An Introduction to Modal Logic III

Soundness of Normal Modal Logics

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Introduction

Soundness proof

- As usual, a proof of completeness is consists of two parts: **Soundness** and **Completeness**,
- the aim of the **Soundness proof** is proving that certain syntactical properties are preserved in the semantics,
- in particular, we will prove that if a modal formula φ has the properties of being a **theorem** or being **deducible from a set of formulas** Γ, then these properties are preserved in the Kripke frame based semantics as the properties of being φ **valid** or a **semantic consequence** of Γ.
- the Soundness proof follows the **usual pattern**, proving that the closure properties of the deducibility operator $\vdash$ are preserved by the logical consequence operator $\vDash$.

Completeness proof

- The aim of the **Completeness proof** is proving that certain semantical properties are preserved in the syntax,
- in particular, we will prove that if a modal formula φ has the properties of being **valid** or being a **semantic consequence of a set of formulas** Γ, then these properties are preserved in the Hilbert-style syntax (or in the closure operator-like syntax) as the properties of being φ a **theorem** or **deducible** from Γ,
- the Completeness proof follows the **usual pattern**, proving that if a formula φ **is not** a theorem or deducible from a set of formulas Γ, then φ **is not** valid or a semantic consequence of Γ,
- like in the propositional case, the Completeness proof is done using a particular kind of **semantical structures built up directly from the syntax**,
- in this case they are called **Canonical frames** and **models**.

Preliminaries

(a reminder)

Syntax

Language

- A countable set of propositional variables $Prop = \{p, q, \dots\}$,
- the classical propositional constants $\top$ and $\bot$,
- the classical propositional connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$,
- two unary modal connectives $\Box$ and $\Diamond$.

Formulas

The set Φ of modal formulas is inductively built from *Prop* in the following way:

- Propositional variables and constants are formulas,
- if φ and ψ are formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\neg\varphi$ are formulas,
- if φ is a formula, then $\Box\varphi$ and $\Diamond\varphi$ are formulas.

Normal Modal Logics

Definition

A *normal modal logic* Λ is a **set of formulas** containing:

- all classical tautologies (in the modal language),
- $\Box(p \to q) \to (\Box p \to \Box q)$ (axiom (K)),
- $\Box\, p \leftrightarrow \neg\Diamond\neg p$,
- $\Diamond p \leftrightarrow \neg\Box\neg p$,

and is closed under:

(MP) Modus Ponens: if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$,

(US) Uniform Substitution: if $\varphi \in \Lambda$, then $\psi \in \Lambda$, where ψ is obtained from φ by replacing propositional variables by arbitrary formulas,

(G) Generalization: if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$.

Theorems and Deducibility

Let Λ be a modal logic $\Gamma \cup \varphi$ be a set of modal formulas, then:

- We say that a formula φ is **deducible** from Γ, if there is a sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi = \varphi_n$ and each φ_i either belongs to Γ or is an axiom of Λ or is obtained from previous formulas by applying (MP), (US) and (G).

$$\Gamma \vdash_\Lambda \varphi.$$

- We say that a formula φ is a **theorem** of Λ it is deducible from the empty set of formulas.

$$\vdash_\Lambda \varphi.$$

Kripke frames and models

A **Kripke frame** is a structure $\mathfrak{F} = \langle W, R \rangle$, where:

- W is a non-empty set of elements, often called **possible worlds**,
- $R \subseteq W \times W$ is a binary relation on W, called the **accessibility relation** of W.

A **Kripke model** is a structure $\mathfrak{M} = \langle W, R, V \rangle$, where:

- $\langle W, R \rangle$ is a Kripke frame,
- $\mathbf{V} : \mathbf{Prop} \times \mathbf{W} \longrightarrow \{\mathbf{0}, \mathbf{1}\}$ is a function that assigns a boolean value to every ordered pair of propositional variables and possible worlds.

Evaluation of formulas

Given a Kripke model $\mathfrak{M} = \langle W, R, V\rangle$ and a world $w \in W$, the evaluation V of propositional variables can be inductively extended to arbitrary formulas in the following way:

- $V(\top, w) = 1$,
- $V(\bot, w) = 0$,
- $V(\varphi \wedge \psi, w) = \min\{V(\varphi, w), V(\psi, w)\}$,
- $V(\varphi \vee \psi, w) = \max\{V(\varphi, w), V(\psi, w)\}$,
- $V(\varphi \to \psi, w) = \max\{1 - V(\varphi, w), V(\psi, w)\}$,
- $V(\neg\varphi, w) = 1 - V(\varphi, w)$,
- $V(\Box\varphi, w) = (\forall v)(R(w, v) \Rightarrow V(\varphi, v))$,
- $V(\Diamond\varphi, w) = (\exists v)(R(w, v) \wedge V(\varphi, v))$.

Local and Global Satisfiability

- We say that a formula φ is **locally satisfiable**, if there exists a model $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$, such that

$$\mathfrak{M}, w \vDash \varphi$$

- We say that a formula φ is **globally satisfiable**, in a model $\mathfrak{M} = \langle W, R, V \rangle$, if φ is (locally) satisfiable in every point $w \in W$. In symbols:

$$\mathfrak{M} \vDash \varphi$$

Validity

- We say that a formula φ is **valid in a frame** $\mathfrak{F} = \langle W, R \rangle$, if for every model $\mathfrak{M} = \langle W, R, V \rangle$ and every $w \in W$, it holds that $\mathfrak{M}, w \vDash \varphi$. In symbols

$$\mathfrak{F} \vDash \varphi$$

- We say that a formula φ is **valid in a class of frames F** if it is valid in every frame $\mathfrak{F} \in \mathbf{F}$. In symbols:

$$\mathbf{F} \vDash \varphi$$

- We say that a formula φ is **valid**, if it is valid in every class of frames $\mathbf{F}$. In symbols:

$$\vDash \varphi$$

Semantic Consequence relations

Let $\Gamma \cup \varphi$ be a set of modal formulas and $\mathbf{M}$ a class of models, then:

- We say that a formula φ is a **local consequence** of Γ over $\mathbf{M}$, if for all models $\mathfrak{M} = \langle W, R, V \rangle \in \mathbf{M}$ and all points $w \in W$, it holds that
  - if $\mathfrak{M}, w \models \Gamma$, then $\mathfrak{M}, w \models \varphi$.

  In symbols: $\Gamma \models^{l}_{\mathbf{M}} \varphi$.

- We say that a formula φ is a **global consequence** of Γ over $\mathbf{M}$, if for all models $\mathfrak{M} = \langle W, R, V \rangle \in \mathbf{M}$ it holds that
  - if $\mathfrak{M} \models \Gamma$, then $\mathfrak{M} \models \varphi$.

  In symbols: $\Gamma \models^{g}_{\mathbf{M}} \varphi$.

Soundness

Propositional formulas and rules

- from the definition of Kripke model, every propositional theorem (in the modal language) is true in every node w of every Kripke model;
- for the same reason, (MP) and (US) are sound rules in every node w of every Kripke model;

Rule (G)

For (G) rule, consider that:

1. if φ is a theorem, then it is true in every node w of every Kripke model;
2. hence, given a node v of a Kripke model, φ is true in every successor of v;
3. therefore, $\Box\varphi$ is true in v.

So, (G) is a sound rule in every node w of every Kripke model.

Axiom $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model and $w \in W$, then:

- formula $\Diamond\varphi$ is true in w iff
- there exists $v \in W$ such that both $R(w, v)$ and φ is true in v, iff
- it is not true that, in every $v \in W$ such that $R(w, v)$, formula φ is false, iff
- it is not true that, in every $v \in W$ such that $R(w, v)$, formula $\neg\varphi$ is true, iff
- it is not true that formula $\Box\neg\varphi$ is true in w, iff
- formula $\Box\neg\varphi$ is false in w, iff
- formula $\neg\Box\neg\varphi$ is true in w.

Axiom (K)

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and $w \in W$, then:

- suppose that formula $\Box(\varphi \rightarrow \psi)$ is true in w,
- then, for every $v \in W$ such that $R(w, v)$ it holds that $\varphi \rightarrow \psi$ is true in v.
- Let formula $\Box\varphi$ be true in w,
- then, for every $v \in W$ such that $R(w, v)$ it holds that φ is true in v,
- since $\varphi \rightarrow \psi$ is true in v, by (MP), we obtain that ψ is true in every v such that $R(w, v)$.
- Hence $\Box\psi$ is true in w too.
- therefore formula $\Box\varphi \rightarrow \Box\psi$ is true in w.