models^1 \varphi \iff \|(\forall x)\, \tau_x(\varphi)\|_{\mathbf{M}_{\mathfrak{M}}} = 1$.

From modal to description logic (and vice-versa)

$$\mathcal{D} = \langle N_A, N_R \rangle \iff \langle \{p_j : j \in J\}, \{\Box_1, \ldots, \Box_m\} \rangle$$

$$\begin{aligned}
f(A_j) &= p_j, \text{ for each } j \in J \\
f(C \boxtimes D) &= f(C) \& f(D) \\
f(\neg C) &= \sim f(C) \\
f(\forall R_k.C) &= \Box_k f(C), 1 \leq k \leq m \\
f(\bot) &= \bar{0} \\
f(\mathfrak{r}) &= \bar{r}
\end{aligned}$$

Model translation

$$\mathbf{M} = \langle W, (A_j^{\mathbf{M}})_{j\in J}, R_1^{\mathbf{M}}, \ldots, R_m^{\mathbf{M}} \rangle \Longrightarrow \mathfrak{M}_{\mathbf{M}} = \langle W, R_1^{\mathfrak{M}_{\mathbf{M}}}, \ldots, R_m^{\mathfrak{M}_{\mathbf{M}}}, V_{\mathfrak{M}_{\mathbf{M}}} \rangle$$

where:

- $R_j^{\mathfrak{M}_{\mathbf{M}}} = R_j^{\mathbf{M}}$
- $V_{\mathfrak{M}_{\mathbf{M}}} : \Phi \times W \to L_n$ such that $V_{\mathfrak{M}_{\mathbf{M}}}(p_j, w) = A_j^{\mathbf{M}}(w)$. Then:

- $\|C(w)\|_{\mathbf{M}} = V_{\mathfrak{M}_{\mathbf{M}}}(f(C), w)$.
- $\|C(w)\|_{\mathbf{M}} = 1 \iff \mathfrak{M}_{\mathbf{M}}, w \models^1 f(C)$.
- $\|(\forall x)C(x)\|_{\mathbf{M}} = 1 \iff \mathfrak{M}_{\mathbf{M}}, \models^1 f(C)$.

Axioms and Knowledge Bases

- graded concept inclusion axiom

$$\langle C \sqsubseteq D, \bar{r} \rangle \quad := \quad \bar{r} \to (\forall x)(C(x) \to D(x))$$

- graded equivalence axiom

$$\langle C \equiv D, \bar{r} \rangle \quad := \quad \bar{r} \to (\forall x)(C(x) \leftrightarrow D(x))$$

- graded concept assertion axiom

$$\langle C(a), \bar{r} \rangle \quad := \quad \bar{r} \to C(a)$$

- graded role assertion axioms

$$\langle R(a,b), \bar{r} \rangle \quad := \quad \bar{r} \to R(a,b)$$

- A TBox $\mathcal{T}$ is a finite set of graded concept inclusion and equivalence axioms.
- An ABox $\mathcal{A}$ is a finite set of graded concept and role assertion axioms.
- A knowledge base (KB) is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$.

Reasoning tasks

Let C and D be concepts:

- A concept C is r-satisfiable w.r.t. a knowledge base $\mathcal{K}$ in a degree greater or equal than r iff there is an $Ł_n^c$-model **M** of $\mathcal{K}$, and an individual $a \in M$ such that

$$\|C(a)\|_{\mathbf{M}} \geq r$$

- C is positively satisfiable when $r = \frac{1}{n}$ (strictly greater than 0)
- C is 1-satisfiable when $r = 1$.
- C is subsumed by a concept D in a degree greater or equal than r w.r.t. a KB $\mathcal{K}$ iff, for every $Ł_n^c$-model **M** of $\mathcal{K}$, it holds that

$$\|(\forall x)(C(x) \to D(x))\|_{\mathbf{M}} \geq r$$

Reductions between reasoning tasks

- C is r-satisfiable w.r.t. $\mathcal{K}$
  iff
  $\bar{r} \sqsupset C$ is 1-satisfiable w.r.t. $\mathcal{K}$.

- C is subsumed in a degree $\geq r$ by D w.r.t. $\mathcal{K}$
  iff
  $\bar{r} \boxtimes C$ is subsumed in degree 1 by D.

- C is subsumed in degree 1 by a D w.r.t. $\mathcal{K}$
  iff
  $\mathcal{K} \cup \{C \boxtimes \sim D\}$ is not positively satisfiable.