An Introduction to Modal Logic VI

Beyond Canonicity

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Introduction

- we have proved that
  - the normal modal logic **T** is complete with respect to the class of **reflexive Kripke frames**;
  - the normal modal logic **K**4 is complete with respect to the class of **transitive Kripke frames**;
  - the normal modal logic **S**4 is complete with respect to the class of **reflexive-transitive Kripke frames**;
- using the canonical frame method it is possible to prove further results.

Frame Completeness results

T	$\rightsquigarrow$	reflexive frames
$K4$	$\rightsquigarrow$	transitive frames
$S4$	$\rightsquigarrow$	reflexive and transitive frames
B	$\rightsquigarrow$	reflexive and symmetric frames
$S5$	$\rightsquigarrow$	reflexive, symmetric and transitive frames
D	$\rightsquigarrow$	serial frames
$D4$	$\rightsquigarrow$	serial and transitive frames
GL	$\rightsquigarrow$	transitive and Noetherian frames

Frame Completeness and Canonicity

- If a logic Λ is **weakly complete with respect to a class of frame**, we say that it is **frame complete**;
- if the theorems of logic Λ are valid in its canonical frame, we say that it is **Canonical**;
- Canonicity implies Frame Completeness.. **but the inverse implication does not hold!**;
- the logic $\mathbf{G}L$ is an example of frame complete logic that **is not Canonical**.

GL is not Canonical

Preliminaries

- The logic **G**L is the extension of K by means of the axiom:

$$\Box(\Box p \to p) \to \Box p$$

- GL is complete with respect to **transitive** and **Noetherian** frames;
- we recall that, a frame is **Noetherian** if there are no infinite ascending R-chains;
- Nevertheless, GL **is not Canonical**.

An indirect proof

- The reason is that the canonical frame $\mathfrak{F}_{GL}$ of GL **is not Noetherian**,
- **for the moment** we have not the tools to prove directly this fact,
- so, we will prove the non-canonicity of GL **indirectly**,
- we will use the fact that the consequence relation defined by frames is not finitary.

Canonicity and Finitarity

- A consequence relation $\models$ is **finitary** if, every time that $\Sigma \models \varphi$, there is a finite subset $\Gamma \subseteq \Sigma$ such that $\Gamma \models \varphi$.
- Given a logic Λ, by definition of deducibility, the operator $\vdash_\Lambda$ is finitary.
- Strong Frame Completeness proves that

$$\Sigma \models^{l}_{Fr(\Lambda)} \varphi \iff \Sigma \vdash_\Lambda \varphi.$$

- Hence, if Λ is strongly frame complete, then $\models^{l}_{Fr(\Lambda)}$ is finitary.
- As we have seen, if Λ is canonical, then it is strongly frame complete.
- Hence, if $\models^{l}_{Fr(\Lambda)}$ is not finitary, then Λ is not canonical.

$\models^l_{GL}$ is not finitary

Consider the set of formulas:

$$\Sigma = \{\Diamond p_0\} \cup \{\Box(p_n \to \Diamond p_{n+1})\colon n \in \mathbb{N}\}$$

We will prove that

- $\Sigma \models^l_{Fr(GL)} \bot$ and
- for every finite $\Gamma \subseteq \Sigma$, we have that $\Gamma \not\models^l_{Fr(GL)} \bot$

$\Sigma \models^{l}_{GL} \bot$

In order to prove that $\Sigma \models^{l}_{GL} \bot$, we have to prove that Σ **is false** at every point w of every model $\mathfrak{M}$ built on a frame in $Fr(GL)$.

- Suppose the contrary: so there exist $\mathfrak{F} \in Fr(GL)$, $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $w_0 \in W$ such that $\mathfrak{M}, w_0 \models^{l}_{GL} \Sigma$;
- since $\Diamond p_0$ is true at w_0, then there exists $w_1 \in W$ such that $R(w_0, w_1)$ and p_0 is true at w_1;
- since $\Box(p_0 \rightarrow \Diamond p_1)$ is true at w_0 and $R(w_0, w_1)$, then $p_0 \rightarrow \Diamond p_1$ is true at w_1;
- hence, by (MP), $\Diamond p_1$ is true at w_1;

- since $\Diamond p_1$ is true at w_1, then there exists $w_2 \in W$ such that $R(w_1, w_2)$ and p_1 is true at w_2;
- since $R(w_0, w_1)$, and $R(w_1, w_2)$, by transitivity we have that $R(w_0, w_2)$;
- since $\Box(p_1 \rightarrow \Diamond p_2)$ is true at w_0 and $R(w_0, w_2)$, then $p_1 \rightarrow \Diamond p_2$ is true at w_2;
- hence, by (MP), $\Diamond p_2$ is true at w_2;
- repeating the process, we obtain an infinite ascending R-chain, which is impossible, since $\mathfrak{F} \in Fr(GL)$.

For every finite $\Gamma \subseteq \Sigma$, we have that $\Gamma \nvDash^{l}_{Fr(GL)} \bot$

In order to prove that for every finite $\Gamma \subseteq \Sigma$, we have that $\Gamma \nvDash^{l}_{Fr(GL)} \bot$, we have to prove that any finite $\Gamma \subseteq \Sigma$ **is satisfiable** in a model $\mathfrak{M}$ built on a frame in $Fr(GL)$.

- Let Γ a finite subset of Σ;
- let n be the maximum number such that p_n occurs in Γ;
- consider the model $\mathfrak{M} = \langle W, R, V \rangle$, where:
  - $W = \{0, 1, \ldots, n+1\}$,
  - $R =<$,
  - for $m \leq n$, we have that $V(p_m, m+1) = 1$ and 0 otherwise;

- $\mathfrak{M}$ is built on a transitive and Noetherian frame;
- hence $\mathfrak{M}$ is a model of GL;
- moreover, the finite subset $\{\Diamond p_0\} \cup \{\Box(p_m \to \Diamond p_{m+1})\colon m < n\}$ of Σ is satisfiable in $\mathfrak{M}$;
- since $\Gamma \subseteq \{\Diamond p_0\} \cup \{\Box(p_m \to \Diamond p_{m+1})\colon m < n\}$, then Γ is satisfiable in $\mathfrak{M}$;
- hence $\Gamma \not\models^{l}_{Fr(GL)} \bot$.

Note: being Noetherian is not first order condition

Through a similar compactness argument it can be proved that the condition of being Noetherian **is not first order definable**.

- Suppose it is, then there is a set of formulas Γ in the first order language with one binary relation R whose models are precisely Noetherian structures;
- consider the set of formulas:

$$\Sigma = \Gamma \cup \{R(c_n, c_{n+1}) \colon n \in \mathbb{N}\}$$

  in the first order language with R and a countable set of new constants $\{c_n \colon n \in \mathbb{N}\}$;
- since Γ defines Noetherian, then every finite subset of Σ is satisfiable;
- by Compactness Theorem in first order logic, Σ is satisfiable;
- hence, in the model that satisfies Σ there is an infinite ascending R-chain, which is impossible.