Many-valued Horn Logic is Hard

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Introduction

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- The **general question** that we address in this seminar is whether a given finite-valued logic-based formalism can be more complex than the same formalism on crisp semantics.
- E.g. every **finite-valued FDL** has been proved to belong to the **same complexity class** as the corresponding crisp DL.
- A question that naturally arises is whether the finite-valued fuzzy framework is **not more complex** than the crisp-valued formalism in general.
- This presentation is based on part of the paper **Many-valued Horn Logic is Hard**, published in the proceedings of the International Workshop PRUV 2014.
- The paper is a joint work with R. Peñaloza and S. Borgwardt and it has been written during the **stay at the TU Dresden** financed in the framework of the POST UP II project.

General idea

- Classical logic-based formalisms are already **highly nondeterministic** when all the connectives are present in the language.
- In this work we are considering **propositional Horn clauses**.
- Reasoning in such formalisms is **polynomial** under classical two-valued semantics.
- We reduce the problem of deciding **classical satisfiability** of propositional formulas to the **satisfiability problem for Horn clauses with finite-valued constraints**.
- In this way, we show that for these clauses a **finite-valued Łukasiewicz** conjunction operator can **induce additional nondeterminism**.

Preliminaries

Propositional formulas

Given:

- a countable set $Var = \{p_1, p_2, \dots\}$ of **propositional variables**,
- the set $\{\wedge, \vee, \rightarrow, \neg, \top, \bot\}$ of **logical connectives**,

the set of **propositional formulas** is inductively defined by:

- every variable $p \in Var$ is a formula, as well as $\top$ and $\bot$,
- if φ and ψ are formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\neg\varphi$, are formulas too.

Classical semantics

A **classical propositional valuation** is a function $v\colon Var \longrightarrow \{0,1\}$ which is **inductively extended** to the set of formulas by:

$$\begin{aligned} v(\bot) &= 0 \\ v(\top) &= 1 \\ v(\varphi \wedge \psi) &= \min\{v(\varphi), v(\psi)\} \\ v(\varphi \vee \psi) &= \max\{v(\varphi), v(\psi)\} \\ v(\varphi \rightarrow \psi) &= \max\{1 - v(\varphi), v(\psi)\} \\ v(\neg\varphi) &= 1 - v(\varphi) \end{aligned}$$

A propositional formula φ is said to be **satisfiable** if there exists a propositional valuation v such that:

$$v(\varphi) = 1$$

Finite Łukasiewicz *t*-norm

- For any natural number $n \geqslant 2$, we consider the set of n truth values

$$Ł_n := \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$$

together with the following operators:

  - The **finite Łukasiewicz t-norm** $*\colon Ł_n \times Ł_n \longrightarrow Ł_n$ defined by

$$x * y := \max\{0, x + y - 1\}.$$

  - The **residuum** of the finite Łukasiewicz t-norm $\Rightarrow\colon Ł_n \times Ł_n \longrightarrow Ł_n$ computed as

$$x \Rightarrow y := \min\{1, 1 - x + y\}.$$

- As usual in fuzzy logic, these two operators satisfy the property:

$$x * y \leqslant z \text{ iff } y \leqslant x \Rightarrow z \text{ for all } x, y, z \in Ł_n.$$

- The algebra $\langle Ł_n, *, \Rightarrow, 0, 1\rangle$ is called the **finite Łukasiewicz chain** of length n.

Fuzzy Horn clauses

- A **fuzzy Horn clause** is an expression of the form
  - $\langle p_1 \,\&\, \ldots \,\&\, p_k \to q_1 \,\&\, \ldots \,\&\, q_m \geqslant r \rangle$,
  - $\langle p_1 \,\&\, \ldots \,\&\, p_k \to \bot \geqslant r \rangle$,

  where
  - $k \geqslant 0,\ m \geqslant 1$,
  - $p_1, \ldots, p_k, q_1, \ldots, q_m \in Var$,
  - $r \in Ł_n$.
- A **fuzzy Horn theory** is a finite set of fuzzy Horn clauses.
- Notice that the **conjunction on the left-hand side of a Horn clause might be empty**; that is, a Horn clause could have the form $\langle \to y_1 \,\&\, \ldots \,\&\, y_m \geqslant p \rangle$.

Semantics

- A **valuation** of the propositional variables is a function $v\colon Var \longrightarrow Ł_n$.

- A fuzzy Horn clause is **satisfied** by v if:

  - $\big(v(p_1) * \cdots * v(p_k)\big) \Rightarrow \big(v(q_1) * \cdots * v(q_m)\big) \geqslant p$,
  - $\big(v(p_1) * \cdots * v(p_k)\big) \Rightarrow 0 \geqslant p$,

  where the operation of the left-hand side of the implication is evaluated to 1 whenever $k = 0$.

- A fuzzy Horn theory $\mathcal{H}$ is **satisfiable** if there is a valuation that satisfies all fuzzy Horn clauses in $\mathcal{H}$.

NP-hardness

m-conjunctive normal form

- Given a set *Var* of variables, the set of **literals** is

$$Var \cup \{\neg p \mid p \in Var\}.$$

- For a natural number m, an m-**clause** is a **disjunction** of m literals:

$$\bigvee_{i=1}^{m} \ell_i .$$

- A propositional formula φ is in m-**conjunctive normal form** (m-CNF) if it is a conjunction of m-clauses:

$$\varphi \bigwedge_{i=1}^{k} C_i$$

  for some $k \geqslant 0$, where each $C_i, 1 \leqslant i \leqslant k$ is an m-clause.

- It is well-known that **deciding the satisfiability** of m-CNF formulae is NP-hard, for any $m \geqslant 3$

Reduction

- Let $n \geqslant 4$ and define $m := n - 1$.
- Given an m-CNF formula

$$\varphi = \bigwedge_{i=1}^{k} C_i,$$

we **construct a fuzzy Horn theory** $\mathcal{H}_\varphi$ such that

$$\varphi \text{ is satisfiable} \quad \text{iff} \quad \mathcal{H}_\varphi \text{ is satisfiable.}$$

- Let $Var(\varphi) \subseteq Var$ be the set of all **propositional variables appearing in** φ.
- For each $p \in Var(\phi)$, we employ a **fresh variable** $p' \in Var$ to **simulate the literal** $\neg p$ in the fuzzy Horn theory $\mathcal{H}_\varphi$.

Simulating the classical negation

- As a first step, we define the Horn theory

$$\mathcal{H}_\ell := \{\langle p \,\&\, p' \to \bot \geqslant \tfrac{1}{m}\rangle \mid p \in Var(\varphi)\} \cup \{\langle \to p \,\&\, p' \geqslant \tfrac{m-1}{m}\rangle \mid p \in Var(\varphi)\}.$$

- **Any valuation** $v\colon Var \longrightarrow Ł_n$ that satisfies the Horn clause

$$\langle \to p \,\&\, p' \geqslant \tfrac{m-1}{m}\rangle$$

must be such that

  - $v(p) \geqslant \frac{m-1}{m}$,
  - $v(p') \geqslant \frac{m-1}{m}$,
  - $\max\{v(x), v(x')\} = 1$.

- Moreover, if v also satisfies

$$\langle p \,\&\, p' \to \bot \geqslant \tfrac{1}{m} \rangle,$$

then it must be the case that

$$\min\{v(p), v(p')\} = \tfrac{m-1}{m}.$$

- For every $p \in Var(\varphi)$ and for all valuations v satisfying $\mathcal{H}_\ell$, **exactly one** of p, p' is evaluated to 1, while **the other** is evaluated to $\frac{m-1}{m}$.

- The **intuition** of this construction is that we will read
  - $v(p) = 1$ as evaluating the variable p to 1, and
  - $v(p) \neq 1$ (and hence $v(p') = 1$) as evaluating p to 0.

Building a fuzzy Horn theory

- Consider now the translation ρ that **maps literals to variables**, defined by

$$\rho(\ell) := \begin{cases} p & \text{if } \ell = p \in Var \\ p' & \text{if } \ell = \neg p, p \in Var. \end{cases}$$

- We **extend this mapping to** m**-clauses** by setting

$$\rho\Big(\bigvee_{i=1}^{m} \ell_i \Big) := \&_{i=1}^{m} \rho(\ell_i).$$

- A valuation **satisfies** the Horn clause

$$\langle \rightarrow \&_{i=1}^{m} \rho(\ell_i) \geqslant \tfrac{1}{m} \rangle$$

  if and only if **at least one of the conjuncts** $\rho(\ell_i)$ is evaluated to 1.

- This conjunct $\rho(\ell_i)$ will correspond to the **literal satisfying the clause** $\bigvee_{i=1}^{m} \ell_i$.

- We thus define the fuzzy Horn theory

$$\mathcal{H}_\varphi := \mathcal{H}_\ell \cup \{\langle \rightarrow \rho(C_i) \geqslant \tfrac{1}{m} \rangle \mid 1 \leqslant i \leqslant k\}.$$

Completeness of the reduction

The m-CNF formula φ is satisfiable iff the fuzzy Horn theory $\mathcal{H}_\varphi$ is satisfiable.

($\Rightarrow$) If φ is satisfiable, then there is a **classical propositional valuation** V satisfying φ. We **define a fuzzy valuation** $v\colon Var \longrightarrow Ł_n$ by setting for every $p \in Var(\varphi)$

$$v(p) := \begin{cases} 1 & \text{if } V(p) = 1 \\ \frac{m-1}{m} & \text{otherwise,} \end{cases}$$

$$v(p') := \frac{2m-1}{m} - v(p).$$

v **satisfies** all the Horn clauses in $\mathcal{H}_\ell$. Consider now the Horn clause $\langle \rightarrow \rho(C) \geqslant \frac{1}{m} \rangle$ for some m-clause $C = \bigvee_{i=1}^{m} \ell_i$ of φ. Since V **is a model of** φ, there exists an $1 \leqslant i \leqslant m$ such that ℓ_i is evaluated to 1 by V. By construction, $v(\rho(\ell_i)) = 1$, and hence v satisfies the Horn clause.

($\Leftarrow$) Conversely, let v be **a model of** $\mathcal{H}_\phi$, and **consider the valuation** V that maps every variable p

- to 1, if $v(p) = 1$ and
- to 0 if $v(p) = \frac{m-1}{m}$.

For any m**-clause in** φ, we know that $v(\rho(C)) \geqslant \frac{1}{m}$. Thus, **there is at least one literal** ℓ appearing in C such that

$$v(\rho(\ell)) = 1.$$

Hence:

- If ℓ is a propositional variable p, then $v(p) = 1$ and V evaluates p to 1; hence V satisfies the clause C.
- Otherwise, $\ell = \neg p$ for some propositional variable p. In this case, $v(p) = \frac{m-1}{m}$ and V evaluates p to 0; i.e., V evaluates $\neg p$ to 1, and hence V satisfies C.

Conclusions

- Using our construction, since $n = m + 1$, we obtain that **satisfiability of fuzzy Horn theories** in fuzzy Horn logic is NP-hard for any $n \geqslant 4$.
- The **upper bound** follows from the fact that one can simply guess a valuation in polynomial time ($\mathcal{O}(n \cdot |Var|)$) and then check satisfaction of all clauses.
- our results prove hardness for **four-or-more-valued** Horn logics.
- For **two-valued Horn logics**, it is well-known that satisfiability is decidable in **linear time**.
- Unfortunately, the case of **three-valued semantics** is not covered by our result.

Thank you for the attention !