Decidability of a Description Logic over Infinite-Valued Product Logic

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Descriptions Logics

- Description Logics (DLs) are knowledge representation languages particularly suited for specifying ontologies, creating knowledge bases and reasoning with them. DLs have been studied extensively over the last two decades.
- The vocabulary of DLs consists of concepts, which denote sets of individuals, e. g.

  Person, Parent, Male, Female,

  and roles, which denote binary relations among individuals, e. g.

  hasChild, hasRelative, hasSister

From atomic concepts and roles and by means of constructors, DL systems allow us to build complex descriptions of concepts, e.g.

$$\text{Person} \sqcap \text{Male},$$

$$\exists \text{hasChild.Female},$$

$$\text{Person} \sqcap \forall \text{hasChild.Male}$$

- These complex descriptions are used to describe a domain through a knowledge base (KB).
- A KB contains a Terminological Box (*TBox*) with the definitions of relevant domain concepts and some hierarchical relationships among them, called inclusion axioms, e. g.

$$\exists \text{hasChild.Female} \sqsubseteq \exists \text{hasRelative.Female},$$

$$\exists \text{hasSister.Male} \equiv \bot,$$

- and an Assertional Box (*ABox*) with specifications of properties of the domain individuals, called assertional axioms or assertions, e. g.

$$\text{Person} \sqcap \text{Male}(\text{John}),$$

$$\exists \text{hasChild}.\text{Female}(\text{Mary}),$$

$$\text{Person} \sqcap \forall \text{hasChild}.\text{Male}(\text{Mary})$$

Classical DLs

Classical description logics are fragments of first order classical logic that are

- expressive enough to represent knowledge,
- decidable and, as much as possible,
- reasonably complex to build efficient reasoning algorithms.

Semantics

An interpretation $\cdot^{\mathcal{I}}$, for a Classical DL consists of:

1. a non-empty set (crisp) $\Delta^{\mathcal{I}}$, e. g.

$$\Delta^{\mathcal{I}} = \{\text{John, Marc, Philip, Mary, Rose}\}$$

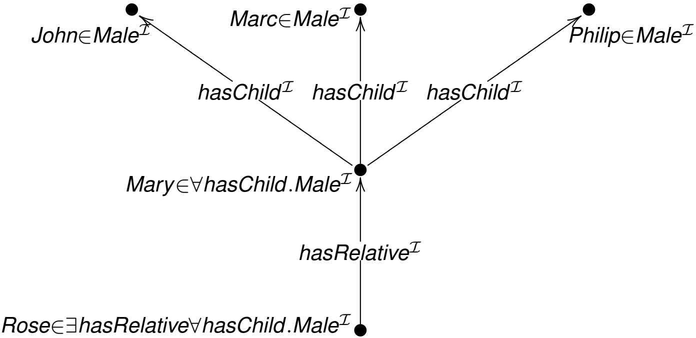
2. an interpretation function $\cdot^{\mathcal{I}}$ which assigns to each concept name A, a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and to each ole name R, a set $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The good behaviour of Classical DLs is due to the fact that the first order fragment related to DLs enjoys Finite Model Property, i.e. checking for satisfiability and validity of assertions or concepts can be limited to finite models.

For example, in order to show that concept

$$\exists \text{hasRelative}.\forall \text{hasChild}.\text{Male}$$

is satisfiable it is enough to provide the following model.



Vague concepts and Fuzzy sets

- From an application viewpoint, vague concepts like

  patient with a high fever

  person living near a pollution source

  have to be considered in Description Languages.
- A natural generalization to cope with vague concepts and relations consists in interpreting concepts and roles as fuzzy sets and fuzzy relations respectively.

Fuzzy sets and Fuzzy logic

- Fuzzy sets and fuzzy logics were born to deal with the problem of approximate reasoning. Nowadays there is a *mathematical logic* framework studying the semantics given by prominent residuated chains (i.e., those over $[0, 1]$), semantics now called standard semantics.
- In recent times, formal logic systems have been developed for such semantics, and the logics based on triangular norms (*t*-norms) have become the central paradigm in fuzzy logic.

Standard algebras

Given a t-norm $*$, Standard $*$ algebra is the algebra $[0,1]_* = \langle [0,1], *, \Rightarrow_*, 1, 0\rangle$, where:

- the domain is the real unit interval $[0,1]$,
- operation $*$ is the given t-norm that,
  - if $*$ is Łukasiewicz t-norm, it is the operation
    $$x * y = \max\{0, x + y - 1\}$$
  - if $*$ is product t-norm, it is the usual product between reals.
- operation $\Rightarrow_*$ is its residuum which:
  - if $*$ is Łukasiewicz t-norm, it is defined as
    $$x \Rightarrow_* y = \min\{1, 1 - x + y\}$$
  - if $*$ is product t-norm, it is defined as:
    $$x \Rightarrow_* y = \min\{1, \frac{y}{x}\}$$
- constants 0 and 1 have their usual values.

Moreover, we have the following definable connectives:

- $\neg x := x \Rightarrow_* 0$,
- $x \Leftrightarrow_* y := (x \Rightarrow_* y) * (y \Rightarrow_* x)$,
- $x \wedge y := x * (x \Rightarrow_* y)$,
- $x \vee y := ((x \Rightarrow_* y) \Rightarrow_* y) \wedge ((y \Rightarrow_* x) \Rightarrow_* x)$.

The results we are going to expose are limited to two basic languages:

- the attributive language with complement based on infinite-valued Łukasiewicz t-norm (Ł-$\mathcal{ALC}$),
- the attributive language with qualified existencial quantification based on infinite-valued product t-norm (Π-$\mathcal{ALE}$).

The basic concept constructors of these two languages are the same and are:

- conjunction $\boxdot$,
- implication $\rightarrow$,
- top and bottom concepts $\top$, $\bot$,
- existential and universal quantificator $\exists$, $\forall$.

Concepts

The set of concepts is the smallest set such that:

- every concept name A is a concept,
- $\bot$ and $\top$ are concepts,
- if C, D are concepts, then $C \boxdot D$ and $C \rightarrow D$ are concepts,
- if C is a concept and R is a role name, then $\forall R.C$ and $\exists R.C$ are concepts.

Semantics

An $*$-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists on a crisp set $\Delta^{\mathcal{I}}$ (called the *domain* of $\mathcal{I}$) and an *interpretation function* $\cdot^{\mathcal{I}}$, which maps every concept C to a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$, every role name R to a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$ and such that, for every concepts C, D, every role name R and every element $a \in \Delta^{\mathcal{I}}$, it holds that:

$$
\begin{aligned}
\bot^{\mathcal{I}}(a) &= 0 \\
\top^{\mathcal{I}}(a) &= 1 \\
(C \boxdot D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) * D^{\mathcal{I}}(a) \\
(C \to D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \Rightarrow_* D^{\mathcal{I}}(a) \\
(\forall R.C)^{\mathcal{I}}(a) &= inf\{R^{\mathcal{I}}(a,b) \Rightarrow_* C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}}(a) &= sup\{R^{\mathcal{I}}(a,b) * C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\}
\end{aligned}
$$

Reasoning Problems

Let C be a concept, $*$ a t-norm and $r \in [0,1]$. Then,

- C is said to be 1-satisfiable in $*$-$\mathcal{ALE}$ if there is some interpretation $\mathcal{I}$ and object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = 1$.
- C is said to be r-satisfiable in $*$-$\mathcal{ALE}$ if there is some interpretation $\mathcal{I}$ and object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = r$.
- C is said to be valid in $*$-$\mathcal{ALE}$ if for every interpretation $\mathcal{I}$ and object $a \in \Delta^{\mathcal{I}}$, $C^{\mathcal{I}}(a) = 1$.

We will write Sat_r^* and Val^* to denote the set of concepts that are, respectively, r-satisfiable and valid in $*$-$\mathcal{ALE}$.
Moreover, if there exists $r \in [0,1]$ such that $C \in \mathrm{Sat}_r^*$, we say that C is positively satisfiable.

Some logical results

- First order standard tautologies are not recursively enumerable for Lukasiewicz logic and not arithmetical for Product Logic. (Hájek)
- In the $\mathcal{ALC}$ description languages over Lukasiewicz logic, satisfiability (validity, subsumption) problem is decidable. (Hájek)
- From Hájek's results follows that the satisfiability (validity, subsumption) problem in $\mathcal{ALC}$ languages over the logic of a finite continuous t-norm is a decidable problem.

Decidability of Ł-$\mathcal{ALC}$

Our reduction is an extension of Hájek's algorithm to prove decidability of satisfiability and validity problems for Ł-$\mathcal{ALC}$.

Theorem (Łukasiewicz Case; P. Hájek, 2005)

For every $r \in [0,1] \cap \mathbb{Q}$, the set Sat_r^L is decidable; and the set Val^L is also decidable.

Witnessed models

Definition (P. Hájek)

An $*$-interpretation $\mathcal{I}$ is witnessed when it satisfies

(wit∃) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$, there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\exists R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a,b) * C^{\mathcal{I}}(b),$$

(wit∀) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$, there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a,b) \Rightarrow_{*} C^{\mathcal{I}}(b).$$

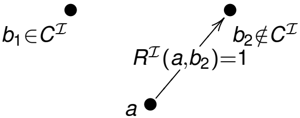
An Example

Claim: In the 2-valued case (also finitely-valued) all interpretations are witnessed.

Let $\cdot^{\mathcal{I}}$ be a bi-valued interpretation $a \in \Delta^{\mathcal{I}}$ R a role name and C a concept name, then:

- if $(\exists R.C)^{\mathcal{I}}(a) = 1$, then there is some $b \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a, b) \wedge C^{\mathcal{I}}(b) = 1$,

In fact, suppose that there is no $b \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a, b) \wedge C^{\mathcal{I}}(b) = 1$,



then we have that

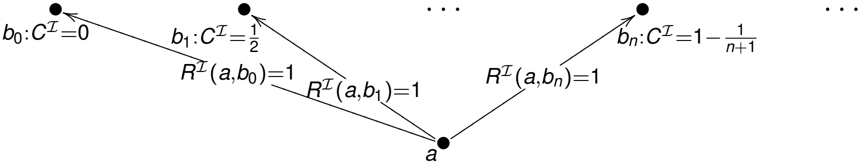
$$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a, b) \wedge C^{\mathcal{I}}(b)\} = 0,$$

a contradiction.

A Counter-example

Let $\cdot^{\mathcal{I}}$ be a $[0,1]_*$-interpretation such that $\Delta^{\mathcal{I}} = \{a\} \cup \{b_n\}_{n\in\mathbb{N}}$, and for a role name R and a concept name C:

- $R^{\mathcal{I}}(a, b_n) * C^{\mathcal{I}}(b_n) = 1 - \frac{1}{n+1}$
- $R^{\mathcal{I}}(a, b) * C^{\mathcal{I}}(b) = 0$ for each other $b \in \Delta^{\mathcal{I}}$,



then we have that:

$$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b\in\Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a, b) * C^{\mathcal{I}}(b)\} = 1,$$

but there is no $b \in \Delta^{\mathcal{I}}$ such that

$$R^{\mathcal{I}}(a, b) * C^{\mathcal{I}}(b) = 1$$

Witnessed Completeness

Nevertheless, P. Hájek proves the following result:

Theorem (Łukasiewicz Case; P. Hájek)

For every concept C, the following are equivalent:

1. *C is true in a $[0, 1]_L$-interpretation,*
2. *C is true in a witnessed $[0, 1]_L$-interpretation,*
3. *C is true in a finite $[0, 1]_L$-interpretation.*

Reduction to propositional satisfiability

- Thanks to the last result it is possible to reduce validity and satisfiability for Ł-$\mathcal{ALC}$ to the semantic consequence problem in the propositional Łukasiewicz Logic, which is known to be decidable.
- We give an informal presentation of this reduction. Given an assertion, say

$$C_0 = \exists R.(\forall R.D \boxdot \forall R.E)$$

1. first we produce a set of formulas T_{C_0} describing a witnessed model which satisfies C_0,
2. second, we provide a translation $pr(\cdot)$ of formulas in T_{C_0} into a propositional language.

The set T_{C_0}

In order to produce the set T_{C_0} we begin from the whole formula C_0 and consider each quantified subformula occurring in it.

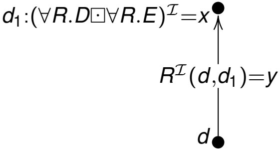
- So, when we meet an assertion:

$$C_0 = \exists R.(\forall R.D \boxdot \forall R.E)(d)$$

  we produce a new constant d_1 and add to T_{C_0} the new formula:

$$\exists R.(\forall R.D \boxdot \forall R.E)(d) \equiv (R(d, d_1) \boxdot (\forall R.D \boxdot \forall R.E)(d_1))$$

which says us that we are building the following interpretation $\mathcal{I}$:



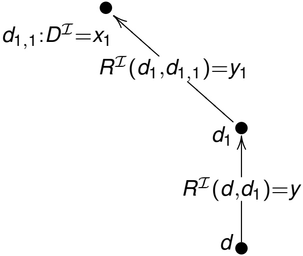
- Next we meet the assertion:

$$\forall R.D(d_1)$$

So, we produce a new constant $d_{1,1}$ and add to T_{C_0} the new formula:

$$\forall R.D(d_1) \equiv (R(d_1, d_{1,1}) \rightarrow D(d_{1,1}))$$

which says us that we are building the following interpretation $\mathcal{I}$:



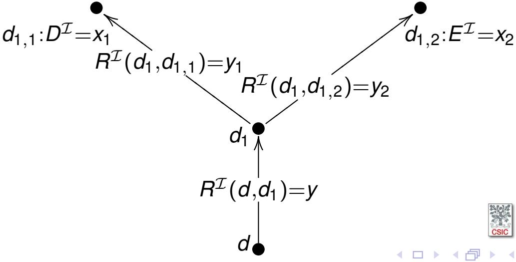
- Then we meet the assertion:

$$\forall R.E(d_1)$$

So, we produce a new constant $d_{1,2}$ and add to T_{C_0} the new formula:

$$\forall R.E(d_1) \equiv (R(d_1, d_{1,2}) \rightarrow E(d_{1,2}))$$

which says us that we are building the following interpretation $\mathcal{I}$:



- Finally we add to T_{C_0} the new formulas:

$$\forall R.D(d_1) \to (R(d_1, d_{1,2}) \to E(d_{1,2}))$$

and

$$\forall R.E(d_1) \to (R(d_1, d_{1,1}) \to E(d_{1,1}))$$

which say us that, in the interpretation $\mathcal{I}$ we have built

$$(\forall R.D)^{\mathcal{I}}(d_1) = \inf_{c \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(d_1, c) \to D^{\mathcal{I}}(c)\}$$

and

$$(\forall R.E)^{\mathcal{I}}(d_1) = \inf_{c \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(d_1, c) \to E^{\mathcal{I}}(c)\}$$

The translation $pr(\cdot)$

The map *pr* associates to every assertion a formula in the propositional logic (with the variables given above) according to the following clauses:

1. $pr(C(a)) = p_{C(a)}$ if C is an atomic or a quantified concept,
2. $pr(R(a,b)) = p_{R(a,b)}$ if R is a role name and a, b are individuals,
3. $pr(\bot(a)) = \bot$,
4. $pr(\top(a)) = \top$
5. $pr((C \boxdot D)(a)) = pr(C(a)) \odot pr(D(a))$,
6. $pr((C \to D)(a)) = pr(C(a)) \to pr(D(a))$.

If T is a set of assertions, then $pr(T)$ is $\{pr(\alpha) \mid \alpha \in T\}$.

Reduction

Proposition (Łukasiewicz Case; P. Hájek, 2005)

A concept C_0 is satisfiable iff *the set $pr(T_{C_0}) \cup pr(C_0)$ is satisfiable*

A concept C_0 is valid iff *$pr(C_0)$ is a propositional consequence of the set $pr(T_{C_0})$*

iff $pr(T_{C_0}) \models pr(C_0)$

Decidability of Π-$\mathcal{ALE}$

What we prove is an analogous theorem for Π-$\mathcal{ALE}$

Theorem (Product Case)

The set of positively satisfiable concepts is decidable; and the set Val *is also decidable.*

Failure of Finite Model Property

In the case of Standard Product *t*-norm, the Finite Model Property fails. Consider the concept

$$C := (\forall R.A \boxdot \neg\forall R.(A \boxdot A))$$

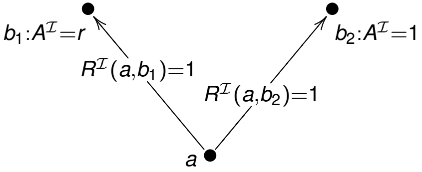
C is unsatisfiable in each finite model, indeed, if there exists a finite $[0,1]_\Pi$-interpretation $\cdot^\mathcal{I}$ and $a \in \Delta^\mathcal{I}$ such that

$$(\forall R.A)^\mathcal{I}(a) = r > 0$$

with $r \in [0,1]$, then, there exists $b \in \Delta^\mathcal{I}$,

$$R^\mathcal{I}(a,b) \Rightarrow A^\mathcal{I}(b) = r > 0$$

and it is the infimum,



but, hence,

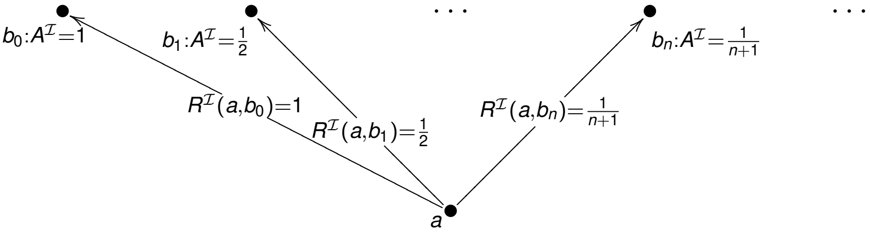
$$\neg(\forall R.(A \odot A))^{\mathcal{I}}(a) = 0$$

and, therefore

$$(\forall R.A \odot \neg\forall R.(A \odot A))^{\mathcal{I}}(a) = 0$$

Nevertheless, let $\cdot^{\mathcal{I}}$ be a $[0,1]_\Pi$-interpretation such that

1. $\Delta^{\mathcal{I}} = \{a\} \cup \{b_n\}_{n\in\mathbb{N}}$,
2. $R^{\mathcal{I}}(a, b_n) = \frac{1}{n+1}$, for every $n \in \mathbb{N}$,
3. $A^{\mathcal{I}}(b_n) = \frac{1}{n+1}$, for every $n \in \mathbb{N}$,



With this interpretation we have that:

1. $(\forall R.A)^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(a,b) \Rightarrow A^{\mathcal{I}}(b)\} = 1$ and
2. $(\forall R.(A \boxdot A))^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(a,b) \Rightarrow (A^{\mathcal{I}} \boxdot A^{\mathcal{I}})(b)\} = 0$,

Hence

$$(\forall R.A)^{\mathcal{I}} \boxdot \neg(\forall R.(A \boxdot A))^{\mathcal{I}}(a) = 1$$

but this interpretation has an infinite domain.

Quasi-witnessed models

Definition (M. C. Laskowski and S. Malekpour)

An $*-$interpretation $\mathcal{I}$ is quasi-witnessed when it satisfies

(wit∃) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\exists R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a,b) * C^{\mathcal{I}}(b),$$

(qwit∀) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$

- either $(\forall R.C)^{\mathcal{I}}(a) = 0$,
- or there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a,b) \Rightarrow_* C^{\mathcal{I}}(b).$$

Quasi-witnessed models and Π-$\mathcal{ALE}$

Theorem

Let C be a concept. The following statements are equivalent:

1. *C is true in all* $[0,1]_\Pi$*-interpretations,*
2. *C is true in all interpretations over a* 1*-generated subalgebra of* $[0,1]_\Pi$*,*
3. *C is true in all quasi-witnessed* $[0,1]_\Pi$*-interpretations.*

Reduction to propositional satisfiability

- Thanks to the last result it is possible to reduce validity and satisfiability for Π-$\mathcal{ALE}$ to the semantic consequence in propositional Product Logic which is known to be a decidable problem
- We will give an informal account of this reduction. Given an assertion, say

$$C_0 := (\forall R.A \boxdot \neg\forall R.(A \boxdot A))(d)$$

1. first we produce a set of formulas T_{C_0} describing a model which satisfies C_0,
2. second we produce a set of formulas Y_{C_0} which constrains the model described by T_{C_0},
3. third, we provide a translation $pr(\cdot)$ of formulas in T_{C_0} and Y_{C_0} into a propositional language in the same way as before.

The set T_{C_0}

The construction of the set T_{C_0} is the same as before but with the difference that, for universally quantified assertions we add to it the formula

$$(\forall R.A(d) \equiv (R(d, d_1) \rightarrow A(d_{1,1}))) \sqcup \neg\forall R.A(d)$$

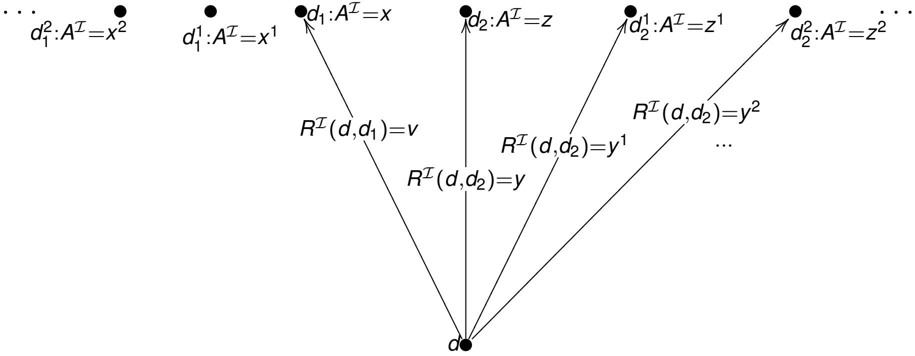
which says us that, in the interpretation $\mathcal{I}$ that we are building, either

$$(\forall R.A(d))^{\mathcal{I}} = R^{\mathcal{I}}(d, d_1) \rightarrow A^{\mathcal{I}}(d_{1,1})$$

or

$$(\forall R.A(d))^{\mathcal{I}} = 0$$

like in the following interpretation $\mathcal{I}$:



The set Y_{C_0}

Moreover, when we meet an universally quantified assertion, we add to the set Y_{C_0} the following formula:

$$\neg\forall R.A(d) \odot (R(d, d_1) \rightarrow A(d_1))$$

which constrains interpretation $\mathcal{I}$ not to verify both

$$(\forall R.A)^{\mathcal{I}}(d) = 0$$

and

$$R^{\mathcal{I}}(d, d_1) \rightarrow A^{\mathcal{I}}(d_1) = 1$$

in order to overcome a problem in an earlier version of this work.

Reduction

Proposition

Let C_0 be a concept, and let T_{C_0} and Y_{C_0} be the two finite sets associated by the algorithm. For every $r \in [0,1]$, the following statements are equivalent:

1. *C_0 is satisfiable with truth value r in a quasi-witnessed Π-interpretation,*
2. *there is some propositional evaluation e over the set $Prop$ such that $e(pr(C(d_0))) = r$, $e[pr(T_{C_0})] = 1$, and $e[\psi] \neq 1$ for every $\psi \in pr(Y_{C_0})$.*

Which is equivalent to say that:

$C \in \mathrm{QSat}_1$ iff $\bigvee pr(Y_{C_0})$ is not a consequence, in the propositional product logic, of the set $\{pr(C(d_0))\} \cup pr(T_{C_0})$

iff $\{pr(C(d_0))\} \cup pr(T_{C_0}) \not\models \bigvee pr(Y_{C_0})$

$C \in \mathrm{QVal}$ iff $pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$ is a consequence, in the propositional product logic, of the set $pr(T_{C_0})$

iff $pr(T_{C_0}) \models pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$

Hence, we have a reduction of these problems to the semantic consequence problem, with a finite number of hypothesis, in the propositional product logic. Hájek, 2006 proves that such problem is in *PSPACE*.