An Introduction to Modal Logic VII

The finite model property

Marco Cerami

Palacký University in Olomouc
Department of Computer Science
Olomouc, Czech Republic

Olomouc, November 21^{st} 2013

european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

Introduction

- Two important properties of normal modal logic are the **finite model property** and the **finite frame property**;
- these properties are strictly related to each other;
- they are related to **frame completeness** and **decidability** too;
- this differentiates modal logic from first order logic;
- many normal modal logics have been proven to **have these properties**;

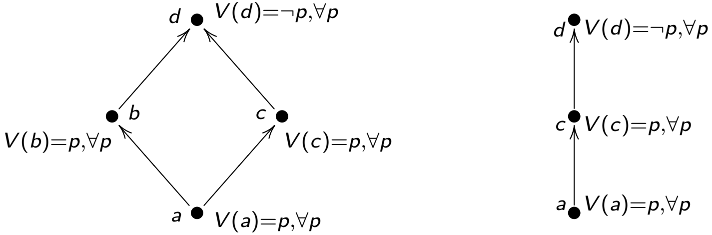
Modally equivalent

and

differentiated models

Modally equivalent models

Two models $\mathfrak{M}$ and $\mathfrak{M}'$ are **modally equivalent** if the same formulas are valid in $\mathfrak{M}$ and $\mathfrak{M}'$.

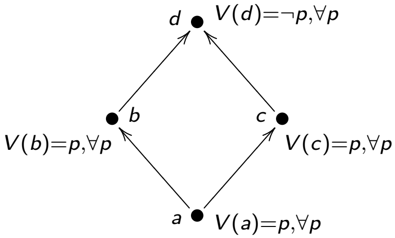


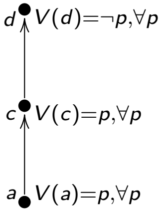
These models are modally equivalent.

Differentiated models

A model $\mathfrak{M} = \langle W, R, V \rangle$ is called **differentiated** if for every two points $w, v \in W$ such that $w \neq v$, there is a formula φ such that

$$\mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, v \not\models \varphi.$$





The first model is not a differentiated one, because, every formula φ is true at b iff it is true at c. The second is indeed differentiated.

Finite differentiated models and Λ-frames

- Let Λ be a normal modal logic and $\mathfrak{M} = \langle W, R, V \rangle$ be a **finite differentiated model** of Λ;
- we want to show that every formula $\varphi \in \Lambda$ is **valid** in the frame $\mathfrak{M} = \langle W, R \rangle$.
- In search of a contradiction, suppose that there is $\alpha \in \Lambda$, a model $\mathfrak{M}' = \langle W, R, V' \rangle$ and $w \in W$ such that

$$\mathfrak{M}', w \not\models \alpha,$$

- since $\mathfrak{M}$ is finite and differentiated, for every $v \in W$ there is a formula ψ_v which is **true just in** v (it is the conjunction of formulas true in v which differentiate v from other points);

- for every propositional variable p, consider the formula

$$\chi_p := \bigvee_{v \in V'(p)} \psi_v,$$

- clearly $V'(p) = V(\chi_p)$.
- Define the substitution:

$$\sigma(p) = \chi_p,$$

  for every propositional variable p;
- it is easy to see that for every formula φ it holds

$$V'(\varphi) = V(\sigma(\varphi)).$$

- By the initial assumption we have that $w \notin V'(\alpha)$,
- then, $w \notin V(\sigma(\alpha))$,
- but $\alpha \in \Lambda$ and, hence $\sigma(\alpha) \in \Lambda$,
- therefore $\mathcal{M}$ is not a model of Λ, a contradiction.
- So, every formula $\varphi \in \Lambda$ is **valid** in the frame $\mathfrak{M} = \langle W, R \rangle$.

The quotient of a model

The relation $\sim$

- Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. For every $w \in W$ let
$$Th_{\mathfrak{M}}(w) = \{\varphi \colon \mathfrak{M}, w \models \varphi\}.$$

- We define the relation $\sim$ on W by:
$$w \sim v \quad \text{iff} \quad Th_{\mathfrak{M}}(w) = Th_{\mathfrak{M}}(v).$$

- Clearly $\sim$ is an **equivalence relation**.

- We denote by $[w]$ the **equivalence class** of w through $\sim$;

The quotient of a model

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. We consider the **quotient model**

$$\mathfrak{M}_\sim = \langle W_\sim, R_\sim, V_\sim \rangle$$

where:

- $W_\sim$ is the set of all the equivalence classes through $\sim$;
- $R_\sim$ is a binary relation on $W_\sim$ defined by:

$$R_\sim([w],[v]) \quad \text{iff} \quad \exists w' \in [w], \exists v' \in [v] \text{ such that } R(w', v');$$

- $V_\sim$ is a valuation of the propositional variables defined by:

$$V_\sim(p,[w]) = 1 \quad \text{iff} \quad V(p, w) = 1;$$

Modal equivalence between $\mathfrak{M}_\sim$ and $\mathfrak{M}$

- Given a model $\mathfrak{M} = \langle W, R, V \rangle$ it is easy to prove that $\mathfrak{M}_\sim$ and $\mathfrak{M}$are **modally equivalent**;
- what we need to prove is that, for every formula φ and every point $w \in W$, it holds that:

$$\mathfrak{M}, w \vDash \varphi \quad \text{iff} \quad \mathfrak{M}_\sim, [w] \vDash \varphi$$

- the proof is made by an easy **induction**.

Proof

Var If φ is a **propositional variable**, then, by definition of $V_\sim$, we have that

$$V_\sim(p, [w]) = 1 \quad \text{iff} \quad V(p, w) = 1;$$

Bool If φ is a **boolean combination** of the formulas ψ and χ, then, suppose, by induction hypothesis, that for every point $w \in W$:

$$V_\sim(\psi, [w]) = 1 \quad \text{iff} \quad V(\psi, w) = 1;$$

and the same for χ. Hence, if e.g. $\varphi = \psi \wedge \chi$

- $V_\sim(\varphi, [w]) = 1$ iff,
- $V_\sim(\psi \wedge \chi, [w]) = 1$ iff,
- $V_\sim(\psi, [w]) = 1$ and $V_\sim(\chi, [w]) = 1$ iff,
- $V(\psi, w) = 1$ and $V(\chi, w) = 1$ iff,
- $V(\psi \wedge \chi, w) = 1$ iff,
- $V(\varphi, w) = 1$.

Mod If φ is **modal formula** $\Diamond\psi$ then suppose, by induction hypothesis, that for every point $v \in W$:

$$V_{\sim}(\psi, [v]) = 1 \quad \text{iff} \quad V(\psi, v) = 1;$$

Hence,

- $V_{\sim}(\Diamond\psi, [w]) = 1$ iff,
- there exists $[v] \in W_{\sim}$ such that
  - ⋆ $R_{\sim}([w], [v])$
  - ⋆ $V_{\sim}(\psi, [v]) = 1$ iff,
- there exists $[v] \in W_{\sim}$ such that
  - ⋆ $\exists w' \in [w], \exists v' \in [v]$ such that $R(w', v')$
  - ⋆ $V(\psi, v) = 1$ iff,
- $\exists w' \in [w], \exists v' \in [v]$ such that $R(w', v')$ and $V(\psi, v') = 1$ iff,
- $V(\Diamond\psi, w') = 1$ iff,
- $V(\Diamond\varphi, w) = 1$.

$\mathfrak{M}_\sim$ is a differentiated model

- For every model $\mathfrak{M}$, the quotient $\mathfrak{M}_\sim$ is a **differentiated model**.
- In order to see it, let $[w], [v] \in W_\sim$ be such that
$$[w] \neq [v],$$
- then, by definition $w \nsim v$,
- hence there exists φ such that, $\varphi \in Th_{\mathcal{M}}(w)$ and $\varphi \notin Th_{\mathcal{M}}(v)$,
- since $\mathcal{M}$ and $\mathcal{M}_\sim$ are modally equivalent, then
$$\mathcal{M}_\sim, w \vDash \varphi \quad \text{and} \quad \mathcal{M}_\sim, v \nvDash \varphi.$$
- Hence $Th_{\mathcal{M}_\sim}([w]) \neq Th_{\mathcal{M}_\sim}([v])$.

The finite model property

and

the finite frame property

The finite model and frame properties

- A normal modal logic Λ has the **finite model property** (f.m.p.) if
  - for every formula φ that **is not a theorem** of Λ,
  - there is a finite model $\mathfrak{M}$ of Λ where φ is **not valid**.

- A normal modal logic Λ has the **finite frame property** (f.f.p.) if
  - for every formula φ that **is not a theorem** of Λ,
  - there is a finite frame $\mathfrak{F}$ where all the formulas of Λ are valid and φ **is not valid**.

Relation between the two properties

- A normal modal logic Λ has the finite model property **if and only if** it has the finite frame property.

- Clearly the f.f.p. implies the f.m.p.
- On the other hand, suppose now that Λ has the f.m.p. and let $\varphi \notin \Lambda$;
- by f.m.p., there is a **finite model** $\mathcal{M}$ where φ is **not valid**;
- consider $\mathcal{M}_\sim$, it is **differentiated** and **modally equivalent** to $\mathcal{M}$, hence φ is **not valid** in $\mathcal{M}_\sim$;
- moreover, since $W_\sim$ is the quotient of W, then $\mathcal{M}_\sim$ is a **finite model**.
- Since $\mathcal{M}_\sim$ is finite and differentiated, then φ is **not valid in its frame**, which is finite.
- So, φ is **not valid in a finite frame**.

f.f.p. and weak frame completeness

- If a normal modal logic Λ has the finite frame property, then the set of its **theorems** is characterized by **the class of its finite frames**.
- In this sense it is **weakly frame complete** with respect to the class of its finite frames.
- As a straightforward consequence of the previous result, we have that **f.m.p. also implies weak frame completeness**.