An Introduction to Modal Logic VIII

Filtrations

Marco Cerami

Palacký University in Olomouc
Department of Computer Science
Olomouc, Czech Republic

Olomouc, November 21^{st} 2013

european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

Introduction

- Canonical models are a very powerful tool;
- they give a really **clear insight** of the relations between syntax and semantics in Modal Logic;
- nevertheless they are **not useful in practice**;
- indeed, with a countably infinite set of propositional variables, there are **too many maximally consistent sets** of formulas;
- this makes canonical models **too huge** to be used in practice;
- so, what we need is a tool that **reduces the size of models** while, at the same time, **preserves the truth of (certain) formulas**;
- this kind of task is offered us by the **filtration** method.

Filtrations

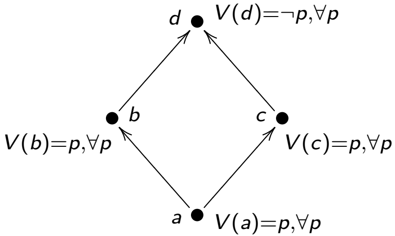
The relation $\sim_\Sigma$

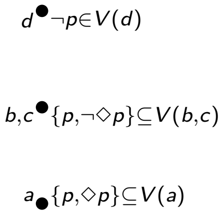
- A set of formulas Σ is **closed under subformulas** if, for any formula φ, ψ it holds that if $\varphi \in \Sigma$ and ψ is a subformula of φ, then $\psi \in \Sigma$.
- For example the set $\Sigma = \{p \wedge q, p, q\}$ is closed under subformulas.
- Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and Σ be a set of formulas closed under subformulas. For every $w \in W$ let
$$Th^{\Sigma}_{\mathfrak{M}}(w) = \{\varphi \in \Sigma \colon \mathfrak{M}, w \models \varphi\}.$$
- We define the relation $\sim_\Sigma$ on W by:
$$w \sim_\Sigma v \text{ iff } Th^{\Sigma}_{\mathfrak{M}}(w) = Th^{\Sigma}_{\mathfrak{M}}(v).$$

  Clearly $\sim_\Sigma$ is an **equivalence relation**.

The quotient of a model through a set Σ

- We denote by $[w]_\Sigma$ the **equivalence class** through $\sim_\Sigma$;
- the **quotient set** of all these equivalence classes will be denoted by W_Σ;
- for example, the sets $\{a\}, \{b, c\}$ and $\{d\}$ are the equivalence classes of the models below through the set of formulas $\{p, \Diamond p\}$:





Σ-appropriate relations

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and Σ be a set of formulas closed under subformulas. We say that a binary relation S on W_Σ is Σ-**appropriate** if for every $w, v \in W$ it holds that:

1. if $R(w, v)$, then $S([w]_\Sigma, [v]_\Sigma)$,
2. if
   - $\Diamond\varphi \in \Sigma$,
   - $S([w]_\Sigma, [v]_\Sigma)$,
   - $\mathcal{M}, v \vDash \varphi$,

   then $\mathcal{M}, w \vDash \Diamond\varphi$.

Σ-appropriate relations: example

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and Σ be a set of formulas closed under subformulas. Define the relation R^s_Σ on W_Σ by:

$$R^s_\Sigma([w]_\Sigma, [v]_\Sigma) \quad \text{iff} \quad \exists w' \in [w]_\Sigma, \exists v' \in [v]_\Sigma \text{ s.t. } R(w', v')$$

Then it is easy to prove that

- R^s_Σ is an appropriate relation,
- R^s_Σ is the smallest appropriate relation.

R^s_Σ is an appropriate relation

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and Σ be a set of formulas closed under subformulas. Consider the relation R^s_Σ on W_Σ. Then

1. On the one end
   - if $R(w, v)$,
   - then $\exists w' \in [w]_\Sigma, \exists v' \in [v]_\Sigma$ s.t. $R(w', v')$,
   - hence $R^s_\Sigma([w]_\Sigma, [v]_\Sigma)$.
2. On the other hand,
   - since $\Diamond\varphi \in \Sigma$, then $\varphi \in \Sigma$,
   - since $R^s_\Sigma([w]_\Sigma, [v]_\Sigma)$, then $\exists w' \in [w]_\Sigma, \exists v' \in [v]_\Sigma$ s.t. $R(w', v')$,
   - since $\mathcal{M}, v \models \varphi$, then $\mathcal{M}, v' \models \varphi$,
   - hence $\mathcal{M}, w' \models \Diamond\varphi$,
   - therefore $\mathcal{M}, w \models \Diamond\varphi$.

R^s_Σ is the smallest appropriate relation

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and Σ be a set of formulas closed under subformulas. Consider the relation R^s_Σ on W_Σ and let S be a Σ-appropriate relation on W_Σ. Then

- Let $[w]_\Sigma, [v]_\Sigma \in W_\Sigma$ be such that $R^s_\Sigma([w]_\Sigma, [v]_\Sigma)$,
- then $\exists w' \in [w]_\Sigma, \exists v' \in [v]_\Sigma$ s.t. $R(w', v')$,
- since S is Σ-appropriate, then $S([w']_\Sigma, [v']_\Sigma)$,
- since $[w']_\Sigma = [w]_\Sigma$ and $[v']_\Sigma = [v]_\Sigma$, then $S([w]_\Sigma, [v]_\Sigma)$.

Filtrations

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and Σ a set of formulas closed under subformulas, then a **filtration** of $\mathfrak{M}$ through Σ is a model

$$\mathfrak{M}_\Sigma = \langle W_\Sigma, R_\Sigma, V_\Sigma \rangle$$

where:

- W_Σ is the **quotient set** of W through Σ,
- R_Σ is a Σ-**appropriate** binary relation on W,
- V_Σ is a **valuation** of the propositional variables defined by:

$$V_\Sigma(p, [w]_\Sigma) = 1 \quad \text{iff} \quad V_\Sigma(p, w) = 1;$$

Modal equivalence between $\mathfrak{M}_\Sigma$ and $\mathfrak{M}$

- Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and Σ a set of formulas closed under subformulas.
- Let $\mathfrak{M}_\Sigma = \langle W_\Sigma, S, V_\Sigma \rangle$ be the filtration of $\mathfrak{M}$ through Σ and S a Σ-appropriate relation on W_Σ.
- It is easy to prove that $\mathfrak{M}_\Sigma$ and $\mathfrak{M}$ are **modally equivalent**;
- what we need to prove is that, **for every formula** $\varphi \in \Sigma$ and every point $w \in W$, it holds that:

$$\mathfrak{M}, w \vDash \varphi \quad \text{iff} \quad \mathfrak{M}_\Sigma, [w]_\Sigma \vDash \varphi$$

- the proof is made by an easy **induction**.

Proof

Var If φ is a **propositional variable** $p \in \Sigma$, then, by definition of V_Σ, we have that

$$V\Sigma(p,[w]) = 1 \quad \text{iff} \quad V(p,w) = 1;$$

Bool If $\varphi \in \Sigma$ is a **boolean combination** of the formulas ψ and χ, then, $\psi, \chi \in \Sigma$, then suppose, by induction hypothesis, that for every point $w \in W$:

$$V\Sigma(\psi,[w]\Sigma) = 1 \quad \text{iff} \quad V(\psi,w) = 1;$$

and the same for χ. Hence, if e.g. $\varphi = \psi \wedge \chi$

- $V_\Sigma(\varphi,[w]_\Sigma) = 1$ iff,
- $V_\Sigma(\psi \wedge \chi,[w]_\Sigma) = 1$ iff,
- $V_\Sigma(\psi,[w]_\Sigma) = 1$ and $V_\Sigma(\chi,[w]_\Sigma) = 1$ iff,
- since Σ is closed under subformulas, by the induct. hyp. iff,
- $V(\psi,w) = 1$ and $V(\chi,w) = 1$ iff,
- $V(\psi \wedge \chi,w) = 1$ iff,
- $V(\varphi,w) = 1$.

Mod If φ is **modal formula** $\Diamond\psi \in \Sigma$ then $\psi \in \Sigma$.

- Suppose, by induction hypothesis, that for every point $v \in W$:

$$V_\Sigma(\psi, [v]_\Sigma) = 1 \quad \text{iff} \quad V(\psi, v) = 1;$$

- On the one hand, assume $\mathcal{M}, w \vDash \Diamond\psi$,
  - then there exists $v \in W$ s.t. $R(w, v)$ and $\mathcal{M}, v \vDash \psi$,
  - then, since S is Σ-appropriate, $S([w]_\Sigma, [v]_\Sigma)$,
  - by i.h., $\mathcal{M}_\Sigma, [v]_\Sigma \vDash \psi$,
  - hence $\mathcal{M}_\Sigma, [w]_\Sigma \vDash \Diamond\psi$.
- On the other hand, assume $\mathcal{M}_\Sigma, [w]_\Sigma \vDash \Diamond\psi$,
  - then there exists $v \in W$ s.t. $S([w]_\Sigma, [v]_\Sigma)$ and $\mathcal{M}_\Sigma, [v]_\Sigma \vDash \psi$,
  - by i.h., $\mathcal{M}, v \vDash \psi$,
  - hence, since S is Σ-appropriate, $\mathcal{M}, w \vDash \Diamond\psi$.

Finiteness of $\mathfrak{M}_\Sigma$ (with Σ finite)

- Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and Σ a **finite** set of formulas closed under subformulas.
- Every filtration $\mathfrak{M}_\Sigma = \langle W_\Sigma, S, V_\Sigma \rangle$ of $\mathfrak{M}$ through Σ is a **finite model**.
- It is **enough** to prove that W_Σ is **finite**;
- to see this, consider that, for every $w \in W$, the set $Th^\Sigma_{\mathcal{M}}(w)$ is a subset of Σ,
- moreover, if $Th^\Sigma_{\mathcal{M}}(w) = Th^\Sigma_{\mathcal{M}}(v)$, then $w \sim_\Sigma v$,
- therefore $[w]_\Sigma = [v]_\Sigma$;
- hence there is an injective map from W_Σ to $\mathcal{P}(\Sigma)$,
- since $\mathcal{P}(\Sigma)$ is finite, so is W_Σ.

Proving the finite model property

Filtrations and the finite model property

- As we have seen, filtrations are **finite**, **differentiated** models which are **modally equivalent** to the original models,
- moreover, since the set of subformulas of a given formula is **finite**, so is the filtration.
- This makes filtration a good candidate to **prove finite model property** for some normal modal logics.
- Indeed filtrations are used in the literature with this aim.
- Often it is not provable that any filtration of a model that has property P, has also property P,
- nevertheless, it is enough to prove that, for every model $\mathfrak{M}$ with property P and every formula φ, **always exists one filtration** of $\mathfrak{M}$ through the set of subformulas of φ, which has property P,
- indeed it is possible to prove the above result for arbitrary finite sets of formulas closed under subformulas.

The Logic K has the finite model property

- The Logic K has the finite model property.
- to see this, let φ be a formula which **is not a theorem** of K,
- let $\Sigma = Sub(\varphi)$ be **the set of subformulas** of φ,
- let $\mathfrak{M}_K$ be the canonical model of K and $\mathfrak{M}_\Sigma$ a filtration of $\mathfrak{M}_K$ through Σ.
- Since $\varphi \notin K$, then there is $\Delta \in W_K$ s.t. $\mathfrak{M}_K, \Delta \not\models \varphi$,
- hence $\mathfrak{M}_\Sigma, [\Delta]_\Sigma \not\models \varphi$.
- Since $\mathfrak{M}_\Sigma$ is a Kripke model, then it is a model of K,
- Since Σ is finite, then $\mathfrak{M}_\Sigma$ is finite,
- hence φ is **not valid in a finite model** of K.

The Logic T has the finite model property

- The Logic T has the finite model property.
- Let $\mathfrak{M} = \langle W, R, V \rangle$ with R **reflexive**,
- let Σ a set of formulas closed under subformulas.
- In the case of logic T, we can prove that **every filtration** of a reflexive model is reflexive,
- it is enough to prove that the **smallest Σ-appropriate relation** R^s_Σ is reflexive;
- indeed, if $R(w, w)$, then $\exists w' \in [w]_\Sigma$ s.t. $R(w', w')$,
- hence $R^s_\Sigma([w]_\Sigma, [w]_\Sigma)$ for every $w \in W$.
- Since R^s_Σ is contained in every Σ-appropriate relation, then **every filtration of a reflexive model is reflexive**.

The Logic $K4$ has the finite model property

- The Logic $K4$ has the finite model property.
- Let $\mathfrak{M} = \langle W, R, V \rangle$ with R **transitive**,
- let Σ a set of formulas closed under subformulas.
- Define the relation T on W_Σ by:

$$T([w]_\Sigma, [v]_\Sigma) \quad \text{iff}$$
$$(\forall \Diamond\varphi \in \Sigma)(\mathfrak{M}, v \models \varphi \vee \Diamond\varphi \;\Rightarrow\; \mathfrak{M}, w \models \Diamond\varphi).$$

Let us see that T is Σ-**appropriate**;

- About the second condition, take $\Diamond\varphi \in \Sigma$,
- by the second premise $T([w]_\Sigma, [v]_\Sigma)$,
- then $(\forall \Diamond\varphi \in \Sigma)(\mathfrak{M}, v \models \varphi \vee \Diamond\varphi \;\Rightarrow\; \mathfrak{M}, w \models \Diamond\varphi)$,
- by the third premise $\mathcal{M}, v \models \varphi$
- then $\mathcal{M}, w \models \Diamond\varphi$.

- About the first condition, suppose that $R(w, v)$, take $\Diamond\varphi \in \Sigma$ be such that either

$$\mathfrak{M}, v \models \varphi \quad \text{or} \quad \mathfrak{M}, v \models \Diamond\varphi,$$

- in the first case, $\mathfrak{M}, w \models \Diamond\varphi$ and we are done,
- in the second case, $\mathfrak{M}, w \models \Diamond\Diamond\varphi$,
- since R is transitive, $\mathfrak{M}, w \models \Diamond\Diamond\varphi \to \Diamond\varphi$,
- hence, by (MP), $\mathfrak{M}, w \models \Diamond\varphi$,
- so, $T([w]_\Sigma, [v]_\Sigma$.

Let us see that T is **transitive**;

- suppose that $T([w]_\Sigma, [v]_\Sigma)$ and $T([v]_\Sigma, [u]_\Sigma)$,
- we want to prove that $T([w]_\Sigma, [u]_\Sigma)$,
- take $\Diamond\varphi \in \Sigma$ be such that either $\mathfrak{M}, u \vDash \varphi \quad$ or $\quad \mathfrak{M}, u \vDash \Diamond\varphi$,
- in the first case, $\mathfrak{M}, v \vDash \Diamond\varphi$ and we are done,
- in the second case, $\mathfrak{M}, v \vDash \Diamond\Diamond\varphi$,
- since R is transitive, $\mathfrak{M}, v \vDash \Diamond\Diamond\varphi \to \Diamond\varphi$,
- hence, by (MP), $\mathfrak{M}, v \vDash \Diamond\varphi$.
- then $\mathfrak{M}, w \vDash \Diamond\Diamond\varphi$,
- since, again, R is transitive, $\mathfrak{M}, w \vDash \Diamond\Diamond\varphi \to \Diamond\varphi$,
- hence, by (MP), $\mathfrak{M}, w \vDash \Diamond\varphi$,
- so, $T([w]_\Sigma, [u]_\Sigma)$.