An Introduction to Modal Logic II

Syntax and Semantics

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Syntax
Language and formulas

Language
- A countable set of propositional variables $Prop = \{p, q, \ldots\}$,
- the classical propositional constants $\top$ and $\bot$,
- the classical propositional connectives $\land$, $\lor$, $\rightarrow$ and $\neg$,
- two unary modal connectives $\Box$ and $\Diamond$.

Formulas
The set $\Phi$ of modal formulas is inductively built from $Prop$ in the following way:

- Propositional variables and constants are formulas,
- if $\varphi$ and $\psi$ are formulas, then $\varphi \land \psi$, $\varphi \lor \psi$, $\varphi \rightarrow \psi$ and $\neg \varphi$ are formulas,
- if $\varphi$ is a formula, then $\Box \varphi$ and $\Diamond \varphi$ are formulas.
Normal Modal Logics

Definition

A normal modal logic $\Lambda$ is a set of formulas containing:

- all classical tautologies (in the modal language),
- $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ (axiom (K)),
- $\Box p \leftrightarrow \neg \Diamond \neg p$,
- $\Diamond p \leftrightarrow \neg \Box \neg p$,

and is closed under:

- (MP) Modus Ponens: if $\varphi \in \Lambda$ and $\varphi \rightarrow \psi \in \Lambda$, then $\psi \in \Lambda$,
- (US) Uniform Substitution: if $\varphi \in \Lambda$, then $\psi \in \Lambda$, where $\psi$ is obtained from $\varphi$ by replacing propositional variables by arbitrary formulas,
- (G) Generalization: if $\varphi \in \Lambda$, then $\Box \varphi \in \Lambda$. 
**Remarks**

- in this way a modal logic is defined as *set of theorems*, rather than as a deducibility operator;
- the *set-like definition* can be equivalently replaced by an Hilbert-style axiomatic system based on the notion of deducibility;
- in this sense every modal logic is the *expansion of CPL* by means of two (or more) modal connectives;
- there exists a **minimal normal modal logic** and it is denoted by K (after S. Kripke);
- note that Lewis’ S1 system is **not** a normal modal logic;
- we are indeed defining what a **uni-modal logic** is, but this framework can be extended to any countable set of modalities;
- nevertheless, normal modal logics are defined semantically.
Axiomatic Extensions: axioms

(4) $\Box p \to \Box \Box p$

(T) $\Box p \to p$

(B) $p \to \Box \Diamond p$

(D) $\Box p \to \Diamond p$

(E) $\Diamond p \to \Box \Diamond p$

(M) $\Box \Diamond p \to \Diamond \Box p$

(G) $\Diamond \Box p \to \Box \Diamond p$

(L) $\Box (\Box p \to p) \to \Box p$
Axiomatic Extensions: logics

\[ T \implies KT \]
\[ K4 \implies K4 \]
\[ S4 \implies KT4 \]
\[ B \implies KTB \]
\[ S5 \implies KT4B \text{ or } KT4E \]
\[ GL \implies KL \]
\[ D \implies KD \]
\[ D4 \implies KD4 \]
Semantics
Kripke frames

A **Kripke frame** is a structure $\mathcal{F} = \langle W, R \rangle$, where:

- $W$ is a non-empty set of elements, often called **possible worlds**,
- $R \subseteq W \times W$ is a binary relation on $W$, called the **accessibility relation** of $W$. 

![Diagram of a Kripke frame]

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Kripke models

A **Kripke model** is a structure $\mathcal{M} = \langle W, R, V \rangle$, where:

- $\langle W, R \rangle$ is a Kripke frame,
- $V : \text{Prop} \times W \rightarrow \{0, 1\}$ is a function that assigns a boolean value to every ordered pair of propositional variables and possible worlds.

The evaluation relation can be also viewed in the two following equivalent ways:

- as a function $V : W \rightarrow \mathcal{P}(\text{Prop})$ such that, given a world $w \in W$ returns the set $V(w)$ of propositional variables true in $w$;
- as a function $V : \text{Prop} \rightarrow \mathcal{P}(W)$ such that, given a propositional variable $p \in \text{Prop}$ returns the set $V(p)$ of worlds where $p$ is true.
**Evaluation of formulas**

Given a Kripke model $\mathcal{M} = \langle W, R, V \rangle$ and a world $w \in W$, the evaluation $V$ of propositional variables can be inductively extended to arbitrary formulas in the following way:

- $V(\top, w) = 1$,
- $V(\bot, w) = 0$,
- $V(\varphi \land \psi, w) = \min\{V(\varphi, w), V(\psi, w)\}$,
- $V(\varphi \lor \psi, w) = \max\{V(\varphi, w), V(\psi, w)\}$,
- $V(\varphi \rightarrow \psi, w) = \max\{1 - V(\varphi, w), V(\psi, w)\}$,
- $V(\neg \varphi, w) = 1 - V(\varphi, w)$,
- $V(\Box \varphi, w) = (\forall v)(R(w, v) \Rightarrow V(\varphi, v))$,
- $V(\Diamond \varphi, w) = (\exists v)(R(w, v) \land V(\varphi, v))$.  

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Semantics of the necessity operator $\Box$

The expression

$$V(\Box \varphi, w) = (\forall v)(R(w, v) \Rightarrow V(\varphi, v))$$

is equivalent to the condition:

formula $\Box \varphi$ is true in world $w$ iff
$\varphi$ is true in every world $v$ accessible from $w$ iff
for every world $w \in W$, if $R(w, v)$, then $V(\varphi, v) = 1$;

$V(b) = \{p, \ldots\}$

$V(c) = \{p, \ldots\}$

$V(\Box p, a) = 1$
The expression
\[ V(\Diamond \varphi, w) = (\exists v)(R(w, v) \land V(\varphi, v)) \]
is equivalent to the condition:

*formula \( \Diamond \varphi \) is true in world \( w \) iff
there exists a world \( v \) accessible from \( w \) and \( \varphi \) is true in \( v \) iff
there exists a world \( v \) such that \( R(w, v) \) and \( V(\varphi, v) = 1 \).*
Example: the theorem $\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi \ (I)$

Let $M = \langle W, R, V \rangle$ be a model and $w \in W$, then:

- formula $\Diamond \varphi$ is true in $w$ iff
- there exists $v \in W$ such that both $R(w, v)$ and $\varphi$ is true in $v$, iff
- it is not true that, in every $v \in W$ such that $R(w, v)$, formula $\varphi$ is false, iff
- it is not true that, in every $v \in W$ such that $R(w, v)$, formula $\neg \varphi$ is true, iff
- it is not true that formula $\Box \neg \varphi$ is true in $w$, iff
- formula $\Box \neg \varphi$ is false in $w$, iff
- formula $\neg \Box \neg \varphi$ is true in $w$. 

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Example: the theorem $\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi \ (\text{II})$

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and $w \in W$, then:

$$V(\Diamond \varphi, w) =$$

$$= (\exists v)(R(w, v) \land V(\varphi, v)) =$$

$$= \neg \neg (\exists v)(R(w, v) \land V(\varphi, v)) =$$

$$= \neg (\forall v)(\neg (R(w, v) \land V(\varphi, v))) =$$

$$= \neg (\forall v)(R(w, v) \Rightarrow \neg V(\varphi, w)) =$$

$$= \neg V(\Box \neg \varphi, w) =$$

$$= V(\neg \Box \neg \varphi, w)$$
Logic
Satisfaction of a formula

Let $M = \langle W, R, V \rangle$ be a model and $w \in W$, then:

- $M, w \models p$ iff $V(p, w) = 1$
- $M, w \models \top$ always
- $M, w \models \bot$ never
- $M, w \models \neg \varphi$ iff $M, w \not\models \varphi$
- $M, w \models \varphi \land \psi$ iff both $M, w \models \varphi$ and $M, w \models \psi$
- $M, w \models \varphi \lor \psi$ iff either $M, w \models \varphi$ or $M, w \models \psi$
- $M, w \models \Box \varphi$ iff for every $v \in W$ s.t. $R(w, v)$, it holds that $M, v \models \varphi$
- $M, w \models \Diamond \varphi$ iff there exists $v \in W$ s.t. $R(w, v)$ and $M, v \models \varphi$
Local and Global Satisfiability

- We say that a formula $\varphi$ is **locally satisfiable**, if there exists a model $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$, such that

$$\mathcal{M}, w \models \varphi$$

- We say that a formula $\varphi$ is **globally satisfiable**, in a model $\mathcal{M} = \langle W, R, V \rangle$, if $\varphi$ is (locally) satisfiable in every point $w \in W$. In symbols:

$$\mathcal{M} \models \varphi$$
Remark

Both notions of local and global satisfiability do not coincide. Consider $M$:

$$V(b) = \{p, \ldots\}$$

$$V(c) = \{\neg p, \ldots\}$$

Then

- $M, b \models \Box p$ and
- $M, c \models \Box p$, but
- $M, a \not\models \Box p$, hence
- $M \not\models \Box p$
Validity

- We say that a formula $\varphi$ is **valid in a frame** $\mathcal{F} = \langle W, R \rangle$, if for every model $\mathcal{M} = \langle W, R, V \rangle$ and every $w \in W$, it holds that $\mathcal{M}, w \models \varphi$. In symbols:

  $\mathcal{F} \models \varphi$

- We say that a formula $\varphi$ is **valid in a class of frames** $F$ if it is valid in every frame $\mathcal{F} \in F$. In symbols:

  $F \models \varphi$

- We say that a formula $\varphi$ is **valid**, if it is valid in every class of frames $F$. In symbols:

  $\models \varphi$
Semantic Consequence relations

Let $\Gamma \cup \varphi$ be a set of modal formulas and $\mathcal{M}$ a class of models, then:

- We say that a formula $\varphi$ is a **local consequence** of $\Gamma$ over $\mathcal{M}$, if for all models $\mathcal{M} = \langle W, R, V \rangle \in \mathcal{M}$ and all points $w \in W$, it holds that
  
  $\triangledown$ if $\mathcal{M}, w \models \Gamma$, then $\mathcal{M}, w \models \varphi$.
  
  In symbols: $\Gamma \vdash_{\mathcal{M}}^{l} \varphi$.

- We say that a formula $\varphi$ is a **global consequence** of $\Gamma$ over $\mathcal{M}$, if for all models $\mathcal{M} = \langle W, R, V \rangle \in \mathcal{M}$ it holds that
  
  $\triangledown$ if $\mathcal{M} \models \Gamma$, then $\mathcal{M} \models \varphi$.
  
  In symbols: $\Gamma \vdash_{\mathcal{M}}^{g} \varphi$. 

Remark

Both notions of local and global consequence do not coincide. Consider $\mathcal{M}$:

Then

- since $\mathcal{M}, a \models p$, but $\mathcal{M}, a \not\models \Box p$, then $\{p\} \not\models^l \mathcal{M} \Box p$;
- since $\mathcal{M} \not\models p$, then $\{p\} \models^g \mathcal{M} \Box p$;
- So, $\Box p$ is a global consequence, but not a local consequence of $p$. 