An Introduction to Modal Logic IV

Canonical Completeness

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General intuition

- like in the propositional case, the trick consists in obtaining **a model from the set of formulas** and their mutual relations;
- in this case it has clearly to be a **Kripke model**;
- each node w of a Kripke model can be seen as a **propositional evaluation** or as a **set of formulas**, indeed the set of all modal formulas that are true in w;
- as a propositional evaluation, w is a **consistent** set of formulas;
- as a consistent set of formulas w is **maximal**;
- so, the general idea consists in building a Kripke model up from **maximally consistent sets of formulas** (MCSs).

Maximally consistent sets of formulas

Maximally consistent sets of formulas

Given a modal logic Λ:

- a set Σ of formulas is Λ-**consistent** if $\Sigma \nvdash_{\Lambda} \bot$;
- a set Σ of formulas is **maximally** Λ-**consistent** if it is Λ-consistent and it is not properly included in any Λ-consistent set.

Properties of MCSs

Given a modal logic Λ, a maximally Λ-consistent set Σ and formulas φ, ψ, then:

- if $\Sigma \vdash_\Lambda \varphi$ then $\varphi \in \Sigma$;
- if $\varphi, \varphi \to \psi \in \Sigma$ then $\psi \in \Sigma$;
- $\bot \notin \Sigma$;
- $\Lambda \in \Sigma$;
- $\neg\varphi \in \Sigma$ iff $\varphi \notin \Sigma$;
- $\varphi \vee \psi \in \Sigma$ iff either $\varphi \in \Sigma$ or $\psi \in \Sigma$;
- for every formula φ, it holds that either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$.

Lindenbaum's Lemma

Lemma

Any Λ-consistent set of formulas can be extended to a maximally Λ-consistent set of formulas.

Proof (sketch)

- Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of the formulas.
- Define:

1. $\Sigma_0 := \Sigma$;
2. $\Sigma_{n+1} := \begin{cases} \Sigma_n \cup \{\varphi_n\} & \text{if it is consistent} \\ \Sigma_n \cup \{\varphi_n\} & \text{otherwise;} \end{cases}$
3. $\Sigma^+ := \bigcup_{n \geq 0} \Sigma_n$;

Since we have that, for every formula φ, either $\varphi \in \Sigma^+$ or $\neg\varphi \in \Sigma^+$, then Σ^+ is a maximally Λ-consistent set of formulas extending Σ.

Canonical Models

Canonical Models

Given a modal logic Λ, its Canonical Model is a triple

$$\mathfrak{M}_\Lambda = \langle W_\Lambda, R_\Lambda, V_\Lambda \rangle,$$

where:

- W_Λ is the set of all maximally Λ-consistent set of formulas Δ,
- R_Λ is the binary relation defined between maximally Λ-consistent sets of formulas by:

$$\Delta R_\Lambda \Delta' \quad \text{iff} \quad \{\varphi \colon \Box\varphi \in \Delta\} \subseteq \Delta',$$

- $V_\Lambda(p, \Delta) = 1$ iff $p \in \Delta$

Remarks

- the set of possible worlds W_Λ is a syntactic notion;
- there is a huge (infinite) number of possible worlds Δ;
- recall that, from the syntactical notion of deduction, for the propositional variables it is obvious that

$$\Delta \vdash_\Lambda p \quad \text{iff} \quad p \in \Delta;$$

- this means that

$$V_\Lambda(p, \Delta) = 1 \quad \text{iff} \quad \Delta \vdash_\Lambda p;$$

- this is a first bridge between the syntactical notion of deduction and the semantical notion of logical consequence, that will be subsequently expanded to the set of all formulas by proving a Truth Lemma.

The canonical relation I

$$\Diamond\psi \in \Delta$$
$$\text{iff}$$
$$\text{there exists } \Delta' \text{ such that } \{\varphi\colon \Box\varphi \in \Delta\} \subseteq \Delta' \text{ and } \psi \in \Delta'.$$

($\Leftarrow$) Suppose that $\Diamond\psi \notin \Delta$,

- then $\neg\Diamond\psi \in \Delta$;
- then $\Box\neg\psi \in \Delta$;
- then for all Δ' such that $\{\varphi\colon \Box\varphi \in \Delta\} \subseteq \Delta'$, it holds that $\neg\psi \in \Delta'$;
- then does not exist any Δ' such that $\{\varphi\colon \Box\varphi \in \Delta\} \subseteq \Delta'$ and $\psi \in \Delta'$.

($\Rightarrow$) Suppose that $\Diamond\psi \in \Delta$,

- let Δ^- be the set $\{\psi\} \cup \{\varphi \colon \Box\varphi \in \Delta\}$;
- Δ^- is consistent:
  - suppose that Δ^- is not consistent,
  - then there are $\{\varphi_1, \dots, \varphi_n\} \subseteq \{\varphi \colon \Box\varphi \in \Delta\}$ such that
    $$\{\varphi_1, \dots, \varphi_n\} \vdash_\Lambda \neg\psi,$$
  - hence $\{\Box\varphi_1, \dots, \Box\varphi_n\} \vdash_\Lambda \Box\neg\psi$;
  - since $\{\Box\varphi_1, \dots, \Box\varphi_n\} \subseteq \Delta$, by (MP) we have that $\Box\neg\psi \in \Delta$;
  - therefore $\neg\Diamond\psi \in \Delta$, against our assumption.
- by Lindenbaum Lemma, there exists a maximally Λ-consistent set of formulas Δ' such that $\Delta^- \subseteq \Delta'$;
- hence both $\{\varphi \colon \Box\varphi \in \Delta\} \subseteq \Delta'$ and $\psi \in \Delta'$, that is what we wanted to prove.

The canonical relation II

$$\Box\psi \in \Delta$$
$$\text{iff}$$
$$\text{for every } \Delta' \text{ such that } \{\varphi \colon \Box\varphi \in \Delta\} \subseteq \Delta' \text{ it holds that } \psi \in \Delta'.$$

- $\Box\psi \in \Delta$ iff
- $\neg\Diamond\neg\psi \in \Delta$ iff
- $\Diamond\neg\psi \notin \Delta$ iff
- does not exist any Δ' such that $\{\varphi \colon \Box\varphi \in \Delta\} \subseteq \Delta'$ and $\neg\psi \in \Delta'$ iff
- for every Δ' such that $\{\varphi \colon \Box\varphi \in \Delta\} \subseteq \Delta'$ it holds that $\psi \in \Delta'$.

Truth Lemma

Truth Lemma

Lemma

For any normal modal logic Λ, any maximally Λ-consistent set Δ and any formula φ,

$$\mathfrak{M}_{\Lambda}, \Delta \vDash \varphi \quad \textit{iff} \quad \varphi \in \Delta.$$

- The Truth Lemma is a key step in the completeness proof.
- Its proof is (as usual) by induction on formulas.

Induction on formulas: propositional part

(*Prop*) If φ is a propositional variable, then the claim holds by definition of V_Λ.

($\neg$) If $\varphi = \neg\psi$, then

- $\mathfrak{M}_\Lambda, \Delta \vDash \neg\psi$ iff
- $\mathfrak{M}_\Lambda, \Delta \nvDash \psi$ iff
- $\psi \notin \Delta$ iff
- $\neg\psi \in \Delta$.

($\vee$) If $\varphi = \psi \vee \chi$, then

- $\mathfrak{M}_\Lambda, \Delta \vDash \psi \vee \chi$ iff
- either $\mathfrak{M}_\Lambda, \Delta \vDash \psi$ or $\mathfrak{M}_\Lambda, \Delta \vDash \chi$ iff
- either $\psi \in \Delta$ or $\chi \in \Delta$ iff
- $\psi \vee \chi \in \Delta$.

Induction on formulas: propositional part

($\Box$) If $\varphi = \Box\psi$, then

- $\mathfrak{M}_\Lambda, \Delta \vDash \Box\psi$ iff
- for all Δ' such that $\Delta R_\Lambda \Delta'$, it holds that $V(\psi, \Delta')$, iff
- for all Δ' such that $\{\varphi\colon \Box\varphi \in \Delta\} \subseteq \Delta'$, it holds that $\psi \in \Delta'$, iff
- $\Box\psi \in \Delta$.

($\Diamond$) If $\varphi = \Diamond\psi$, then

- $\mathfrak{M}_\Lambda, \Delta \vDash \Diamond\psi$ iff
- there exists Δ' such that $\Delta R_\Lambda \Delta'$ and $V(\psi, \Delta')$, iff
- there exists Δ' such that $\{\varphi\colon \Box\varphi \in \Delta\} \subseteq \Delta'$ and $\psi \in \Delta'$, iff
- $\Diamond\psi \in \Delta$.

Completeness

Completeness Theorem

Theorem

For any set of formulas $\Sigma \cup \{\varphi\}$,

$$\textit{if } \Sigma \models^{l}_{\Lambda} \varphi, \textit{ then } \Sigma \vdash_{\Lambda} \varphi.$$

- as usual, suppose that $\Sigma \nvdash_{\Lambda} \varphi$,
- then the set $\Sigma \cup \{\neg\varphi\}$ is consistent,
- by Lindenbaum Lemma, there is a maximally Λ-consistent set Δ such that $\Sigma \cup \{\neg\varphi\} \subseteq \Delta$,
- hence $\mathfrak{M}_{\Lambda}, \Delta \models_{\Lambda} \Sigma$, but $\mathfrak{M}_{\Lambda}, \Delta \not\models_{\Lambda} \varphi$,
- therefore, $\Sigma \not\models^{l}_{\Lambda} \varphi$.