Finite-valued Łukasiewicz Modal Logic is PSPACE-complete

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Finite-valued Łukasiewicz Modal Logic

Finite-valued Łukasiewicz Modal Logic is obtained by adding two modal operators $\Box$ and $\Diamond$ to the propositional language.
Further expansions by means of canonical truth constants and/or the Delta operator can be considered.
For every $n \in \mathbb{N}$ the set of truth values is:

$$L_n := \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$$

Modal Syntax	FDL Syntax	arity	Semantics
$\&$	$\boxtimes$	2	$a * b := \max\{0, a+b-1\}$
$\rightarrow$	$\sqsupset$	2	$a \Rightarrow b := \min\{1, 1-a+b\}$
$\bot$	$\bot$	0	0
$\Box$	$\forall R.$	1	$V(\Box\varphi, w) = \min\{R(w,w') \Rightarrow V(\varphi, w') : w' \in W\}$
$\Diamond$	$\exists R.$	1	$V(\Diamond\varphi, w) = \max\{R(w,w') \& V(\varphi, w') : w' \in W\}$
$\frac{x}{n-1}$	$\frac{x}{n-1}$	0	$\frac{x}{n-1}$
Δ		1	$\delta a = \begin{cases} 1, & \text{if } a = 1 \\ 0, & \text{otherwise.} \end{cases}$

MAIN RESULT

For every $n \in \mathbb{N}$ and every $m \in L_n$,

- the set of modally m-satisfiable formulas over Kripke Ł$_n$-models is PSPACE-complete,
- the set of modally valid formulas over Kripke Ł$_n$-models is PSPACE-complete.

The same complexity result is attained when we add the Delta operator and/or the canonical truth constants.
And also we get the same complexity when we only deal with crisp Kripke models.

Closure of a set of formulas

Let Σ be a set of modal formulas, and $Sub(\Sigma)$ be the set of its subformulas. We define the closure of Σ, in symbols $Cl(\Sigma)$, as the set

$$(Sub(\Sigma) \cup \{\Box\neg\sigma : \Diamond\sigma \in Sub(\Sigma)\} \cup \{\Diamond\neg\sigma : \Box\sigma \in Sub(\Sigma)\})^+,$$

where the superscript $^+$ refers to the process of deleting two consecutive negations.

The Family of Modal Levels

Let Σ be a closed set of modal formulas. We define the sequence $(\Sigma_0, \Sigma_1, \ldots, \Sigma_{deg(\Sigma)})$ by the recurrence

- $\Sigma_0 := \Sigma$,
- $\Sigma_{r+1} := \{\psi : \Diamond\psi \in BD(\Sigma_r)\} \cup \{\psi : \Box\psi \in BD(\Sigma_r)\}$.

The family of modal levels of Σ is the set $\Sigma^\circ := \{\Sigma_0, \Sigma_1, \ldots, \Sigma_{deg(\Sigma)}\}$.

The Algorithm $Witness(H, \Sigma)$

if H is a Hintikka function and $\Sigma = dom(H)$
  and for each subformula $\Diamond\psi \in dom(H)$ there are $k \in L_n$ and a Hintikka function $I \in H_{\Diamond\psi,k}$ such that
  $Witness(I, dom(I))$
 then
  return **true**
else
  return **false**
end if

Witness Sets

We say that $\mathcal{H}$ is a witness set generated by H on Σ when

- $H \in \mathcal{H}$,
- if $I \in \mathcal{H}$ and $\Diamond\psi \in dom(I)$, then there is some $k \in L_n$ and some $J \in I_{\Diamond\psi,k}$ such that $J \in \mathcal{H}$,
- if $J \in \mathcal{H}$ and $J \neq H$, then there are $I^0, \ldots, I^r \in \mathcal{H}$ satisfying $I^0 = H$, $I^r = J$, and for each $0 \leq i < r$, there are a formula $\Diamond\psi \in dom(I^i)$ and an element $k \in L_n$ such that $I^{i+1} \in I^i_{\Diamond\psi,k}$.

Hintikka Functions

Let Σ be a closed set of formulas. A Hintikka function over some $\Sigma_r \in \Sigma^\circ$ is a mapping $H : \Sigma_r \to L_n$ such that

- H is a homomorphism of non modal connectives,
- $H(\Diamond\psi) = 1 - H(\Box\neg\psi)$, for each $\Diamond\psi \in \Sigma_r$,
- $H(\Box\psi) = 1 - H(\Diamond\neg\psi)$, for each $\Box\psi \in \Sigma_r$.

It is said that H is an atom if there exists a Kripke model $\mathfrak{M} = \langle W, R, V\rangle$ and a world $w \in W$ such that $H(\psi) = V(\psi, w)$, for each formula $\psi \in \Sigma$.

k-related Hintikka functions

Let $H : \Sigma_r \to L_n$ be a Hintikka function, $k \in L_n$ and $\Diamond\psi \in \Sigma_r$. We say that a Hintikka function $H' : \Sigma_{r+1} \to L_n$ is induced by $\Diamond\psi$ and k-related to H (in symbols, $H' \in H_{\Diamond\psi,k}$) if the following conditions hold:

- $H(\Diamond\psi) = k * H'(\psi)$,
- for each $\Box\vartheta \in \Sigma_r$, $H(\Box\vartheta) \leq k \Rightarrow H'(\vartheta)$.

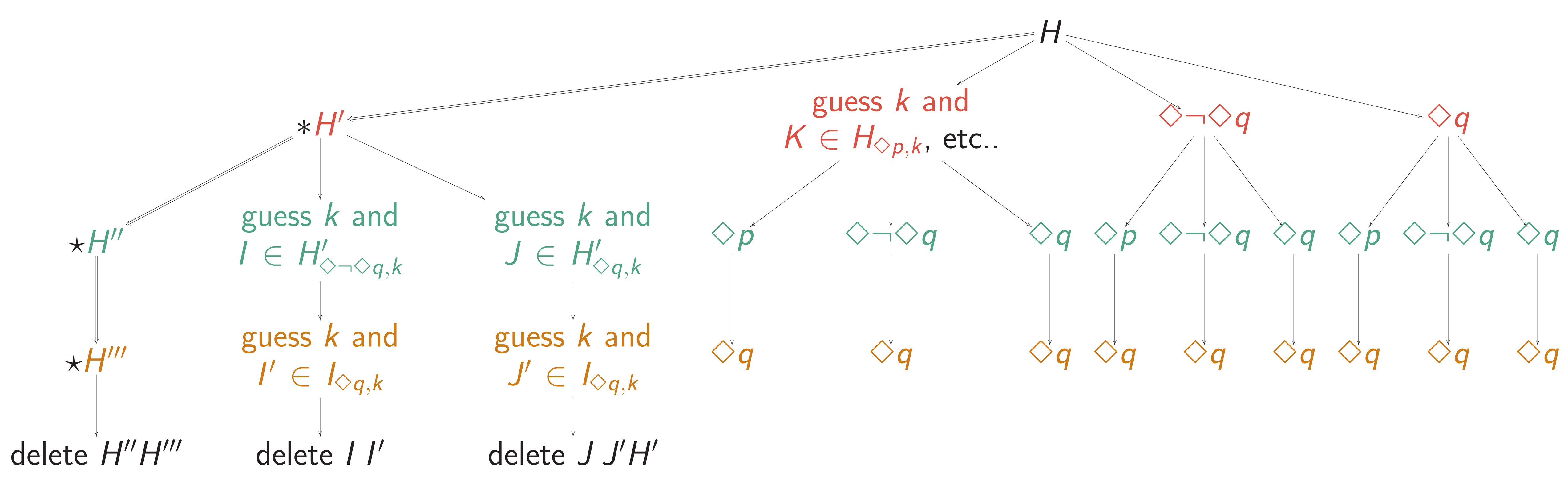
Soundness and Completeness of $Witness(H, Cl(\varphi))$

A modal formula φ is m-satisfiable $\iff$ there is a Hintikka function H with $H(\varphi) = m$ and a Witness set $\mathcal{H}$ generated by H on $Cl(\varphi)$ $\iff$ $Witness(H, Cl(\varphi))$ returns **true**

Running $Witness(H, \Sigma)$ for an example input

For $n = 3$ and $\Sigma = Cl(\{\Box(\Diamond p \to \Box\Diamond q)\})$

Σ°							
$Cl(\Sigma_0)$	H	$Cl(\Sigma_1)$	$H' \in H_{\Diamond\neg(\Diamond p \to \Box\Diamond q),1}$	$Cl(\Sigma_2)$	$H'' \in H'_{\Diamond p,0.5}$	$Cl(\Sigma_3)$	$H''' \in H''_{\Diamond q,1}$
$\Box(\Diamond p \to \Box\Diamond q)$	1	$\Diamond p \to \Box\Diamond q$	1				
$*\Diamond\neg(\Diamond p \to \Box\Diamond q)$	0	$\neg(\Diamond p \to \Box\Diamond q)$	0				
		$\star\Diamond p$	0.5	p	1		
		$\Box\neg p$	0.5	$\neg p$	0		
		$\Box\Diamond q$	0.5	$\star\Diamond q$	0	q	0
		$\Diamond\neg\Diamond q$	0.5	$\neg\Diamond q$	1		
$\Diamond p \to \Box\Diamond q$	0.5						
$\Diamond p$	1	p	1				
$\Box\neg p$	0	$\neg p$	0				
$\Box\Diamond q$	0.5	$\Diamond q$	0.5	q	1		
$\Diamond\neg\Diamond q$	0.5	$\neg\Diamond q$	0.5				
$\Diamond q$	0.5	q	0.5				
$\Box\neg q$	0.5	$\neg q$	0.5				
p	0.5						
q	0						



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