An Introduction to Modal Logic II

Syntax and Semantics

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Syntax

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Language and formulas

Language

- A countable set of propositional variables $Prop = \{p, q, \ldots\}$,
- the classical propositional constants $\top$ and $\bot$,
- the classical propositional connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$,
- two unary modal connectives $\Box$ and $\Diamond$.

Formulas

The set Φ of modal formulas is inductively built from *Prop* in the following way:

- Propositional variables and constants are formulas,
- if φ and ψ are formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\neg\varphi$ are formulas,
- if φ is a formula, then $\Box\varphi$ and $\Diamond\varphi$ are formulas.

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Normal Modal Logics

Definition

A *normal modal logic* Λ is a **set of formulas** containing:

- all classical tautologies (in the modal language),
- $\Box(p \to q) \to (\Box p \to \Box q)$ (axiom (K)),
- $\Box\ p \leftrightarrow \neg\Diamond\neg p$,
- $\Diamond p \leftrightarrow \neg\Box\neg p$,

and is closed under:

(MP) Modus Ponens: if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$,

(US) Uniform Substitution: if $\varphi \in \Lambda$, then $\psi \in \Lambda$, where ψ is obtained from φ by replacing propositional variables by arbitrary formulas,

(G) Generalization: if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$.

Remarks

- in this way a modal logic is defined as **set of theorems**, rather than as a deducibility operator;
- the **set-like definition** can be equivalently replaced by an Hilbert-style axiomatic system based on the notion of deducibility;
- in this sense every modal logic is the **expansion of CPL** by means of two (or more) modal connectives;
- there exists a **minimal normal modal logic** and it is denoted by K (after S. Kripke);
- note that Lewis' S1 system is **not** a normal modal logic;
- we are indeed defining what a **uni-modal logic** is, but this framework can be extended to any countable set of modalities;
- nevertheless, normal modal logics are **defined semantically**.

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Axiomatic Extensions: axioms

(4)	$\Box p \to \Box\Box p$
(T)	$\Box p \to p$
(B)	$p \to \Box\Diamond p$
(D)	$\Box p \to \Diamond p$
(E)	$\Diamond p \to \Box\Diamond p$
(M)	$\Box\Diamond p \to \Diamond\Box p$
(G)	$\Diamond\Box p \to \Box\Diamond p$
(L)	$\Box(\Box p \to p) \to \Box p$

Axiomatic Extensions: logics

T	$\rightsquigarrow$	KT
$K4$	$\rightsquigarrow$	$K4$
$S4$	$\rightsquigarrow$	$KT4$
B	$\rightsquigarrow$	KTB
$S5$	$\rightsquigarrow$	$KT4B$ or $KT4E$
GL	$\rightsquigarrow$	KL
D	$\rightsquigarrow$	KD
$D4$	$\rightsquigarrow$	$KD4$

Semantics

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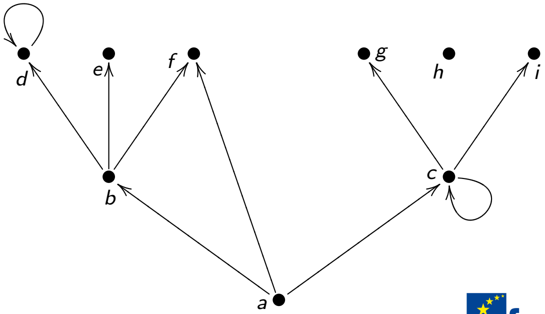
Kripke frames

A **Kripke frame** is a structure $\mathfrak{F} = \langle W, R \rangle$, where:

- W is a non-empty set of elements, often called **possible worlds**,
- $R \subseteq W \times W$ is a binary relation on W, called the **accessibility relation** of W.



Kripke models

A **Kripke model** is a structure $\mathfrak{M} = \langle W, R, V \rangle$, where:

- $\langle W, R \rangle$ is a Kripke frame,
- $\mathbf{V}\colon \mathbf{Prop} \times \mathbf{W} \longrightarrow \{\mathbf{0}, \mathbf{1}\}$ is a function that assigns a boolean value to every ordered pair of propositional variables and possible worlds.

The evaluation relation can be also viewed in the two following equivalent ways:

- as a function $\mathbf{V}\colon \mathbf{W} \longrightarrow \mathcal{P}(\mathbf{Prop})$ such that, given a world $w \in W$ returns the set $V(w)$ of propositional variables true in w;
- as a function $\mathbf{V}\colon \mathbf{Prop} \longrightarrow \mathcal{P}(\mathbf{W})$ such that, given a propositional variable $p \in Prop$ returns the set $V(p)$ of worlds where p is true.

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Evaluation of formulas

Given a Kripke model $\mathfrak{M} = \langle W, R, V \rangle$ and a world $w \in W$, the evaluation V of propositional variables can be inductively extended to arbitrary formulas in the following way:

- $V(\top, w) = 1$,
- $V(\bot, w) = 0$,
- $V(\varphi \wedge \psi, w) = \min\{V(\varphi, w), V(\psi, w)\}$,
- $V(\varphi \vee \psi, w) = \max\{V(\varphi, w), V(\psi, w)\}$,
- $V(\varphi \rightarrow \psi, w) = \max\{1 - V(\varphi, w), V(\psi, w)\}$,
- $V(\neg\varphi, w) = 1 - V(\varphi, w)$,
- $V(\Box\varphi, w) = (\forall v)(R(w, v) \Rightarrow V(\varphi, v))$,
- $V(\Diamond\varphi, w) = (\exists v)(R(w, v) \wedge V(\varphi, v))$.

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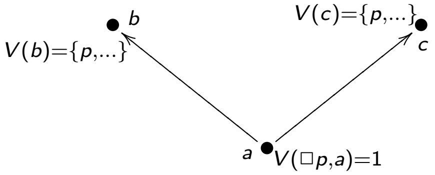
Semantics of the necessity operator $\Box$

The expression

$$V(\Box\varphi, w) = (\forall v)(R(w, v) \Rightarrow V(\varphi, v))$$

is equivalent to the condition:

formula $\Box\varphi$ *is true in world* w *iff*
φ *is true in every world* v *accessible from* w *iff*
for every world $w \in W$, *if* $R(w, v)$, *then* $V(\varphi, v) = 1$;



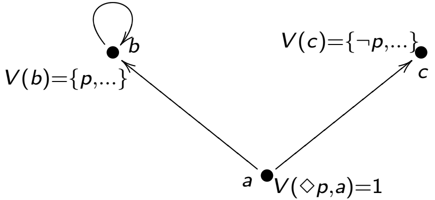
Semantics of the possibility operator $\Diamond$

The expression

$$V(\Diamond\varphi, w) = (\exists v)(R(w, v) \wedge V(\varphi, v))$$

is equivalent to the condition:

formula $\Diamond\varphi$ is true in world w iff
there exists a world v accessible from w and φ is true in v iff
there exists a world v such that $R(w, v)$ and $V(\varphi, v) = 1$;



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Example: the theorem $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ (I)

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and $w \in W$, then:

- formula $\Diamond\varphi$ is true in w iff
- there exists $v \in W$ such that both $R(w, v)$ and φ is true in v, iff
- it is not true that, in every $v \in W$ such that $R(w, v)$, formula φ is false, iff
- it is not true that, in every $v \in W$ such that $R(w, v)$, formula $\neg\varphi$ is true, iff
- it is not true that formula $\Box\neg\varphi$ is true in w, iff
- formula $\Box\neg\varphi$ is false in w, iff
- formula $\neg\Box\neg\varphi$ is true in w.

Example: the theorem $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ (II)

Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model and $w \in W$, then:

$$
\begin{aligned}
& V(\Diamond\varphi, w) = \\
= \; & (\exists v)(R(w,v) \wedge V(\varphi, v)) = \\
= \; & \neg\neg(\exists v)(R(w,v) \wedge V(\varphi, v)) = \\
= \; & \neg(\forall v)(\neg(R(w,v) \wedge V(\varphi, v))) = \\
= \; & \neg(\forall v)(R(w,v) \Rightarrow \neg V(\varphi, w)) = \\
= \; & \neg V(\Box\neg\varphi, w) = \\
= \; & V(\neg\Box\neg\varphi, w)
\end{aligned}
$$

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Logic

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Satisfaction of a formula

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and $w \in W$, then:

$\mathfrak{M}, w \vDash p$	iff	$V(p, w) = 1$
$\mathfrak{M}, w \vDash \top$		always
$\mathfrak{M}, w \vDash \bot$		never
$\mathfrak{M}, w \vDash \neg\varphi$	iff	$\mathfrak{M}, w \nvDash \varphi$
$\mathfrak{M}, w \vDash \varphi \wedge \psi$	iff	both $\mathfrak{M}, w \vDash \varphi$ and $\mathfrak{M}, w \vDash \psi$
$\mathfrak{M}, w \vDash \varphi \vee \psi$	iff	either $\mathfrak{M}, w \vDash \varphi$ or $\mathfrak{M}, w \vDash \psi$
$\mathfrak{M}, w \vDash \Box\varphi$	iff	for every $v \in W$ s.t. $R(w, v)$, it holds that $\mathfrak{M}, v \vDash \varphi$
$\mathfrak{M}, w \vDash \Diamond\varphi$	iff	there exists $v \in W$ s.t. $R(w, v)$ and $\mathfrak{M}, v \vDash \varphi$

Local and Global Satisfiability

- We say that a formula φ is **locally satisfiable**, if there exists a model $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$, such that

$$\mathfrak{M}, w \models \varphi$$

- We say that a formula φ is **globally satisfiable**, in a model $\mathfrak{M} = \langle W, R, V \rangle$, if φ is (locally) satisfiable in every point $w \in W$. In symbols:

$$\mathfrak{M} \models \varphi$$

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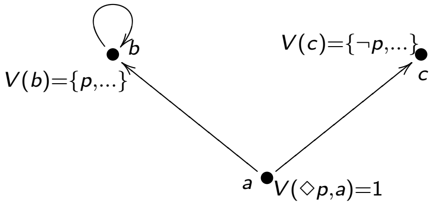
Remark

Both notions of local and global satisfiability **do not coincide**.

Consider $\mathfrak{M}$:



Then

- $\mathfrak{M}, b \models \Box p$ and
- $\mathfrak{M}, c \models \Box p$, but
- $\mathfrak{M}, a \not\models \Box p$, hence
- $\mathfrak{M} \not\models \Box p$

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Validity

- We say that a formula φ is **valid in a frame** $\mathfrak{F} = \langle W, R \rangle$, if for every model $\mathfrak{M} = \langle W, R, V \rangle$ and every $w \in W$, it holds that $\mathfrak{M}, w \models \varphi$. In symbols

$$\mathfrak{F} \models \varphi$$

- We say that a formula φ is **valid in a class of frames F** if it is valid in every frame $\mathfrak{F} \in \mathbf{F}$. In symbols:

$$\mathbf{F} \models \varphi$$

- We say that a formula φ is **valid**, if it is valid in every class of frames $\mathbf{F}$. In symbols:

$$\models \varphi$$

Semantic Consequence relations

Let $\Gamma \cup \varphi$ be a set of modal formulas and **M** a class of models, then:

- We say that a formula φ is a **local consequence** of Γ over **M**, if for all models $\mathfrak{M} = \langle W, R, V \rangle \in \mathbf{M}$ and all points $w \in W$, it holds that
  - if $\mathfrak{M}, w \models \Gamma$, then $\mathfrak{M}, w \models \varphi$.

  In symbols: $\Gamma \models^{l}_{\mathbf{M}} \varphi$.

- We say that a formula φ is a **global consequence** of Γ over **M**, if for all models $\mathfrak{M} = \langle W, R, V \rangle \in \mathbf{M}$ it holds that
  - if $\mathfrak{M} \models \Gamma$, then $\mathfrak{M} \models \varphi$.

  In symbols: $\Gamma \models^{g}_{\mathbf{M}} \varphi$.

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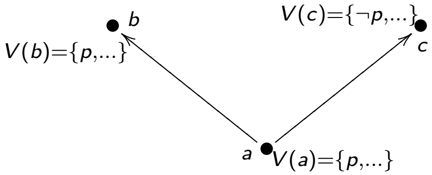
Remark

Both notions of local and global consequence **do not coincide**.
Consider $\mathfrak{M}$:



Then

- since $\mathfrak{M}, a \models p$, but $\mathfrak{M}, a \not\models \Box p$, then $\{p\} \not\models^{l}_{\{\mathfrak{M}\}} \Box p$;
- since $\mathfrak{M} \not\models p$, then $\{p\} \models^{g}_{\{\mathfrak{M}\}} \Box p$;
- So, $\Box p$ is a global consequence, but not a local consequence of p.

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