An Introduction to Modal Logic V

Axiomatic Extensions and Classes of Frames

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INVESTMENTS IN EDUCATION DEVELOPMENT

Introduction

- we have proved that the minimal normal modal logic K is complete with respect to the class of all Kripke frames;
- we can do the same with some particular **axiomatic extensions** of K;
- in this case we consider particular **classes of frames**;
- these axiomatic extensions have usually been **already considered in the literature** before of being studied in the framework of Kripke semantics;
- the classes of frames are defined by particular **properties of the accessibility relation**;
- by means of the methods of canonical models, it is possible to prove a **correspondence** between axiomatic extensions and classes of frames.

Axiomatic Extensions of K

The Logic K

Definition

A *normal modal logic* Λ is a **set of formulas** containing:

- all classical tautologies (in the modal language),
- $\Box(p \to q) \to (\Box p \to \Box q)$ (axiom (K)),
- $\Box\ p \leftrightarrow \neg\Diamond\neg p$,
- $\Diamond p \leftrightarrow \neg\Box\neg p$,

and is closed under:

(MP) Modus Ponens: if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$,

(US) Uniform Substitution: if $\varphi \in \Lambda$, then $\psi \in \Lambda$, where ψ is obtained from φ by replacing propositional variables by arbitrary formulas,

(G) Generalization: if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$.

Axiomatic Extensions: axioms

(4)	$\Box p \to \Box\Box p$
(T)	$\Box p \to p$
(B)	$p \to \Box\Diamond p$
(D)	$\Box p \to \Diamond p$
(E)	$\Diamond p \to \Box\Diamond p$
(M)	$\Box\Diamond p \to \Diamond\Box p$
(G)	$\Diamond\Box p \to \Box\Diamond p$
(L)	$\Box(\Box p \to p) \to \Box p$

Axiomatic Extensions: logics

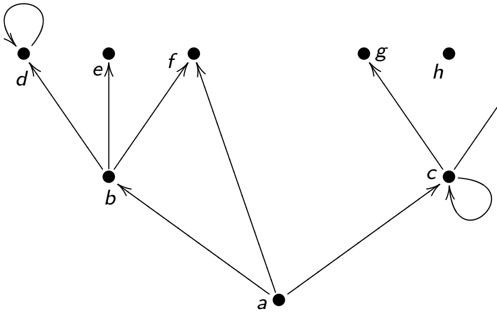
T	$\rightsquigarrow$	KT
$K4$	$\rightsquigarrow$	$K4$
$S4$	$\rightsquigarrow$	$KT4$
B	$\rightsquigarrow$	KTB
$S5$	$\rightsquigarrow$	$KT4B$ or $KT4E$
GL	$\rightsquigarrow$	KL
D	$\rightsquigarrow$	KD
$D4$	$\rightsquigarrow$	$KD4$

Classes of Frames

Kripke frames

A **Kripke frame** is a structure $\mathfrak{F} = \langle W, R \rangle$, where:

- W is a non-empty set of elements, often called **possible worlds**,
- $R \subseteq W \times W$ is a binary relation on W, called the **accessibility relation** of W.



Classes of frames

R is reflexive	$(\forall x)R(x,x)$
R is symmetric	$(\forall x\forall y)(R(x,y) \Rightarrow R(y,x))$
R is transitive	$(\forall x\forall y\forall z)(R(x,y) \wedge R(y,z) \Rightarrow R(x,z))$
R is euclidean	$(\forall x\forall y\forall z)(R(x,y) \wedge R(x,z) \Rightarrow R(y,z))$
R is serial	$(\forall x\exists y)R(x,y)$
R is a partial function	$(\forall x\forall y\forall z)(R(x,y) \wedge R(x,z) \Rightarrow y = z)$
R is a function	R is a function and is serial
R is dense	$(\forall x\forall y)(R(x,y) \Rightarrow (\exists z)(R(x,z) \wedge R(z,y)))$
R is Noetherian	there are no infinite ascending R-chains

Frame Completeness

and

Canonical Frames

Frame Completeness

- We say that a formula φ is a **(local) consequence in a class of frames F** of a set of formulas Σ if for every point w of every model $\mathfrak{M}$ on every frame $\mathfrak{F} \in \mathbf{F}$ it happens that

$$\mathfrak{M}, w \models \Sigma \quad \Rightarrow \quad \mathfrak{M}, w \models \varphi$$

in symbols:

$$\Sigma \models_{\mathbf{F}} \varphi$$

- For any normal modal logic Λ, we say that it is **sound** with respect to a class of frames $\mathbf{F}$ if, for any set of formulas $\Sigma \cup \varphi$:

$$\Sigma \vdash_{\Lambda} \varphi \quad \Rightarrow \quad \Sigma \models_{\mathbf{F}} \varphi$$

- For any normal modal logic Λ, we say that it is **(strongly) complete** with respect to a class of frames $\mathbf{F}$ if, for any set of formulas $\Sigma \cup \varphi$:

$$\Sigma \models_{\mathbf{F}} \varphi \quad \Rightarrow \quad \Sigma \vdash_{\Lambda} \varphi$$

Remarks

- the soundness proof consists in proving that **the axiom that extends** K **is sound** in the class of frames considered;
- about completeness, consider that:

$$\Sigma \models_{\mathbf{F}} \varphi \quad \Rightarrow \quad \Sigma \vdash_{\Lambda} \varphi$$

if and only if

$$\Sigma \nvdash_{\Lambda} \varphi \quad \Rightarrow \quad \Sigma \not\models_{\mathbf{F}} \varphi$$

  So, we have to find a frame $\mathfrak{F} \in \mathbf{F}$ such that $\mathfrak{F} \models \Sigma$ but $\mathfrak{F} \not\models \varphi$, under the assumption that φ is not deducible from Σ;

- if Λ is complete with respect to its canonical frame $\mathfrak{F}_{\Lambda}$, it is enough to prove that $\mathfrak{F}_{\Lambda} \in \mathbf{F}$.

Example I:

the logic T and

the class of reflexive frames

T and the class of reflexive frames

T is the axiomatic extension of K by means of axiom:

$(T): \quad \Box p \to p$

- About soundness:
  - we have already proved that the duality axioms and (K) are sound in every frame;
  - in the same way we have proved that deductive rules (MP), (US) and (G) preserve the truth in every frame;
  - we only need to prove that (T) **is sound in every reflexive frame**.
- About completeness:
  - we need to build **a reflexive frame** which, under the supposition that $\Sigma \nvdash_T \varphi$, satisfies Σ but not φ;
  - we only need to prove that **the canonical frame $\mathfrak{F}_T$ is reflexive**.

Soundness

- Let $\mathfrak{F} = \langle W, R \rangle$ be a frame and assume that R is reflexive,
- let $\mathfrak{M} = \langle W, R, V \rangle$ be any model on $\mathfrak{F}$ and $w \in W$,
- suppose that $\mathfrak{M}, w \vDash \Box p$,
- hence $(\forall v)(R(w, v) \Rightarrow V(p, v))$,
- then p is true in every point v such that $R(w, v)$,
- since R is reflexive, we have that $R(w, w)$,
- then p is true at point w,
- so, $\Box p \to p$ is true at w.

Completeness

- Assume that $\Box p \rightarrow p$ is a valid formula of Λ,
- then $\Box p \rightarrow p \in \Delta$, for every maximally Λ-consistent set Δ,
- hence $\Box p \rightarrow p$ is true at every point of the canonical frame $\mathfrak{F}_\Lambda = \langle W_\Lambda, R_\Lambda \rangle$;
- let $\Delta \in W_\Lambda$ and let $\Box \varphi \in \Delta$,
- by (MP), we have that $\varphi \in \Delta$,
- by definition of R_Λ, we have that $R(\Delta, \Delta')$ iff $\{\psi \colon \Box \psi \in \Delta\} \subseteq \Delta'$
- hence $R(\Delta, \Delta)$,
- so, $\mathfrak{F}_\Lambda = \langle W_\Lambda, R_\Lambda \rangle$ is reflexive.

Example II:

the logic $K4$ and

the class of transitive frames

$K4$ and the class of transitive frames

$K4$ is the axiomatic extension of K by means of axiom:

(4) : $\Box p \rightarrow \Box\Box p$

- About soundness:
  - we have already proved that the duality axioms and (K) are sound in every frame;
  - in the same way we have proved that deductive rules (MP), (US) and (G) preserve the truth in every frame;
  - we only need to prove that (4) **is sound in every transitive frame**.
- About completeness:
  - we need to build **a transitive frame** which, under the supposition that $\Sigma \nvdash_T \varphi$, satisfies Σ but not φ;
  - we only need to prove that **the canonical frame $\mathfrak{F}_4$ is transitive**.

Soundness

- Let $\mathfrak{F} = \langle W, R \rangle$ be a frame and assume that R is transitive,
- let $\mathfrak{M} = \langle W, R, V \rangle$ be any model on $\mathfrak{F}$ and $w \in W$,
- suppose that $\mathfrak{M}, w \vDash \Box p$,
- hence $(\forall v)(R(w, v) \Rightarrow V(p, v))$,
- then p is true in every point x such that $R(w, x)$,
- let $v, u \in W$ be any points such that $R(w, v)$ and $R(v, u)$, by transitivity we have that $R(w, u)$
- hence p is true at point u,
- then $\Box p$ is true at point v,
- therefore $\Box\Box p$ is true at point w,
- so, $\Box p \rightarrow \Box\Box p$ is true at w.

Completeness

- Assume that $\Box p \to \Box\Box p$ is a valid formula of Λ,
- then $\Box p \to \Box\Box p \in \Delta$, for every maximally Λ-consistent set Δ,
- hence $\Box p \to \Box\Box p$ is true at every point of the canonical frame $\mathfrak{F}_\Lambda = \langle W_\Lambda, R_\Lambda \rangle$;
- let $\Delta, \Delta', \Delta'' \in W_\Lambda$ be such that $R_\Lambda(\Delta, \Delta')$ and $R_\Lambda(\Delta', \Delta'')$,
- if $\Box\varphi \in \Delta$, by (MP), we have that $\Box\Box\varphi \in \Delta$,
- by definition of R_Λ, we have that $R(\Delta, \Gamma)$ iff $\{\psi \colon \Box\psi \in \Delta\} \subseteq \Gamma$
- hence $\Box\varphi \in \Delta'$ and $\varphi \in \Delta''$,
- therefore $R_\Lambda(\Delta, \Delta'')$,
- so, $\mathfrak{F}_\Lambda = \langle W_\Lambda, R_\Lambda \rangle$ is transitive.