Decidability of a Description Logic over Infinite-Valued Łukasiewicz and Product Logic

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Outline

- In this talk we give an account of the proofs that:
  - Description Logic $\mathcal{ALC}$ over the standard Łukasiewicz algebra $[0,1]_Ł$ is decidable.
  - validity and positive satisfiability problems for Description Logic $\mathcal{ALE}$ over the standard product algebra $[0,1]_\Pi$ are decidable.
- This is done by providing a recursive reduction of such problems to the satisfiability and the semantic consequence in propositional Łukasiewicz and Product Logics.
- The result then follows from the fact that satisfiability and semantic consequence in propositional Łukasiewicz and Product Logics are decidable problems.

FDL Languages

$$\Pi\text{-}\mathcal{ALE} \quad : \quad A \mid \top \mid \bot \mid C \boxdot D \mid C \rightarrow D \mid \neg C \mid \forall R.C \mid \exists R.C$$

$$\text{Ł-}\mathcal{ALC} \quad : \quad A \mid \top \mid \bot \mid C \boxdot D \mid C \rightarrow D \mid \sim C \mid \forall R.C \mid \exists R.C$$

Semantics

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of:

- a crisp set $\Delta^{\mathcal{I}}$ (called the domain of $\mathcal{I}$),
- an interpretation function $\cdot^{\mathcal{I}}$, such that:

$A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$ and $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$,

$$
\begin{aligned}
\bot^{\mathcal{I}}(a) &= 0 \\
\top^{\mathcal{I}}(a) &= 1 \\
(C \boxtimes D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \otimes D^{\mathcal{I}}(a) \\
(C \to D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \Rightarrow D^{\mathcal{I}}(a) \\
(\neg C)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \Rightarrow \bot^{\mathcal{I}}(a) \\
(\sim C)^{\mathcal{I}}(a) &= 1 - C^{\mathcal{I}}(a) \\
(\forall R.C)^{\mathcal{I}}(a) &= \inf\{R^{\mathcal{I}}(a,b) \Rightarrow C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}}(a) &= \sup\{R^{\mathcal{I}}(a,b) \otimes C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\}
\end{aligned}
$$

Witnessed models and infinite-valued Logic

First order Łukasiewicz infinite-valued Logic $[0,1]_Ł\forall$ is complete with respect witnessed models. [Hájek, 1998,2005]

φ is true in a $[0,1]_Ł\forall$-model	$\Longleftrightarrow$	there exists an integer $n > 1$ such that φ is true in a $Ł_n\forall$-model
	$\Longleftrightarrow$	φ is true in a witnessed $Ł\forall$-model

Reduction to propositional logic

- Thanks to the last result it is possible to provide a reduction of Ł-$\mathcal{ALC}$ to propositional Łukasiewicz Logic, which is known to be a decidable problem.
- It is done in two steps, given an assertion $C(d)$:
  1. first we produce a set of formulas T_C describing a witnessed model which (possibly) satisfies $C(d)$,
  2. second, we provide a translation $pr(\cdot)$ of formulas in T_C into a propositional language.

The set T_{C_0}

We will give an informal account of this reduction. Given an assertion, say

$$C_0 = \exists R.(\forall R.D \boxdot \forall R.E)(d)$$

for each quantified subformula occurring in it we produce a new constant and a couple of formulas are added to T_{C_0}:

$\exists R.(\forall R.D \boxtimes \forall R.E)(d)$ $\quad d_1 \quad$ $\exists R.(\forall R.D \boxtimes \forall R.E)(d) \equiv (R(d, d_1) \boxtimes (\forall R.D \boxtimes \forall R.E)(d_1))$

$\forall R.D(d_1)$ $\quad d_{1,1} \quad$ $\forall R.D(d_1) \equiv (R(d_1, d_{1,1}) \rightarrow D(d_{1,1}))$

$\forall R.D(d_1) \rightarrow (R(d_1, d_{1,2}) \rightarrow E(d_{1,2}))$

$\forall R.E(d_1)$ $\quad d_{1,2} \quad$ $\forall R.E(d_1) \equiv (R(d_1, d_{1,2}) \rightarrow E(d_{1,2}))$

$\forall R.E(d_1) \rightarrow (R(d_1, d_{1,1}) \rightarrow E(d_{1,1}))$

The translation $pr(\cdot)$

The mapping *pr* associates to every assertion occurring in a formula in T_{C_0} and Y_{C_0} a propositional variable, according to the following clauses:

- $pr(C(a))=P_{C(a)}$ if C is an atomic or a quantified concept,
- $pr(R(a,b))=P_{R(a,b)}$ if R is a role name,
- $pr(\bot(a))=\bot$,
- $pr(\top(a))=\top$
- $pr((C \boxdot D)(a))=pr(C(a)) \odot pr(D(a))$,
- $pr((C \to D)(a))=pr(C(a)) \to pr(D(a))$.

If T is a set of assertions, then $pr(T)$ is $\{pr(\alpha) \mid \alpha \in T\}$.

So, the elements of the set $pr(T_{C_0})$ are:

$$\exists R.(\forall R.D \boxtimes \forall R.E)(d) \equiv (R(d, d_1) \boxtimes (\forall R.D \boxtimes \forall R.E)(d_1))$$

$$P_{\exists R.(\forall R.D \boxtimes \forall R.E)(d)} \equiv (P_{R(d,d_1)} \otimes (P_{\forall R.D(d_1)} \otimes P_{\forall R.E(d_1)}))$$

$$\forall R.D(d_1) \equiv (R(d_1, d_{1,1}) \rightarrow D(d_{1,1}))$$

$$P_{\forall R.D(d_1)} \equiv (P_{R(d_1,d_{1,1})} \rightarrow P_{D(d_{1,1})})$$

$$\forall R.D(d_1) \rightarrow (R(d_1, d_{1,2}) \rightarrow E(d_{1,2}))$$

$$P_{\forall R.D(d_1)} \rightarrow (P_{R(d_1,d_{1,2})} \rightarrow P_{E(d_{1,2})})$$

$$\forall R.E(d_1) \equiv (R(d_1, d_{1,2}) \rightarrow E(d_{1,2}))$$

$$P_{\forall R.E(d_1)} \equiv (P_{R(d_1,d_{1,2})} \rightarrow P_{E(d_{1,2})})$$

$$\forall R.E(d_1) \rightarrow (R(d_1, d_{1,1}) \rightarrow E(d_{1,1}))$$

$$P_{\forall R.E(d_1)} \rightarrow (P_{R(d_1,d_{1,1})} \rightarrow P_{E(d_{1,1})})$$

Propositional evaluations

It is proved that, for each $r \in [0, 1]$:

there is an individual *d* and a witnessed interpretation $\mathcal{I}$ such that $C^{\mathcal{I}}(d) = r$	$\iff$	there is an individual *d* and a propositional evaluation *e* such that $e(pr(T_C)) = 1$ and $e(pr(C(d))) = r$

Proof: from FDL interpretations to propositional evaluations

Given a witnessed interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} = r$, define the propositional evaluation $e_{\mathcal{I}}$ such that, for every concept and role assertion $D(a)$ and $R(a, b)$, occuring in a formula in T_C,

$$e_{\mathcal{I}}(pr(D(a)))=D^{\mathcal{I}}(a)$$

and

$$e_{\mathcal{I}}(pr(R(a,b)))=R^{\mathcal{I}}(a,b)$$

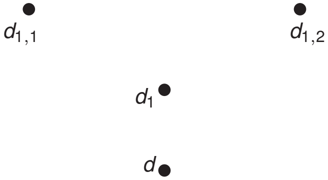
Hence, it is a simple task to check that $e_{\mathcal{I}}(pr(T_C)) = 1$ and $e_{\mathcal{I}}(pr(C(d))) = r$.

Proof: from propositional evaluations to FDL interpretations

We give an sketch by means of the example assertion C_0 before:

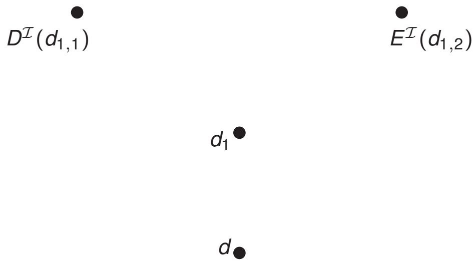
Given the set T_{C_0} and a propositional evaluation e such that $e(pr(T_{C_0})) = 1$ and $e(pr(C_0(d))) = r$, we define how to build a quasi-witnessed interpretation $\mathcal{I}_e$:

- The elements of the domain $\Delta^{\mathcal{I}}$ are the constant occurring in T_{C_0}:



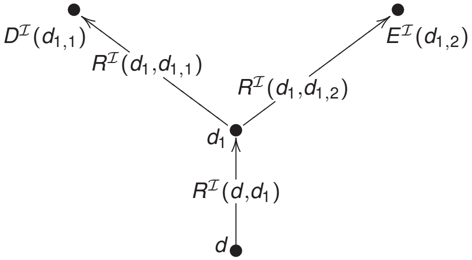
- For each atomic concept A and each constant d, d_n occurring in T_{C_0}, let:

$$A^{\mathcal{I}_e}(d_n) = e(pr(A(d_n)))$$



- For each role name R and each constant d_n occurring in T_{C_0},let:

$$R^{\mathcal{I}_e}(d, d_n) = e(pr(R(d, d_n)))$$



Quasi-witnessed models [Laskowski and Malekpour, 2007]

A Π-interpretation $\mathcal{I}$ is quasi-witnessed when it satisfies that for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$:

(wit∃) there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\exists R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a, b) \cdot C^{\mathcal{I}}(b)$$

(qwit∀)

- either there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a, b) \Rightarrow C^{\mathcal{I}}(b)$$

- or $(\forall R.C)^{\mathcal{I}}(a) = 0$

Quasi-witnessed models and standard semantics

Tautologies and positively satisfiable formulas in $[0,1]_\Pi\forall$ are the same of those in quasi-witnessed standard models. [Cerami, Esteva and Bou, 2010]

$$\varphi \in [0,1]_\Pi\forall\text{-}Taut \iff \varphi \in [0,1]_\Pi\forall\text{-}Taut^{qw}$$

$$\varphi \in [0,1]_\Pi\forall\text{-}pos\text{-}Sat \iff \varphi \in [0,1]_\Pi\forall\text{-}pos\text{-}Sat^{qw}$$

Previous related results

- First order standard tautologies are not recursively axiomatizable and, worst, not arithmetical for Product Logic. [Montagna, 2001]
- Satisfiability (validity, subsumption) problem in the $\mathcal{ALC}$ description language over Lukasiewicz Logic is decidable. [Hájek, 2005]
- Satisfiability (validity, subsumption) in witnessed models for the $\mathcal{ALCE}$ description language over Product Logic is decidable. [Bobillo and Straccia, 2009]
- $\mathcal{ALE}$ description language over Product Logic does not enjoy finite model property. [Hájek, 2005]

Reduction to propositional satisfiability

- We provide a reduction of validity and satisfiability for Π-$\mathcal{ALE}$ to the semantic consequence in propositional Product Logic which is known to be a decidable problem.
- It is done in three steps, given an assertion $C(d)$:
  1. first we produce a set of formulas T_C, which provides positive constraints to build the model that (possibly) satisfies $C(d)$,
  2. second we produce a set of formulas Y_C, which provides negative constraints to build the model that (possibly) satisfies $C(d)$,
  3. third, we provide a translation $pr(\cdot)$ of formulas in T_C and Y_C into a propositional language.

Example: the set T_{C_0}

We will give an informal account of this reduction. Given an assertion, say

$$C_0(d) = (\neg\forall R.A \odot \neg\exists R.\neg A)(d)$$

for each quantified subformula occurring in it we produce a new constant and a couple of formulas are added to T_{C_0}:

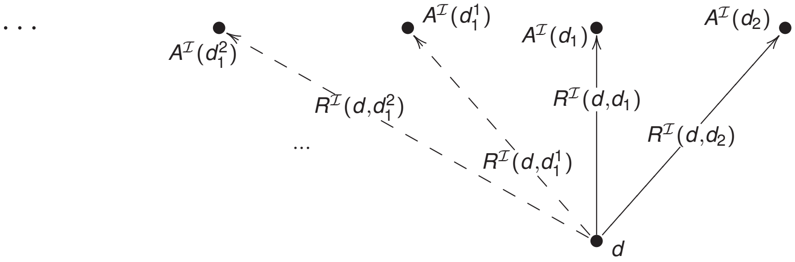
$$\forall R.A(d) \quad d_1 \quad (\forall R.A(d) \equiv (R(d,d_1) \to A(d_1))) \sqcup \neg\forall R.A(d)$$

$$\forall R.A(d) \to (R(d,d_2) \to A(d_2))$$

$$\exists R.\neg A(d) \quad d_2 \quad \exists R.\neg A(d) \equiv (R(d,d_2) \odot \neg A(d_2))$$

$$(R(d,d_1) \odot \neg A(d_1)) \to \exists R.\neg A(d)$$

which says us that we are building the following interpretation $\mathcal{I}$:



Example: the set Y_{C_0}

Moreover, for the universally quantified subformula, we add to the set Y_{C_0} the following formula:

$$\neg\forall R.A(d) \boxdot (R(d, d_1) \rightarrow A(d_1))$$

which constrains interpretation $\mathcal{I}$ not to verify both

$$(\forall R.A)^{\mathcal{I}}(d) = 0$$

and

$$R^{\mathcal{I}}(d, d_1) \rightarrow A^{\mathcal{I}}(d_1) = 1$$

in order to overcome a problem in an earlier version of this work.

So, the elements of the set $pr(T_{C_0})$ are:

$$(\forall R.A(d) \equiv (R(d,d_1) \rightarrow A(d_1))) \sqcup \neg\forall R.A(d)$$

$$(P_{\forall R.A(d)} \equiv (P_{R(d,d_1)} \rightarrow P_{A(d_1)})) \vee \neg P_{\forall R.A(d)}$$

$$\forall R.A(d) \rightarrow (R(d,d_2) \rightarrow A(d_2)) \quad P_{\forall R.A(d)} \rightarrow (P_{R(d,d_1)} \rightarrow P_{A(d_2)})$$

$$\exists R.\neg A(d) \equiv (R(d,d_2) \boxdot \neg A(d_2)) \quad P_{\exists R.\neg A(d)} \equiv (P_{R(d,d_2)} \odot P_{A(d_2)})$$

$$(R(d,d_1) \boxdot \neg A(d_1)) \rightarrow \exists R.\neg A(d) \quad (P_{R(d_1,d_1)} \odot P_{A(d_1)}) \rightarrow P_{\exists R.A(d)}$$

and the element of the set $pr(Y_{C_0})$ is:

$$\neg\forall R.A(d) \boxdot (R(d,d_1) \rightarrow A(d_1)) \quad \neg P_{\forall R.A(d)} \odot (P_{R(d,d_1)} \rightarrow P_{A(d_1)})$$

Propositional evaluations

We say that a propositional evaluation e is quasi-witnessing for an assertion C if:

- $e(\varphi) = 1$, for every $\varphi \in pr(T_C)$ and
- $e(\psi) \neq 1$, for every $\psi \in pr(Y_C)$

and prove that, for each $r \in [0, 1]$:

there is an individual d and a quasi-witnessed interpretation $\mathcal{I}$ such that $C^{\mathcal{I}}(d) = r$	$\iff$	there is an individual d and a quasi-witnessing propositional evaluation e such that $e(pr(C(d))) = r$

Proof: from FDL interpretations to propositional evaluations

Given a quasi-witnessed interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} = r$, define the propositional evaluation $e_{\mathcal{I}}$ such that, for every concept and role assertion $D(a)$ and $R(a, b)$, occuring in a formula in $T_C \cup Y_C$,

$$e_{\mathcal{I}}(pr(D(a))) = D^{\mathcal{I}}(a)$$

and

$$e_{\mathcal{I}}(pr(R(a, b))) = R^{\mathcal{I}}(a, b)$$

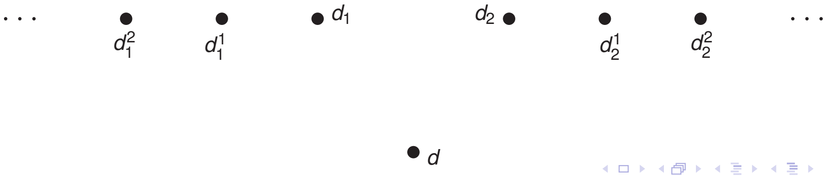
Hence, it is a simple task to check that $e_{\mathcal{I}}$ is a quasi-witnessing propositional evaluation and $e_{\mathcal{I}}(C(d)) = r$.

Proof: from propositional evaluations to FDL interpretations

We give an sketch by means of the example assertion C_0 before:

Given the sets T_{C_0}, Y_{C_0} and a quasi-witnessing propositional evaluation e such that $e(pr(C_0(d))) = r$, we define how to build a quasi-witnessed interpretation $\mathcal{I}_e$:

- The elements of the domain $\Delta^{\mathcal{I}}$ are the constant occurring in $T_{C_0} \cup Y_{C_0}$, plus a countable infinite set of new elements $\{d_n^i : n \in \omega \backslash 0\}$ for each constant d_n occurring in $T_{C_0} \cup Y_{C_0}$ and different from the root d:

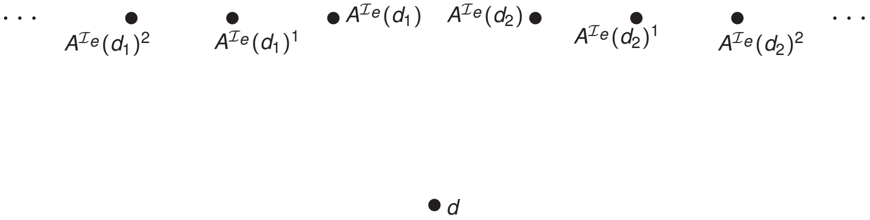


- For each atomic concept A and each constant d, d_n occurring in $T_{C_0} \cup Y_{C_0}$, define:

$$A^{\mathcal{I}_e}(d_n) = e(pr(A(d_n)))$$

and for each new element $d_n^i \in \Delta^{\mathcal{I}_e}$, define:

$$A^{\mathcal{I}_e}(d_n^i) = (e(pr(A(d_n))))^i$$

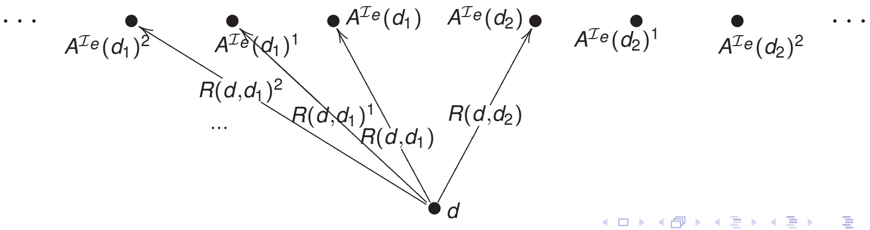


- For each role name R and each constant d_n occurring in $T_{C_0} \cup Y_{C_0}$, define:

$$R^{\mathcal{I}_e}(d, d_n) = e(pr(R(d, d_n)))$$

and for each new element $d_n^i \in \Delta^{\mathcal{I}_e}$, define:

$$R^{\mathcal{I}_e}(d, d_n^i) = \begin{cases} (e(pr(R(d, d_n))))^i, & \text{if } R(d, d_n) \rightarrow A(d_n)) \text{ occurrs in } T_{C_0} \text{ and } e(pr(\forall R.A(d))) = 0 \\ 0, & \text{otherwise} \end{cases}$$



Main Proposition

Proposition

Let C be a concept, and let T_C and Y_C be the two finite sets associated by the algorithm. For every $r \in [0, 1]$, the following statements are equivalent:

1. *C is satisfiable with truth value r in a quasi-witnessed $[0, 1]_\Pi$-interpretation,*
2. *there is a propositional evaluation e such that:*
   - $e(\varphi) = 1$, *for every* $\varphi \in pr(T_C)$ *and*
   - $e(\psi) \neq 1$, *for every* $\psi \in pr(Y_C)$
   - $e(pr(C(d_0))) = r$.

Reduction

$C \in 1\text{-}Sat^{qw}$ iff $\bigvee pr(Y_{C_0})$ is not a consequence, in propositional Product Logic, of the set $\{pr(C(d_0))\} \cup pr(T_{C_0})$

iff $\{pr(C(d_0))\} \cup pr(T_{C_0}) \not\models \bigvee pr(Y_{C_0})$

$C \in Taut^{qw}$ iff $pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$ is a consequence, in propositional Product Logic, of the set $pr(T_{C_0})$

iff $pr(T_{C_0}) \models pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$

Hence, we have a reduction of these problems to the semantic consequence problem, with a finite number of hypothesis, in the propositional Product Logic. Hájek, 2006 proves that such problem is in *PSPACE*.

Moreover, by the result before mentioned about the completeness of $[0,1]_\Pi\forall$ with repect to quasi-witnessed models we have that:

$$C \in Taut \quad \text{iff} \quad pr(T_{C_0}) \models pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$$

$$C \in pos\text{-}Sat \quad \text{iff} \quad pr(T_{C_0}) \not\models \neg pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$$

1-*Sat* is still an open problem.

Thanks!