Decidability of a Description Logic over Infinite-Valued Product Logic

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Outline

- We prove that the validity and positive satisfiability problems for Description Logic $\mathcal{ALE}$ over the standard product algebra $[0, 1]_\Pi$ are decidable.
- We prove it by providing a recursive reduction of such problems to the semantic consequence in propositional Product Logic.
- The result then follows from the fact that semantic consequence in propositional Product Logic is a decidable problem.
- Notice that we are not considering satisfiability with respect to a knowledge base.

Standard Π algebra

Standard Π algebra is the algebra $[0,1]_\Pi = \langle [0,1], \cdot, \Rightarrow, 1, 0 \rangle$, where:

- the domain is the real unit interval $[0,1]$,
- operation $\cdot$ is the usual product between reals.
- operation $\Rightarrow$ is its residuum which is defined as $\min\{1, \frac{y}{x}\}$
- constants 0 and 1 have their usual values.
- moreover it is definable a residuated negation $\neg$, whose truth value function is:

$$\neg x = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$

The Language Π-$\mathcal{ALE}$

- The rules of concept formation are:

$$C, D \leadsto A \mid \top \mid \bot \mid C \boxdot D \mid C \to D \mid \forall R.C \mid \exists R.D$$

- A Π-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of:
  - a crisp set $\Delta^{\mathcal{I}}$ (called the domain of $\mathcal{I}$),
  - an interpretation function $\cdot^{\mathcal{I}}$, such that:

1. $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$ and $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$,
2. $$\begin{aligned} \bot^{\mathcal{I}}(a) &= 0 \\ \top^{\mathcal{I}}(a) &= 1 \\ (C \boxdot D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \cdot D^{\mathcal{I}}(a) \\ (C \to D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \Rightarrow D^{\mathcal{I}}(a) \\ (\forall R.C)^{\mathcal{I}}(a) &= \inf\{R^{\mathcal{I}}(a,b) \Rightarrow C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\} \\ (\exists R.C)^{\mathcal{I}}(a) &= \sup\{R^{\mathcal{I}}(a,b) \cdot C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\} \end{aligned}$$

Quasi-witnessed models [Laskowski and Malekpour, 2007]

An Π-interpretation $\mathcal{I}$ is quasi-witnessed when it satisfies that for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$:

(wit∃) there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\exists R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a,b) \cdot C^{\mathcal{I}}(b)$$

(qwit∀)

- either there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a,b) \Rightarrow C^{\mathcal{I}}(b)$$

- or $(\forall R.C)^{\mathcal{I}}(a) = 0$

Quasi-witnessed models and standard semantics

Tautologies and positively satisfiable formulas in $[0,1]_\Pi\forall$ are the same of those in quasi-witnessed standard models. [Cerami, Esteva and Bou, 2010]

$$\varphi \in [0,1]_\Pi\forall\text{-}Taut \iff \varphi \in [0,1]_\Pi\forall\text{-}Taut^{qw}$$

$$\varphi \in [0,1]_\Pi\forall\text{-}pos\text{-}Sat \iff \varphi \in [0,1]_\Pi\forall\text{-}pos\text{-}Sat^{qw}$$

Previous related results

- First order standard tautologies are not arithmetical for Product Logic. [Montagna, 2001]
- Satisfiability (validity, subsumption) problem in the $\mathcal{ALC}$ description language over Lukasiewicz Logic is decidable. [Hájek, 2005]
- Satisfiability (validity, subsumption) in witnessed models for the $\mathcal{ALCE}$ description language over Product Logic is decidable. [Bobillo and Straccia, 2009]

Reduction to propositional satisfiability

- We provide a reduction of validity and satisfiability for Π-$\mathcal{ALE}$ to the semantic consequence in propositional Product Logic which is known to be a decidable problem.
- It is done in three steps:
  1. first we produce a set of formulas T_{C_0}, which provides positive constraints to build the model that (possibly) satisfies $C_0(d)$,
  2. second we produce a set of formulas Y_{C_0}, which provides negative constraints to build the model that (possibly) satisfies $C_0(d)$,
  3. third, we provide a translation $pr(\cdot)$ of formulas in T_{C_0} and Y_{C_0} into a propositional language.

Example: the set T_{C_0}

We will give an informal account of this reduction. Given an assertion, say

$$C_0(d) = (\neg\forall R.A \boxdot \neg\exists R.\neg A)(d)$$

for each quantified subformula occurring in it we produce a new constant and a couple of formulas are added to T_{C_0}:

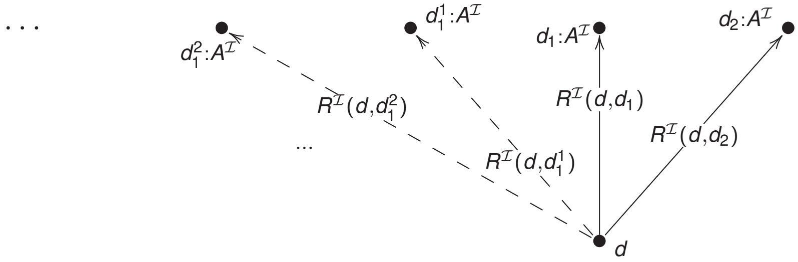
$$\forall R.A(d) \quad d_1 \quad (\forall R.A(d) \equiv (R(d,d_1) \to A(d_1))) \sqcup \neg\forall R.A(d)$$

$$\forall R.A(d) \to (R(d,d_2) \to A(d_2))$$

$$\exists R.\neg A(d) \quad d_2 \quad \exists R.\neg A(d) \equiv (R(d,d_2) \boxdot \neg A(d_2))$$

$$(R(d,d_1) \boxdot \neg A(d_1)) \to \exists R.\neg A(d)$$

which say us that we are building the following interpretation $\mathcal{I}$:



Example: the set Y_{C_0}

Moreover, for the universally quantified subformula, we add to the set Y_{C_0} the following formula:

$$\neg\forall R.A(d) \odot (R(d, d_1) \to A(d_1))$$

which constrains interpretation $\mathcal{I}$ not to verify both

$$(\forall R.A)^{\mathcal{I}}(d) = 0$$

and

$$R^{\mathcal{I}}(d, d_1) \to A^{\mathcal{I}}(d_1) = 1$$

in order to overcome a problem in an earlier version of this work.

The translation $pr(\cdot)$

The mapping *pr* associates to every assertion occurring in a formula in T_{C_0} and Y_{C_0} a propositional variable, according to the following clauses:

1. $pr(C(a))=P_{C(a)}$ if C is an atomic or a quantified concept,
2. $pr(R(a,b))=P_{R(a,b)}$ if R is a role name,
3. $pr(\bot(a))=\bot$,
4. $pr(\top(a))=\top$
5. $pr((C \boxdot D)(a))=pr(C(a)) \odot pr(D(a))$,
6. $pr((C \to D)(a))=pr(C(a)) \to pr(D(a))$.

If T is a set of assertions, then $pr(T)$ is $\{pr(\alpha) \mid \alpha \in T\}$.

So, the elements of the set $pr(T_{C_0})$ are:

$$(\forall R.A(d) \equiv (R(d,d_1) \to A(d_1))) \sqcup \neg\forall R.A(d)$$

$$(P_{\forall R.A(d)} \equiv (P_{R(d,d_1)} \to P_{A(d_1)})) \vee \neg P_{\forall R.A(d)}$$

$$\forall R.A(d) \to (R(d,d_2) \to A(d_2)) \quad P_{\forall R.A(d)} \to (P_{R(d,d_1)} \to P_{A(d_2)})$$

$$\exists R.\neg A(d) \equiv (R(d,d_2) \boxdot \neg A(d_2)) \quad P_{\exists R.\neg A(d)} \equiv (P_{R(d,d_2)} \odot P_{A(d_2)})$$

$$(R(d,d_1) \boxdot \neg A(d_1)) \to \exists R.\neg A(d) \quad (P_{R(d_1,d_1)} \odot P_{A(d_1)}) \to P_{\exists R.A(d)}$$

and the element of the set $pr(Y_{C_0})$ is:

$$\neg\forall R.A(d) \boxdot (R(d,d_1) \to A(d_1)) \quad \neg P_{\forall R.A(d)} \odot (P_{R(d,d_1)} \to P_{A(d_1)})$$

Propositional evaluations

We say that a propositional evaluation e is quasi-witnessing for an assertion C if:

- $e(\varphi) = 1$, for every $\varphi \in T_C$ and
- $e(\psi) \neq 1$, for every $\psi \in Y_C$

and prove that, there is an individual d such that, for each $r \in [0,1]$:

there exists a quasi-witnessed interpretation $\mathcal{I}$ such that $C^{\mathcal{I}}(d) = r$	$\Longleftrightarrow$	there exists a quasi-witnessing propositional evaluation e such that $e(pr(C(d))) = r$

Proof: from FDL interpretations to propositional evaluation

Given a quasi-witnessed interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} = r$, define the propositional evaluation $e_{\mathcal{I}}$ such that, for every concept and role assertion $D(a)$ and $R(a, b)$, occuring in a formula in $T_C \cup Y_C$,

$$e_{\mathcal{I}}(pr(D(a))) = D^{\mathcal{I}}(a)$$

and

$$e_{\mathcal{I}}(pr(R(a, b))) = R^{\mathcal{I}}(a, b)$$

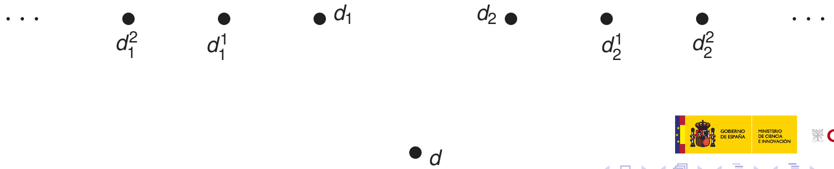
Hence, it is a simple task to check that $e_{\mathcal{I}}$ is a quasi-witnessing propositional evaluation and $e_{\mathcal{I}}(C(d)) = r$.

Proof: from propositional evaluation to FDL interpretations

We give an sketch by means of the example assertion C_0 above:

Given the sets T_{C_0}, Y_{C_0} and a quasi-witnessing propositional evaluation e such that $e(pr(C_0(d))) = r$, we define how to build a quasi-witnessed interpretation $\mathcal{I}_e$:

- The elements of the domain $\Delta^{\mathcal{I}}$ are the constant occurring in $T_{C_0} \cup Y_{C_0}$, plus a countable infinite set of new elements $\{d_n^i \ : \ n \in \omega \backslash 0\}$ for each constant d_n occurring in $T_{C_0} \cup Y_{C_0}$ and different from the root d:

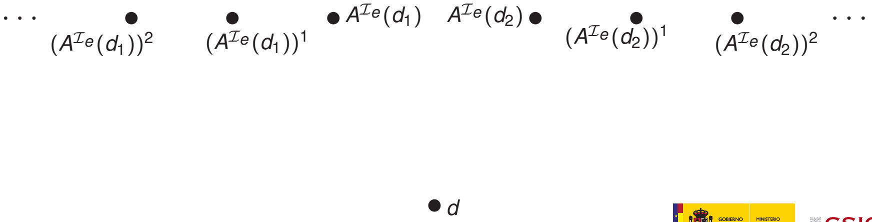


- For each atomic concept A and each constant d, d_n occurring in $T_{C_0} \cup Y_{C_0}$, define:

$$A^{\mathcal{I}_e}(d_n) = e(pr(A(d_n)))$$

and for each new element $d_n^i \in \Delta^{\mathcal{I}_e}$, define:

$$A^{\mathcal{I}_e}(d_n^i) = (e(pr(A(d_n))))^i$$

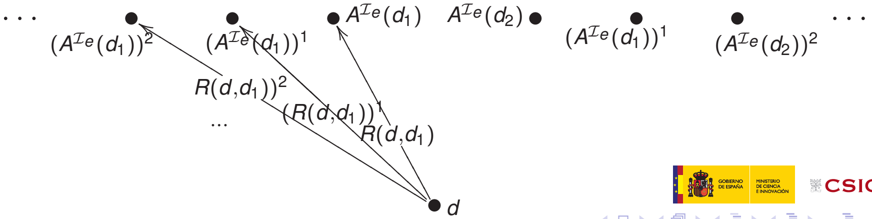


- For each role name R and each constant d_n occurring in $T_{C_0} \cup Y_{C_0}$, define:

$$R^{\mathcal{I}_e}(d, d_n) = e(pr(R(d, d_n)))$$

and for each new element $d_n^i \in \Delta^{\mathcal{I}_e}$, define:

$$R^{\mathcal{I}_e}(d, d_n^i) = \begin{cases} (e(pr(R(d, d_n))))^i, & \text{if } R(d, d_n) \to A(d_n))) \\ & \text{ occurrs in } T_{C_0} \\ & \text{and } e(pr(\forall R.A(d))) = 0 \\ R^{\mathcal{I}_e}(d, d_n^i) = 0, & \text{otherwise} \end{cases}$$



Reduction

Proposition

Let C_0 be a concept, and let T_{C_0} and Y_{C_0} be the two finite sets associated by the algorithm. For every $r \in [0, 1]$, the following statements are equivalent:

1. *C_0 is satisfiable with truth value r in a quasi-witnessed Π-interpretation,*
2. *there is some propositional evaluation e over the set Prop such that $e(pr(C(d_0))) = r$, $e[pr(T_{C_0})] = 1$, and $e[\psi] \neq 1$ for every $\psi \in pr(Y_{C_0})$.*

Which is equivalent to say that:

$C \in \mathrm{QSat}_1$ iff $\bigvee pr(Y_{C_0})$ is not a consequence, in the propositional product logic, of the set $\{pr(C(d_0))\} \cup pr(T_{C_0})$

iff $\{pr(C(d_0))\} \cup pr(T_{C_0}) \not\models \bigvee pr(Y_{C_0})$

$C \in \mathrm{QVal}$ iff $pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$ is a consequence, in the propositional product logic, of the set $pr(T_{C_0})$

iff $pr(T_{C_0}) \models pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$

Hence, we have a reduction of these problems to the semantic consequence problem, with a finite number of hypothesis, in the propositional product logic. Hájek, 2006 proves that such problem is in *PSPACE*.