Vague Counterfactuals

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Lewis' System of Spheres for Classical Counterfactuals

- As commonly understood, a *counterfactual* is a conditional sentence whose antecedent is false.
- Material implication (from e.g. classical logic) makes conditional sentences with a false antecedent unconditionally true.
- In contrast with material implication sentences, a false antecedent does not make the counterfactual conditional automatically true.

Lewis' Semantics for Classical Counterfactuals

Lewis (1973) provided a semantics for classical logic counterfactuals based on a type of structures called *Systems of Spheres*.

Definition

A *system of spheres* on a set W, denoted by \$, is a subset of $\mathcal{P}(W)$ such that \$ is

1. totally ordered by inclusion, and
2. closed under arbitrary unions and non-empty intersections.

Let $[\varphi] \subseteq W$ denote the set of possible worlds in which φ is true.

Definition

Let W be a set of possible worlds, \$ a system of spheres on W, $i \in W$ and, for every formula φ, then:
A counterfactual $\varphi \mathrel{\Box\!\!\longrightarrow} \psi$ is true in a world i, iff

1. either $[\varphi] = \emptyset$,
2. or $w \models \psi$, for every $w \in [\varphi]$ which belongs to the sphere $S \in \$^i$ closest to i, such that $S \cap [\varphi] \neq \emptyset$.

A counterfactual $\varphi \mathrel{\Diamond\!\!\longrightarrow} \psi$ is true in a world i, iff

1. either $[\varphi] = \emptyset$
2. or $w \models \psi$, for some $w \in [\varphi]$ which belongs to the sphere $S \in \$^i$ closest to i, such that $S \cap [\varphi] \neq \emptyset$.

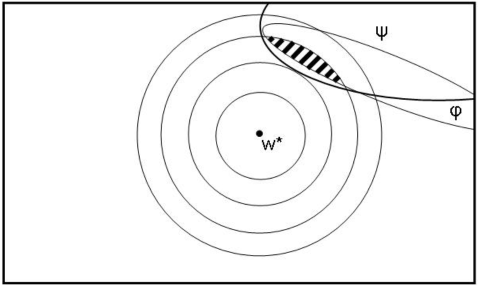


Figure: Semantics for *would*-counterfactuals

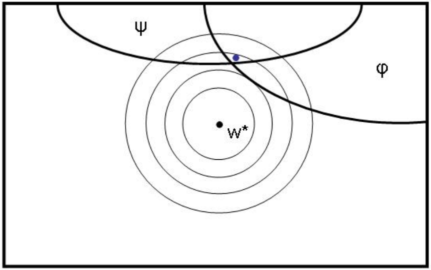


Figure: Semantics for *might*-counterfactuals

Would-might interdefinability in the classical case.

In the classical framework, like in Modal Logic, it is usually sufficient to assume as primitive only one of the above symbols and to define the other by means of negation.

Proposition

Let φ, ψ be arbitrary formulas, then:

1. $\varphi \mathrel{\Box\!\!\!\rightarrow} \psi \equiv \neg(\varphi \mathrel{\Diamond\!\!\!\rightarrow} \neg\psi)$
2. $\varphi \mathrel{\Diamond\!\!\!\rightarrow} \psi \equiv \neg(\varphi \mathrel{\Box\!\!\!\rightarrow} \neg\psi)$

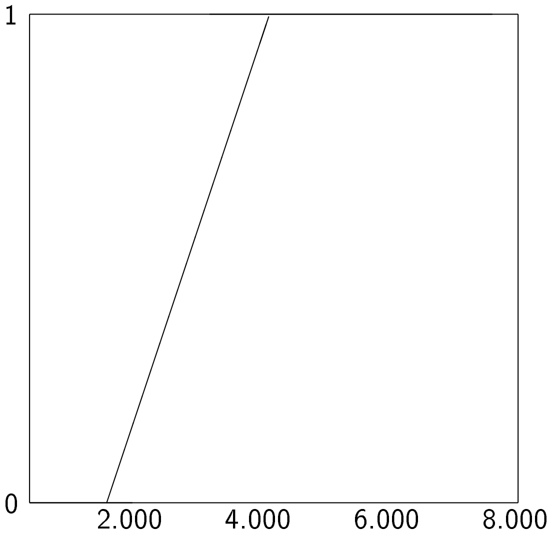
Vague Sentences

- We consider a *vague sentence* as a sentence that, by the nature of the meaning involved cannot be understood as merely true or false.
- As an example, if we fix that a tall man is a man whose height is greater or equal to 1.80 m, we cannot consider a man who is 1.79 m tall as a short man, even if he is not tall. Considerations like these drove to the ancient Sorites paradox; a modern way to overcome such paradox has been to consider *fuzzy sets*.

- Zadeh (1965) defined a fuzzy set M as a set whose characteristic function χ_M, is a function which returns a real value between 0 and 1, i.e. $\chi_M(x) \in [0, 1]$.
- A *characteristic function*, in the classical framework, is a function χ such that $\chi_M(x) = 1$ if an individual x is element of a set M and $\chi_M(x) = 0$ otherwise.
- Intuitively, if M is the set of tall men and x is a man who is 1,79 m tall, then we may have, say, $\chi_M(x) = 0.95$.

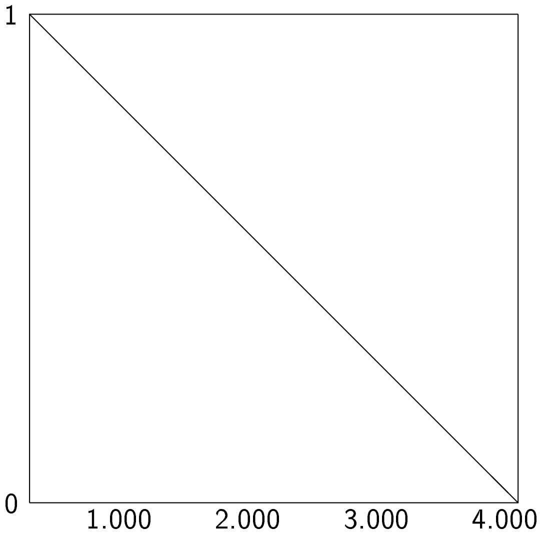
Example of vague sentences (i)

The mountain x is height.



Example of vague sentences (ii)

The place x is near from Salzburg.



t-norms

Definition

A *t-norm* is a binary operation $* : [0,1]^2 \longrightarrow [0,1]$ such that:

1. $*$ is commutative and associative.
2. $*$ is non-decreasing in both arguments.
3. For every $x \in [0,1]$, it holds that $1 * x = x$ and $0 * x = 0$.

If, moreover, $*$ is a continuous mapping from $[0,1]^2$ to $[0,1]$, we talk about a *continuous* t-norm.

The main examples of continuous t-norms are:

1. *Łukasiewicz* t-norm (denoted by Ł), defined by the function: $x * y = max(0, x + y - 1)$,
2. *Gödel* t-norm (denoted by G), defined by the function: $x * y = min(x, y)$,
3. *Product* t-norm (denoted by Π), defined by the function: $x * y = x \cdot y$.

Residua

Definition

Let $*$ be a t-norm, then its residuum is a binary operation $\Rightarrow_* : [0,1]^2 \longrightarrow [0,1]$ such that, for every $x, y \in [0,1]$:

$$x \Rightarrow_* y = sup\{z \in [0,1] \mid x * z \leq y\}$$

Intuitively, residua are the semantics of implications, indeed, it the framework of Fuzzy Logic, the expression *material implication* is substituted by the more general *residuated implication*.

The main examples of residua are:

1. *Łukasiewicz*, defined by the function: $x \Rightarrow_* y = min(1, 1 - x + y)$,
2. *Gödel*, defined by the function: $x \Rightarrow_* y = 1$, if $x \leq y$, y, otherwise,
3. *Product*, defined by the function: $x \Rightarrow_* y = min(1, \frac{y}{x})$.

Quantitative vs meta-linguistic criteria

- In a classical framework, it is the same to say that a given sentence φ *is true*, *holds* or that it has value 1.
- So, in this case, it is possible to express the truth value of a sentence by means of a meta-linguistic expression.
- For this reason, Lewis does not consider important to give a quantitative definition of his truth conditions for counterfactuals.

- In a context of multi-valued sentences, we deal with a different situation: as a simple remark, to give meta-linguistic account for each truth value, we would need an infinite set of adjectives ranging between *true* and *false*...
- ...and we are not sure that there exists any natural language that posses all these expressions.
- The natural choice seems then to be using numbers.

Vague counterfactuals

- By a *vague counterfactual* we understand a counterfactual involving vague sentences, i.e., sentences, that are not merely true or false, but can be evaluated in $[0, 1]$.
- This implies that the counterfactual, as a formula, can be evaluated in $[0, 1]$ as well.
- The most widely accepted definition of a (classical) counterfactual is to be a conditional with a false (within the actual world) antecedent, but...
- while in the classical framework there is no difference between speaking about a *false* antecedent and about an antecedent *that is not true*, in a multi-valued framework we have the chance of smarter specifications which allow a wider expressivity.

The value of the antecedent

As an example, consider the sentence:

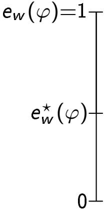
"If I was tall, I would touch the roof"

- This sentence assumes that, in the actual world, I am not tall in degree 1... not that I am tall in degree 0,
- in other words, I can think that I'm not tall even without thinking that I'm short.

- A first choice would be simply to consider worlds where the antecedent φ takes value 1;
- more generally, we could consider worlds where φ takes a value higher than, say, r;
- a particular case of the latter would consist in setting r to be the actual value for φ.
- In what follows we consider these accounts and present a possible way to formalize them.

Definition of 1-semantics

- We define first a simple extension of the semantics for the crisp case, now defining φ-worlds as worlds w where φ is 1-true: $e^{\$}_{w}(\varphi) = 1$.
- Intuitively, our definition tries to select those worlds w such that $e_w(\varphi) = 1$ and belong to the nearest sphere where there is some world w' such that $e_{w'}(\varphi) = 1$.



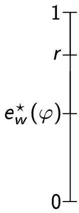
Definition

The 1-semantics of *would* counterfactuals $\varphi \mathrel{\Box\!\!\rightarrow} \psi$ and *might* $\varphi \mathrel{\Diamond\!\!\rightarrow} \psi$ is defined by:

- $e^1_{w^\star}(\varphi \mathrel{\Box\!\!\rightarrow} \psi) = \inf\{e_w(\psi)\}$, where $e_w(\varphi) = 1$ and w belongs to the sphere $S \in \$^i$ closest to i which contains a world w' such that $e_{w'}(\varphi) = 1$.
- $e^1_{w^\star}(\varphi \mathrel{\Diamond\!\!\rightarrow} \psi) = \sup\{e_w(\psi)\}$ where $e_w(\varphi) = 1$ and w belongs to the sphere $S \in \$^i$ closest to i which contains a world w' such that $e_{w'}(\varphi) = 1$

A troubling example

- It is indeed possible to further generalize such a semantics to the $> r$-case, in order to obtain a more refined tool with respect to vague counterfactual.



- The task, however, seems to be non-trivial: observe first that a simple (further) generalization of the previous 1-semantics to the $> r$-case will have the counterintuitive result of making $\varphi \mathrel{\Box\!\!\rightarrow} \varphi$ non-tautological (i.e. not 1-true).

Example

- For a fixed $r < 1$, replace (in the 1-semantics above) in each $e_{(\cdot)}(\varphi) = 1$ clause, the expression $= 1$ by expression $\geq r$.
- Also assume the actual world assigns φ a degree $< r$.
- Now, say the \$-system contains a φ-world w (i.e. $e^{\$}_{w}(\varphi) \geq r$) in the closest sphere S.
- Then, according to this semantics, we would have $e^{\geq r}_{w^{\star}}(\varphi \mathrel{\Box\!\!\rightarrow} \psi) = \inf\{e_w(\psi)\}$ where $e_w(\varphi) \geq r$ and w belongs to the sphere $S \in \$^{i}$ closest to i which contains a world w' such that $e_{w'}(\varphi) \geq r$.
- With such a definition, $\inf\{e_w(\psi)\}$ in the desired world w turns out to be equal to $r < 1$
- Hence $e^{\geq r}_{w^{\star}}(\varphi \mathrel{\Box\!\!\rightarrow} \varphi) < 1$, i.e. the sentence "if I was tall, then I would be tall" is not a tautology.

Definition of $\geq r$-semantics

A simple way to overcome such a problem, inspired on Modal Logic, is to define the value of the counterfactual from the value of the corresponding residuated implication in the possible world w, such that $e_w(\varphi) \geq r$ and w is in the closest sphere to the actual world.

A formal account of this condition is given in the following definition:

Definition

For a given $r \in [0,1]0$, let

$$\mathbb{K}^r = \{w \in W : e_w(\varphi) \geq r \text{ and } w \text{ belongs to the sphere } S \in \$^i \text{ closest to } i \text{ which contains a world } w' \text{ such that } e_{w'}(\varphi) \geq r\}$$

Then we define the $\geq r$-semantics of $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$ as follows:

$$\begin{aligned} e^{\geq r}_{w^\star}(\varphi \,\Box\!\!\rightarrow \psi) &= \inf\{\, e_w(\varphi \rightarrow_* \psi) \mid w \in \mathbb{K}^r\} \\ e^{\geq r}_{w^\star}(\varphi \,\Diamond\!\!\rightarrow \psi) &= \sup\{\, e_w(\varphi * \psi) \mid w \in \mathbb{K}^r\} \end{aligned}$$

Intuitive interpretation

Intuitively, this kind of implication is appropriate when we can give an exact degree (lower bound) to the expected value of the antecedent

Example

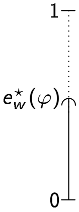
- Assume we interpret truth at degree (at least) 0.8 as *very* true (correspondingly, *very [humanely] tall*, *very [mountainly] tall*, $\ldots$).
- Then we can formalize the following sentences by means of a $\geq r$-semantics:

  (1) If I was very rich, I would be happy. $(\varphi, 0.8) \Box\!\!\longrightarrow \psi$

  (2) If I was very rich I might be happy. $(\varphi, 0.8) \Diamond\!\!\longrightarrow \psi$

Definition of more-than-actual Semantics

The main goal of a semantics for vague counterfactual is, however, to define the truth value of a counterfactual in the case when the antecedent has a value greater than the actual one.



Taking as a starting point the last definition, the weakest condition to be imposed on antecedents of vague counterfactuals will consider worlds where its truth-value is (at least) just slightly higher than in the actual world.

Definition

Let W be a set of possible worlds, $w^\star \in W$ and $ a system of spheres, then:

$$\begin{array}{lcl}
(1)\ e_{w^\star}^{>w^\star -}(\varphi \mathrel{\Box\!\!\rightarrow} \psi) & = & inf_{r > e_{w^\star}(\varphi)}\{e^{r}_{w^\star}(\varphi \mathrel{\Box\!\!\rightarrow} \psi)\} \\
(2)\ e_{w^\star}^{>w^\star +}(\varphi \mathrel{\Box\!\!\rightarrow} \psi) & = & sup_{r > e_{w^\star}(\varphi)}\{e^{r}_{w^\star}(\varphi \mathrel{\Box\!\!\rightarrow} \psi)\}
\end{array}$$

Intuitive interpretation (i)

- As we have seen above, starting from the same $\geq r$-semantics, we can obtain two different kind of definition.
- By means of such a difference it is possible to give account of different natural language counterfactuals.
- By (1) it is possible to give truth conditions for counterfactuals with antecedents that, in the possible world considered, have a value just higher than the actual one.

An example can be:

If Salzburg was nearer, there would be a tube line to arrive there.

However, even if this seems to be the proper semantics for vague would counterfactual, it make sense with a finite set of truth values, since, with a dense one, the value of the antecedent seems to collapse to the actual one.

An example

To show that, in finitely-valued fuzzy logics, the $> w^\star$-semantics is a particular case of $\geq r$-semantics consider:

$$e^{>w^\star}_{w^\star}(\varphi \Box\!\!\longrightarrow \psi) = inf\{\, e_w(\varphi \rightarrow \psi) \mid e_w(\varphi) \geq r, \text{ and } w \in \mathbb{K}^r\}$$

where r is the least value higher than $e_{w^\star}(\varphi)$ in a finite set of truth values.

The more-than-actual semantics in the dense-valued case, typically $[0,1] \cap \mathbb{Q}$, leads to a different situation, since, this time, definition (1) is not a particular case of the $\geq r$-semantics.

Example

- To see this fact, let φ, ψ be two vague sentences such that, for each possible world $w \in W$, $e_w(\psi) = 1 - e_w(\varphi)$ and suppose that, for each $S, S' \in \$$, it holds that if S' is outer than S, then there exist $w \in S$ and $w' \in S'$, such that $e_w(\varphi) < e_{w'}(\varphi)$.
- So, if we consider the counterfactual $\varphi \Box\!\!\longrightarrow \psi$, we have that, for each $r \geq e_{w^\star}(\varphi)$, it holds that $e^r_{w^\star}(\varphi \Box\!\!\longrightarrow \psi) > e^{>w^\star -}_{w^\star}(\varphi \Box\!\!\longrightarrow \psi)$.

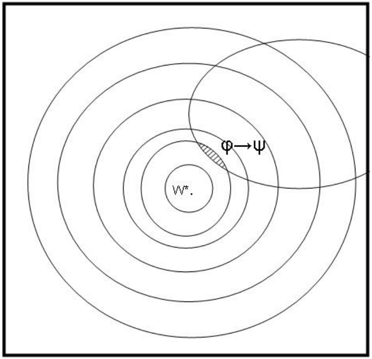


Figure: $\geq r$-semantics when $e_w(\varphi) > e_{w^\star}(\varphi)$

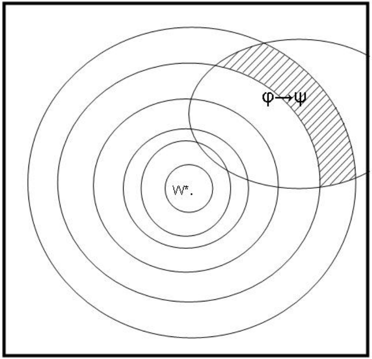


Figure: More-than-actual semantics

Intuitive interpretation (ii)

- By (2) it is possible to give truth conditions for counterfactuals with antecedents that, in the possible world considered, have a value enough higher than the actual one.
- An example can be:

  If Salzburg was near enough, we would come each weekend.
- This settling works also with a dense set of truth values, but is indeed too narrow to give an account for each kind of counterfactual.

Dual counterfactuals

In the Łukasiewicz framework we have that the dual of the above definitions are:

$$\begin{array}{lcl}
(1)\ e_{w^\star}^{>w^\star -}(\varphi \diamond\!\!\longrightarrow \psi) & = & sup_{r > e_{w^\star}(\varphi)}\{e^r_{w^\star}(\varphi \diamond\!\!\longrightarrow \psi)\} \\
(2)\ e_{w^\star}^{>w^\star +}(\varphi \diamond\!\!\longrightarrow \psi) & = & inf_{r > e_{w^\star}(\varphi)}\{e^r_{w^\star}(\varphi \diamond\!\!\longrightarrow \psi)\}
\end{array}$$

An example of (1) can be:

If Salzburg was near enough, we might come each weekend.

An example of (2) can be:

If Prague was nearer, there might be a tube line to arrive there.

Counterfactuals in semantics without an involutive negation

- For the cases of Gödel and Product, we have $v(\varphi) > 0$ implies $v(\neg\varphi) = 0$, and $v(\varphi) = 0$ implies $v(\neg\varphi) = 1$.
- This causes a problem when naively adopting the semantics of the classical case. Suppose we want to evaluate the counterfactual:

$$(\varphi) \quad = \quad \text{If I was tall, I would reach the roof.}$$

- Suppose in the actual world, I'm tall with degree 0.4, i.e. $e_{w^*}(\varphi) = 0.4$.
- By the properties of Gödel negation, this implies that $e_{w^*}(\neg\varphi) = 0$.
- If we look at possible worlds w where the antecedent is true, that is, worlds w where $e_w(\varphi) = 1$ we find that in all these worlds w, $e_w(\neg\varphi) = 0$...
- ...just like in the actual one.

Reduction of the classical framework to ours

- The restriction to crisp systems makes the inf- and $\forall$- clauses (and, hence, the respective semantics) equivalent.
- In other words, it is possible to prove that Lewis' semantics for counterfactuals is a particular case of the preceding semantics under the condition that the evaluations are restricted to $\{0, 1\}$, as in classical logic.

Proposition

Let $e^c_w(\cdot)$ denote Lewis' semantics and W a set of classical possible worlds (i.e. $W \subseteq \{e^c_w : \mathsf{Var} \to \{0,1\}\}$). For any classical system of spheres $\$$ and any world $w \in W$, the 1-semantics $e^1_w(\cdot)$ definition gives:

$$e^c_w(\varphi \mathrel{\Box\!\!\rightarrow} \psi) = e^1_w(\varphi \mathrel{\Box\!\!\rightarrow} \psi)$$

and

$$e^c_w(\varphi \mathrel{\Diamond\!\!\rightarrow} \psi) = e^1_w(\varphi \mathrel{\Diamond\!\!\rightarrow} \psi)$$

Interdefinability

- In the particular case of Łukasiewicz, we also have that classical inter-definability of *would* and *might* counterfactuals is preserved (this is due to the fact that $*_{\text{Ł}}$ is the only t-norm whose negation $e(\varphi) \Rightarrow \overline{0}$ is involutive, and hence behaves well with inf).
- In the next result, in Ł logic, we assume non-counterfactual connectives are evaluated according to Łukasiewicz semantics: $e(\varphi * \psi) = \max\{0, e(\varphi) + e(\psi) - 1\}$, and $e(\varphi \rightarrow_{\text{Ł}} \psi) = \min\{1, 1 - e(\varphi) + e(\psi)\}$.

Proposition

Let W be a set of possible worlds, $w \in W$ and let $\neg$ denote Łukasiewicz negation, then:

$$e^1_w(\varphi \mathrel{\Diamond\!\!\longrightarrow} \psi) = e^1_w(\neg(\varphi \mathrel{\Box\!\!\longrightarrow} \neg\psi))$$

and

$$e^1_w(\varphi \mathrel{\Box\!\!\longrightarrow} \psi) = e^1_w(\neg(\varphi \mathrel{\Diamond\!\!\longrightarrow} \neg\psi))$$

Reduction of Lewis' semantics to ours

This last kind of definition, does not violate Lewis' truth condition in the classical case, but it is a generalization of it.
Indeed, if we restrict the truth values of the sentences involved in the counterfactual to $\{0, 1\}$, we obtain Lewis' truth conditions:

Proposition

Setting $r = 1$ for a given system of spheres $\$$ and world $w^\star$, $\geq$ r-semantics (in classical valuations only) collapses to Lewis'.

$$\begin{array}{rcl} e_{w^\star}(\varphi \,\Box\!\!\rightarrow \psi) & = & e^{r=1}_{w^\star}(\varphi \,\Box\!\!\rightarrow \psi) \\ e_{w^\star}(\varphi \,\Diamond\!\!\rightarrow \psi) & = & e^{r=1}_{w^\star}(\varphi \,\Diamond\!\!\rightarrow \psi) \end{array}$$

Interdefinability

In this case, too, Łukasiewicz semantics allows us to obtain an interdefinability result of the kind of Lewis.

Proposition

Let W be a set of possible worlds, $w \in W$ and let $\neg$ denote Łukasiewicz negation, then:

$$e_w^{\geq r}(\varphi \mathrel{\Diamond\!\!\longrightarrow} \psi) = e_w^{\geq r}(\neg(\varphi \mathrel{\Box\!\!\longrightarrow} \neg\psi))$$

and

$$e_w^{\geq r}(\varphi \mathrel{\Box\!\!\longrightarrow} \psi) = e_w^{\geq r}(\neg(\varphi \mathrel{\Diamond\!\!\longrightarrow} \neg\psi))$$