An Introduction to Modal Logic I

Introduction and Historical remarks

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Introduction

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What is Modal Logic?

"Ask three modal logician what modal logic is, and you are likely to get at least three different answers"

form the Preface of the book *Modal Logic*
by P. Blackburn, M. de Rijke and I. Venema

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What is a “modality”?

“A modality is an expression that, when applied to a sentence S, provides a new sentence about the mode in which S is true or about the mode in which it is accepted”

form the Introduction of the textbook *Lógica Modal*
by R. Jansana and F. Bou

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Examples of modalities

- **Aletic**: necessary, possible, impossible;
- **Deontic**: obligatory, allowed, prohibited, legal, illegal;
- **Doxastic**: it is believed that;
- **Epistemic**: it is known that, everybody knows that;
- **Temporal**: always in the past, sometimes in the past, never in the past, always in the future, sometimes in the future, never in the future;
- **Spatial**: everywhere, somewhere, nowhere;
- **Metalogic**: valid, satisfiable, provable, consistent;
- **Computational**: in every state accessible from the present state, in some state accessible from the present state, in no state accessible from the present state;

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Two main notions of Modal Logic

- the logic of **modal sentences** and modal operators,
- the logic of **relational structures** (meant as Kripke frames).

the two notions **do not coincide**!!

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Historical Remarks

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The ancient era

Period : IV century b.C. (Aristotelian school) and XII century a.C. (Scholastic).

Systems: modal sentences.

Modalities: aletic, epistemic.

Problem: derivation of modal sentences.

Method: Square of opposition.

Strengths: definition (discovery) of the notion.

Weaknesses: no axiomatic system,

no formal semantics in general,

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The square of opposition

$$
\begin{tikzcd}
\textit{Necessary} \arrow[rr, leftrightarrow, "\neg M \neg"] \arrow[dd, leftrightarrow, "M\neg"] \arrow[ddrr, leftrightarrow] & & \textit{Possible} \arrow[dd, leftrightarrow, "M\neg"] \arrow[ddll, leftrightarrow, "\neg M"'] \\
& & \\
\textit{Impossible} \arrow[rr, leftrightarrow, "\neg M \neg"] & & \textit{Contingent}
\end{tikzcd}
$$

The syntactic era

- **Period**: 1918-1959.
- **Systems**: axiomatic systems of modal logics.
- **Modalities**: aletic, temporal.
- **Problem**: proving system distinctness.
- **Method**: Syntactic derivation.
  algebraic methods are often used to prove system distinctness but not systematically (no completeness result).
- **Strengths**: definition of systems and methods for further development.
- **Weaknesses**: no formal semantics in general,
  systems not systematically related to propositional or first order logic,
  incomplete and undecidable systems.

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Lewis' systems

- The original motivation overcoming the unintuitive behavior of the classical material implication;
- to this end, Lewis in 1918 defines the **strict implication**;
- he defines five **Hilbert-style** system S1-S5;
- the notation includes just a conjunction $\wedge$ and the strict implication $\multimap$;
- at the beginning there is **no semantics** defined;
- Lewis' systems are not defined as expansion of Classical Propositional Logic (CPL), rather as alternative **intensional logics**;
- Lewis' systems give a great impulse to the revival of Modal Logic.

Lewis' systems S1

Axioms:

$$
\begin{array}{ll}
(A1) & (p \wedge q) \prec (q \wedge p) \\
(A2) & (p \wedge q) \prec p \\
(A3) & p \prec (p \wedge p) \\
(A4) & ((p \wedge q) \wedge r) \prec (p \wedge (q \wedge r)) \\
(A5) & ((p \prec q) \wedge (q \prec r)) \prec (p \prec r) \\
(A6) & (p \wedge (p \prec q)) \prec q
\end{array}
$$

Rules:

$$
\begin{array}{ll}
(US) & \vdash_{S1} \varphi(p) \text{ implies } \vdash_{S1} \varphi(\psi) \\
(SSE) & \vdash_{S1} \varphi(\chi) \text{ and } \vdash_{S1} \chi \equiv \psi \text{ imply } \vdash_{S1} \varphi(\psi) \\
(AD) & \vdash_{S1} \varphi \text{ and } \vdash_{S1} \psi \text{ imply } \vdash_{S1} \varphi \wedge \psi \\
(SMP) & \vdash_{S1} \varphi \text{ and } \vdash_{S1} \varphi \prec \psi \text{ imply } \vdash_{S1} \psi
\end{array}
$$

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Results

- Study of Lewis' systems from a **syntactical** point of view.
- Mainly results of **distinctness** between Lewis' systems.
- Gödel (1933) defines Lewis' systems using the **modern notation**, proves that Intuitionistic Propositional Logic (IPL) can be translated into S4.
- McKinsey and Tarski (1944-1948) define some kinds of **algebraic semantics** (*"algebras with operators"*) for Lewis' systems and show decidability for S2 and S4.

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Beyond the syntactic era

- Carnap (1946) defines a semantics based on **state descriptions** (sets of formulas), an ancestor of Kripke frames.
- Jónsson and Tarski (1952) show how to **represent algebras with operators** as relational structures.
- Prior (1955) defines **temporal logic** and interprets it in $(\omega, <)$, a particular kind of Kripke frame.

The classical era

Period : 1959-1972.

Systems: modal logics as semantically defined systems.

Modalities: aletic, temporal, doxastic, deontic.

Problem: completeness, model theory.

Method: relational structures.

Strengths: clear insight of several modal systems,

clear relations to propositional and first order logic.

Weaknesses: excessive trust on relational structures.

Kripke frames

- In 1959 Kripke publishes the paper *A completeness theorem in Modal Logic*;
- frames and models are defined and applied to prove completeness of modal axiomatic systems;
- Other authors (Hintikka, Kanger, Prior) previously worked with relational structures;
- Kripke presentation was the most clear and systematic;
- its application to Modal Logic is very natural;
- axiomatic systems of modal logics can be defined semantically from classes of frames (and vice-versa);
- authentic revolution in Modal Logic.

Results

- New questions, methods and perspective.
- Central notion of **normal modal logics**.
- Classification of modal logics through **canonical models**.
- Many completeness results.
- Firts results on **finite model property**.
- Lemmon and Scott (1966) **conjecture** that every modal system is frame complete.

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Beyond the classical era

- Lemmon (1966) systematically defines **algebraic semantics** (*"modal algebras"*) for modal logics.
- Thomason (1972) shows that there are **frame incomplete temporal logics**.
- Thomason and Prior (1974) show that there are **frame incomplete modal logics**.

The modern era

Period : 1972-present.

Systems: modal logics in a wide sense.

Modalities: every kind of modality, introduction of the computational modalities.

Problem: algebraic completeness, model theory, expressivity, complexity.

Method: algebraic and relational structures, computational methods (automata, Turing machines).

Strengths: application of modal logics to other fields, in particular Theoretical Computer Science

Weaknesses: ?

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Epistemic Algebras (Lemmon, 1966)

Definition

A structure $\mathbf{A} = \langle A, \cap, -, P, 0, 1\rangle$ of type $\langle 2, 1, 1, 0, 0\rangle$ is said to be an *Epistemic Algebra* iff

(B) $\langle A, \cap, -, 0, 1\rangle$ is a Boolean Algebra

(E1) For every $a, b, c \in A$, $P(a \cap b) \leq P(a \cap c) \cup P(b \cap -c)$.

(E2) For every $a \in A$, $a \leq Pa$.

(E3) For every $a, b \in A$, $P(a \cup b) = Pa \cup Pb$.

We denote by EA the class of Epistemic Algebras.

Results

- **Frame incompleteness** results (Thomason, Prior, 1974).
- Revival of **algebraic semantics** (Lemmon, 1966; Goldblatt, Thomason, 1974).
- Definition of **Propositional Dynamic Logic** as the logic of computer programs (Pratt, 1976).
- Study of the **computational properties** of several modal logics (Ladner, 1977).
- The study of the **expressive capabilities** of modal logics (Gabbay, 1983).
- **Application** to Knowledge Representation (Schild, 1990) and many other areas of Computer Science.

Propositional Dynamic Logic (Pratt, 1976)

properties: $\varphi, \psi, \ldots$ (propositional formulas)

atomic programs: $\pi_1, \pi_2, \ldots$

union of programs: $\cup$

composition of programs: ;

iteration of programs: *

modality: $\langle \pi \rangle$

The intended meaning of the formula $\langle \pi \rangle \varphi$ is:

"some terminating execution of program π leads to a state where property φ holds"