Undecidability of Fuzzy Description Logics with GCIs under Łukasiewicz semantics

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- Description Logics (DLs) are logic-based knowledge representation languages.
- In their classical version they are used to infer hidden information in knowledge-based systems.
- They are also used as the underlying formalism for the Semantic Web.
- Since 1991 began the effort to generalize the classical version to the fuzzy case.
- The first work on Fuzzy Description Logic (FDL) considered a semantics based on Fuzzy Set Theory.
- In [Hájek, 2005] it is proposed a *t*-norm-based semantics for FDL.
- Since then some works on decidability and complexity of *t*-norm-based FDL have been produced.
- Our work belongs to this framework.

Syntax of concepts

Let **A** be a set of concept names, **R** be a set of role names. The set of Ł-$\mathcal{ALC}$ concepts are built from concept names A using connectives and quantification constructs over roles R

$$
\begin{array}{lcl}
C & \to & \top \\
 & & \bot \\
 & & A \\
 & & C_1 \sqcap C_2 \\
 & & C_1 \sqcup C_2 \\
 & & \neg C \\
 & & \exists R.C \\
 & & \forall R.C
\end{array}
$$

Axioms and knowledge bases

- A concept assertion axiom is an expression of the form $\langle a{:}C, n\rangle$
- A role assertion axiom is an expression of the form $\langle (a_1, a_2){:}R, n\rangle$

where a, a_1, a_2 are individual names, C is a concept, R is a role name and $n \in (0, 1]$ is a rational (a truth value). An ABox $\mathcal{A}$ consists of a finite set of assertion axioms.

- A General Concept Inclusion (GCI) axiom is of the form $\langle C_1 \sqsubseteq C_2, n\rangle$

where C_i is a concept and $n \in (0, 1]$ is a rational. A concept hierarchy $\mathcal{T}$, also called TBox, is a finite set of GCIs.
Finally, a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A}\rangle$ consists of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$.

Łukasiewicz semantics

In the following, we use $\otimes, \oplus, \ominus$ and $\Rightarrow$ to denote Łukasiewicz *t*-norm, *t*-conorm, negation function, and implication function, respectively. They are defined as operations in $[0,1]$ by means of the following functions:

$$
\begin{aligned}
a \otimes b &:= \max\{0, a + b - 1\} \\
a \oplus b &:= \min\{1, a + b\} \\
\ominus a &:= 1 - a \\
a \Rightarrow b &:= \min\{1, 1 - a + b\} \ ,
\end{aligned}
$$

where a and b are arbitrary elements in $[0,1]$.

Semantics of atomic concepts and roles

A fuzzy interpretation is a pair

$$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$$

consisting of a nonempty (crisp) set

$$\Delta^{\mathcal{I}}$$

the domain and of a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns:

1. to each atomic concept A a function $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0, 1]$,
2. to each role R a function $R^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$,
3. to each individual a an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ such that $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ if $a \neq b$

Semantics of complex concepts

The fuzzy interpretation function is extended to complex concepts as follows where $x, y \in \Delta^{\mathcal{I}}$ are elements of the domain. Hence, for every complex concept C we get a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$.

$$
\begin{array}{rcl}
\bot^{\mathcal{I}}(x) & = & 0 \\
\top^{\mathcal{I}}(x) & = & 1 \\
(C \sqcap D)^{\mathcal{I}}(x) & = & C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x) \\
(C \sqcup D)^{\mathcal{I}}(x) & = & C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x) \\
(\neg C)^{\mathcal{I}}(x) & = & \ominus C^{\mathcal{I}}(x) \\
(\forall R.C)^{\mathcal{I}}(x) & = & \inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)\} \\
(\exists R.C)^{\mathcal{I}}(x) & = & \sup_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y)\}
\end{array}
$$

Semantics of axioms

The satisfiability of axioms is then defined by the following conditions:

1. $\mathcal{I}$ satisfies an axiom $\langle a{:}C, n \rangle$ if
$$C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n,$$
2. $\mathcal{I}$ satisfies an axiom $\langle (a, b){:}R, n \rangle$ if
$$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n,$$
3. $\mathcal{I}$ satisfies an axiom $\langle C \sqsubseteq D, n \rangle$ if
$$(C \sqsubseteq D)^{\mathcal{I}} \geq n$$
where
$$(C \sqsubseteq D)^{\mathcal{I}} = \inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} .$$

Witnessed interpretations

- A fuzzy interpretation $\mathcal{I}$ is witnessed iff for every complex concept C, every role R, and every $x \in \Delta^{\mathcal{I}}$ there is some
  - $y \in \Delta^{\mathcal{I}}$ such that $(\exists R.C)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y)$.
  - $y \in \Delta^{\mathcal{I}}$ such that $(\forall R.C)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)$.
- A fuzzy interpretation $\mathcal{I}$ is strongly witnessed iff for every complex concepts C, D, there is some
  - $y \in \Delta^{\mathcal{I}}$ such that $(C \sqsubseteq D)^{\mathcal{I}} = C^{\mathcal{I}}(y) \Rightarrow D^{\mathcal{I}}(y)$

Reasoning tasks

- We say that a fuzzy interpretation $\mathcal{I}$ r-satisfies (is a model of) a concept C, for $r \in [0,1]$, if there is $a \in \Delta^{\mathcal{I}}$ such that

$$C^{\mathcal{I}}(a) = r$$

- We say that a fuzzy interpretation $\mathcal{I}$ satisfies (is a model of) a KB $\mathcal{K}$ in case that it satisfies all axioms in $\mathcal{K}$.

Related work: concept satisfiability

- In [Hájek, 2005] it is proved that:
  - concept satisfiability w.r.t. witnessed models coincides with concept satisfiability w.r.t. finite models under infinite-valued Łukasiewicz semantics,
  - concept satisfiability w.r.t. witnessed models is decidable under infinite-valued Łukasiewicz semantics.
- In [Hájek, 2007] it is proved that:
  - concept satisfiability w.r.t. witnessed models coincides with unrestricted concept satisfiability under infinite-valued Łukasiewicz semantics.

Related work: knowledge base satisfiability

- In [Bobillo, Bou, Straccia, 2011] it is proved that:
  - KB satisfiability w.r.t. witnessed models does not coincides with concept satisfiability w.r.t. finite models under infinite-valued Łukasiewicz semantics.
- In [Baader, Peñaloza, 2011] it is proved that:
  - KB satisfiability w.r.t. witnessed models is undecidable under infinite-valued product semantics,
  - KB satisfiability w.r.t. strongly witnessed models is undecidable under infinite-valued product semantics,

Our result

- KB satisfiability w.r.t. witnessed models is undecidable under infinite-valued Łukasiewicz semantics,
- KB satisfiability w.r.t. finite models is undecidable under infinite-valued Łukasiewicz semantics,

Reverse Post Correspondence Problem (RPCP)

Let $v_1, \ldots, v_p$ and $w_1, \ldots, w_p$ be two finite lists of words over an alphabet

$$\Sigma = \{1, \ldots, s\}.$$

The Reverse Post Correspondence Problem (RPCP) asks whether there is a non-empty sequence

$$i_1, i_2, \ldots, i_k,$$

with $1 \leq i_j \leq p$ such that

$$v_{i_k} v_{i_{k-1}} \ldots v_{i_1} = w_{i_k} w_{i_{k-1}} \ldots w_{i_1}.$$

Such a sequence, if it exists, is called a solution of the problem instance.

The reduction: witnessed models

Define the following TBox:

$$\mathcal{T} := \{ \quad V \equiv V_1 \sqcup V_2, \quad W \equiv W_1 \sqcup W_2 \quad \}$$

and the ABox $\mathcal{A}$ as follows:

$$\mathcal{A} := \{a : \neg V, \ a : \neg W, \ \langle a : A, 0.01 \rangle, \ \langle a : \neg A, 0.99 \rangle\} \ .$$

For $1 \leq i \leq p$

$$\begin{aligned}
\mathcal{T}^i_\varphi := \{\ & \top \sqsubseteq \exists R_i.\top, \\
& V \sqsubseteq (s+1)^{|v_i|} \cdot \forall R_i.V_1, \\
& (s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq V, \\
& W \sqsubseteq (s+1)^{|w_i|} \cdot \forall R_i.W_1, \\
& (s+1)^{|w_i|} \cdot \exists R_i.W_1 \sqsubseteq W \\
& \langle \top \sqsubseteq \forall R_i.V_2, 0.v_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.\neg V_2, 1 - 0.v_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.W_2, 0.w_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.\neg W_2, 1 - 0.w_i \rangle, \\
& A \sqsubseteq (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \forall R_i.A \\
& (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \exists R_i.A \sqsubseteq A \quad \}.
\end{aligned}$$

Now, let

$$\mathcal{T}_\varphi = \mathcal{T} \cup \bigcup_{i=1}^{p} \mathcal{T}_\varphi^i \ .$$

Finally, we define

$$\mathcal{O}_\varphi := \langle \mathcal{T}_\varphi, \mathcal{A} \rangle \ .$$

Intuitively, $\mathcal{O}_\varphi$ is built in such a way that every interpretation $\mathcal{I}$ satisfying it has to contain a search tree for φ.

Consider

$$\mathcal{O}'_\varphi := \langle \mathcal{T}'_\varphi, \mathcal{A} \rangle \ ,$$

where

$$\mathcal{T}'_\varphi := \mathcal{T}_\varphi \cup \bigcup_{1 \leq i \leq p} \{\top \sqsubseteq \forall R_i.(\neg(V \leftrightarrow W) \sqcup \neg A)\} \ .$$

Proposition

The instance φ of the RPCP has a solution iff the ontology $\mathcal{O}'_\varphi$ is not witnessed satisfiable.

Thanks!