An Introduction to Modal Logic XI

PSPACE completeness (part I)

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Olomouc, December 5^{th} 2013

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Introduction

- We have proved that the satisfiability problem of the minimal normal modal logic K is $\textsc{PSpace}$-**hard**;
- this means that every problem that is in $\textsc{PSpace}$ **can be polynomially reduced** to the satisfiability problem of K;
- but this still **does not mean** that the same problem can be solved using an amount of space that is polynomial on the size of the instance.
- Now, we are going to prove that **this problem is in PSpace**.
- In order to achieve this result, we are going to prove that this problem can be solved by a **non-deterministic** Turing machine that runs in $\textsc{PSpace}$;
- the desired result will then follow from the fact that $\textsc{PSpace} = \textsc{NPSpace}$.

Structure of the proof

- We will prove that the algorithm **Witness** is sound and complete with respect to the satisfiability problem for K.
- The proof consists of two parts:
  1. a modal formula φ is K-satisfiable if and only if there exists a structure called **Witness set** for φ;
  2. there exists a Witness set if and only if algorithm *Witness* outputs **true** as answer.
- For each part **both** completeness and soundness will be proved.

a modal formula φ is K-satisfiable	$\Longleftrightarrow$	there exists a Witness set for φ	$\Longleftrightarrow$	algorithm Witness answers **true**

Hintikka Sets

Intuition

- A **Witness set** for a modal formula φ is a **syntactical structure**;
- it is built up from suitable **sets of subformulas** of φ, called **Hintikka sets**;
- the idea is building **possible worlds** of a Kripke model from subformulas of φ;
- intuitively, the **successor** in the accessibility relation of a given Hintikka set H contains some of the formulas ψ such that ψ appears with a modality in H;
- a Witness set is essentially a kind of **tableau** for φ.

Closed sets

- A set of formulas Σ is said to be **closed** if it is closed under **subformulas** and **single negations**, that is:
  1. if $\sigma \in \Sigma$ and θ is a **subformula** of σ, then $\theta \in \Sigma$,
  2. if $\sigma \in \Sigma$ and $\sigma \neq \neg\theta$ for any formula θ, then $\neg\sigma \in \Sigma$.
- If Γ is a set of formulas, then $Cl(\Gamma)$ the **closure** of Γ is the smallest closed set of formulas containing Γ;
- a set of formulas Γ is closed if $Cl(\Gamma) = \Gamma$;
- if Γ is a **finite** set of formulas, so is $Cl(\Gamma)$.

Closed sets: example

Consider the set of formulas:

$$\Sigma \ := \ \{\Box(p \wedge q), \neg\Box p, \neg\Box q\}$$

then $Cl(\Sigma)$ contains the following formulas:

$$\begin{array}{ll}
\Box(p \wedge q), & \neg\Box(p \wedge q), \\
p \wedge q, & \neg(p \wedge q), \\
\neg\Box p, & \Box p \\
\neg\Box q, & \Box q, \\
p, & \neg p, \\
q, & \neg q.
\end{array}$$

Hintikka sets

Let Σ be a closed set of formulas. A **Hintikka set** H over Σ is a **maximal** subset of Σ that satisfies the following conditions:

1. $\bot \notin H$,
2. if $\neg\sigma \in \Sigma$, then $\neg\sigma \in H$ if and only if $\sigma \notin H$,
3. if $\sigma \wedge \theta \in \Sigma$, then $\sigma \wedge \theta \in H$ if and only if $\sigma \in H$ and $\theta \in H$,
4. if $\sigma \vee \theta \in \Sigma$, then $\sigma \vee \theta \in H$ if and only if $\sigma \in H$ or $\theta \in H$,
5. all formulas in Σ are in Negation Normal Form.

- Hintikka sets **do not contain any propositional inconsistencies**;
- nevertheless they **are not necessarily** modally satisfiable;
- when a Hintikka set is modally satisfiable, we call it **atom**.

Hintikka sets: example

Consider again the set of formulas:

$$\Sigma \ := \ \{\Box(p \wedge q), \neg\Box p, \neg\Box q\}$$

and its closure $Cl(\Sigma)$. Then we can obtain a Hintikka set H by dropping the red formulas:

$$\begin{array}{llll}
\Box(p \wedge q), & \neg\Box(p \wedge q), & & \\
p \wedge q, & \neg(p \wedge q), & \rightsquigarrow & \neg p \vee \neg q, \\
\neg\Box p, & \Box p & \rightsquigarrow & \Diamond\neg p, \\
\neg\Box q, & \Box q, & \rightsquigarrow & \Diamond\neg q \\
p, & \neg p, & & \\
q, & \neg q. & &
\end{array}$$

Nevertheless the above Hintikka set H **is not an atom** because the set $\{\Box(p \wedge q), \Diamond\neg p\}$ is inconsistent.

Demands

- Let Σ be a closed set, H a Hintikka set over Σ and $\Diamond\psi \in H$. Then the **demand** that $\Diamond\psi$ creates in H is:

$$Dem(H, \Diamond\psi) \ := \ \{\psi\} \cup \{\theta \colon \Box\theta \in H\}.$$

- We will denote by $H_{\Diamond\psi}$ **the set of Hintikka sets over** $Cl(Dem(H, \Diamond\psi))$ **that contain** $Dem(H, \Diamond\psi)$.
- The operation of creating a demand, differently from chosing a Hintikka set, is **deterministic**.
- For every finite Hintikka set, the number of demands that can be created is **finite**.

Demands: example

Consider again the set of formulas:

$$\Sigma \;:=\; \{\Box(p \wedge q), \neg\Box p, \neg\Box q\}$$

its closure $Cl(\Sigma)$ and the Hintikka set H formerly chosen:

$$\Box(p \wedge q), \quad \Diamond\neg p, \quad \neg\Diamond q, \quad \neg p \vee \neg q, \quad \neg p, \quad q.$$

Consider the demand $Dem(H, \Diamond\neg p)$ created in H by formula $\Diamond\neg p$:

$$\Box(p \wedge q), \quad \Diamond\neg p, \quad \neg\Diamond q, \quad \neg p \vee \neg q, \quad \neg p, \quad q.$$

Clearly, the set $\{p \wedge q, \neg p\}$ **is not satisfiable**, hence the set $H_{\neg\Diamond p}$ of Hintikka sets over $Cl(Dem(H, \neg\Diamond\psi))$ **that contain** $Dem(H, \neg\Diamond\psi)$ is **empty**.

How to use these tools

- Intuitively:
  - the **closure** of a set plays the role of a point in the model,
  - the **Hintikka set** on a closure plays the role of a propositional valuation on that point,
  - the **demand** in a Hintikka set plays the role of the relation between a point and an its successor.
- The idea is to check **all possible** demands on **all possible** Hintikka sets until either a satisfiable family is found or the search space has been fully checked.
- Clearly, for every formula, **the search space is finite**.

K-satisfiability

and

Witness sets

Witness Sets

Let Σ be a finite closed set, H a Hintikka set over Σ. Then $\mathcal{H} \subseteq \mathcal{P}(\Sigma)$ is a **witness set generated by** H **on** Σ if:

1. $H \in \mathcal{H}$,
2. if $I \in \mathcal{H}$, then for each $\Diamond\psi \in I$, there is $J \in I_{\Diamond\psi}$ such that $J \in \mathcal{H}$,
3. if
   - $J \in \mathcal{H}$,
   - $J \neq H$,

   then for some $n > 0$ there are $I^0, \dots, I^n$ such that:
   - $H = I^0$,
   - $J = I^n$,
   - for each $0 \leq i < n$ there is a formula $\Diamond\psi \in I^i$ such that $I^{i+1} \in I^i_{\Diamond\psi}$.

Atoms and Witness Sets

Let Σ be a finite closed set of formulas and H a Hintikka set over Σ, then:

H is an **atom** $\Longleftrightarrow$ there is a Witness set generated by H on Σ

- The left to right direction is proved **by induction** on the modal degree $deg(\Sigma)$ of Σ,
- the **modal degree** of a set of formulas Σ is the maximum of the modal degrees of the formulas belonging to Σ.

From models to Witness Sets

Suppose that H **is an atom**.

0 If $deg(\Sigma) = 0$, then it is a set of propositional formulas. Hence $\mathcal{H} = H$ is trivially a witness set.

d Let $deg(\Sigma) = d$ and suppose that for every Σ' s.t. $deg(\Sigma') < d$, every atom H' over Σ generates a Witness set over Σ.

- Since H is an atom, then there is a model $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \vDash H$,
- hence, for each $\Diamond\psi \in H$ there is $v \in W$ such that $R(w, v)$ and $\mathcal{M}, v \vDash Dem(H, \Diamond\psi)$.

- Let Ψ be the set of formulas satisfied in v,
- then the set

$$I^{\psi} := \Psi \cap Cl(Dem(H, \Diamond\psi))$$

is an atom that contains $Dem(H, \Diamond\psi)$, that is $I^{\psi} \in H_{\Diamond\psi}$.

- By definition, $deg(Cl(Dem(H, \Diamond\psi)) = I^{\psi} < d$ for every $\Diamond\psi \in H$,
- hence, for every $\Diamond\psi \in H$, by h.i., I^{ψ} generates a Witness set $\mathcal{I}^{\psi}$ on $Cl(Dem(H, \Diamond\psi))$.
- Therefore, the set:

$$\mathcal{H} = \{H\} \cup \bigcup_{\Diamond\psi \in H} \mathcal{I}^{\psi}$$

is a Witness set generated by H on Σ.

From Witness Sets to models: building the model

Suppose that H **generates a Witness set** $\mathcal{H}$ on Σ. We will build a model inductively.

Let $\{w_0, w_1, \ldots\}$ a countable set of points. Define:

0 $W_0 = \{w_0\}$, $R_0 = \emptyset$, $f_0(w_0) = H$.

n+1 Suppose that W_n, R_n and $f_n(w_n)$ have been already defined, then:

- if for all $w \in W_n$ such that $\Diamond\psi \in f_n(w)$ there exists $w' \in W_n$ such that
  1. $\psi \in f_n(w')$,
  2. $f_n(w') \in (f_n(w))_{\Diamond\psi}$

  then halt the construction.

- Otherwise, if there is $w \in W_n$ such that $\Diamond\psi \in f_n(w)$, but does not exist $w' \in W_n$ such that the above condition are satisfied, define:
  - ★ $W_{n+1} = W_n \cup \{w_{n+1}\}$,
  - ★ $R_{n+1} = R_n \cup \{(w, w_{n+1})\}$,
  - ★ $f_{n+1} = f_n \cup \{(w_{n+1}, I)\}$,

  where $I \in (f_n(w))_{\Diamond\psi}$ (remind that there exists a Witness set $\mathcal{H}$ and it always exists).

- Since $deg(H)$ is finite, the construction halts at some finite m,
- Once the construction halted, define a propositional valuation V on every $w \in W_m$ as:

$$V(w) \; := \; f_m(w) \cap Prop.$$

From Witness Sets to models: $\mathfrak{M}_m$ is a model of H

Now we have to prove that $\mathfrak{M}_m, w_0 \models H$.

In order to achieve this result, we will prove, by induction on the modal degree of Hintikka sets I, that for every point $w \in W_m$ such that $f_m(w, I)$, it holds that

$$\mathfrak{M}_m, w \models I.$$

So, let $w \in W_m$ and $I \in \mathcal{H}$, then:

0 if $deg(I) = 0$, then it is straightforward from the definition of f_m and the fact that $f_m(w)$ is a Hintikka set.

d Let $deg(I) = d$ and suppose, by h.i., that for every $J \in \mathcal{H}$ with $deg(J) < d$ and every point $w \in W_m$ such that $f_m(w, J)$, the statement holds. Then

- for every the formulas $\theta \in I$ with $deg(\theta) = 0$ the result is straightforward again from the definition of f_m and the fact that $f_m(w)$ is a Hintikka set.
- Let $\Diamond\psi \in I$,
- by definition, there is $v \in W_m$ such that $R_m(w, v)$ and $f_m(v) \in I_{\Diamond\psi}$;
- since $f_m(v) \in I_{\Diamond\psi}$, then $deg(f_m(v)) < d$ and $\psi \in f_m(v)$;
- by i.h. $\mathfrak{M}_m, v \models f_m(v)$, hence $\psi \in V(v)$.
- Hence $\mathfrak{M}_m, w \models \Diamond\psi$,
- therefore $\mathfrak{M}_m, w \models I$.

In particular $\mathfrak{M}_m, w_0 \models H$.

Conclusion of the proof

- We have proved that a Hintikka set H over a closed set Σ is an atom if and only if there is a Witness set generated by H on Σ.
- In particular, if we take $\Sigma = Cl(\{\varphi\})$ for a given modal formula φ, we have that φ is **satisfiable** if and only if there is a Hintikka set H over $Cl(\{\varphi\})$ which generates a Witness set on $Cl(\{\varphi\})$.
- Moreover, the proof shows that if φ is satisfiable, it is in a model of φ that is:
    - **tree-shaped**,
    - **shallow**, since every path in the tree has at most length $deg(\varphi)$.
- This information will be useful later on when proving that the algorithm *Witness* can be implemented in $\mathrm{PS{\small PACE}}$.