Fuzzy Description Logics

Marco Cerami

Palacký University in Olomouc
Department of Computer Science
Olomouc, Czech Republic

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INVESTMENTS IN EDUCATION DEVELOPMENT

Introduction

Introduction

- **Description Logics** (DLs) are logic-based knowledge representation languages.
- They are an attempt to find a **fair trade-off** between expressivity and computational complexity in KR
- In the early 90's it began an effort to **generalize** the classical DLs to the **fuzzy case**.
- The **first works** on Fuzzy Description Logic (FDL) considered a semantics based on **Fuzzy Set Theory**.
- In [Hájek, 2005] it is proposed a t-**norm-based semantics** for FDL.
- Since then some works on t-norm-based FDL have been produced.

Historical Remarks

The first works on FDL

- The first works on FDL begin in the **early 90's**.
- In these first works the generalization to the fuzzy case consisted in
  - generalizing the semantics of **atomic concepts** to fuzzy sets
    $$A^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \longrightarrow [0,1]$$
  - generalizing the semantics of **atomic roles** to fuzzy relations
    $$R^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \longrightarrow [0,1]$$
- The truth functions adopted as a **semantics of complex concepts** are:

$\min\{x,y\}$	for the conjunction $\sqcap$
$\max\{x,y\}$	for the disjunction $\sqcup$
$1-x$	for the negation $\neg$
$\max\{1-x,y\}$	for the implication $\rightarrow$

Non intuitive behavior: an example

- Due to the **absence of a residuated implication**, this semantics can lead to **counter-intuitive consequences**.
- This fact has been pointed out in [Hájek, 2005],
- here is presented the example of the assertion "all hotels near to the main square are expensive"

$$\forall \texttt{hasNear.Expensive(MainSquare)}$$

- Consider, indeed, the situation where:
  - $\texttt{hasNear(MainSquare,Hotel\_1)} = 0.9$,
  - $\texttt{hasNear(MainSquare,Hotel\_2)} = 0.5$,
  - $\texttt{hasNear(MainSquare,Hotel\_3)} = 0.1$,
  - $\texttt{Expensive(Hotel\_1)} = 0.9$,
  - $\texttt{Expensive(Hotel\_2)} = 0.5$,
  - $\texttt{Expensive(Hotel\_3)} = 0.1$,

In this ideal situation we should have that:

$$\forall \texttt{hasNear.Expensive(MainSquare)} = 1$$

because hotels are at least as expensive as they lie near the main square. Nevertheless, using the truth function of Kleene-Dienes implication, the result is different.

$$\begin{aligned}
&(\forall \texttt{hasNear.Expensive(MainSquare)})^{\mathcal{I}} = \\
= \ & \inf_{v \in \Delta^{\mathcal{I}}} \{\texttt{Near}^{\mathcal{I}}(\texttt{MainSquare}^{\mathcal{I}}, v) \Rightarrow \texttt{Expensive}^{\mathcal{I}}(v)\} \leq \\
\leq \ & \inf\{\max\{1 - 0.9, 0.9\}, \max\{1 - 0.5, 0.5\}, \max\{1 - 0.1, 0.1\}\} = \\
= \ & \inf\{0.9, 0.5, 0.9\} \\
= \ & 0.5
\end{aligned}$$

A more general framework for FDL

- [Hájek, 1998] considers formal calculi of many-valued logic to be the kernel of fuzzy logic.
- This framework is nowadays called **Mathematical Fuzzy Logic** (MFL).
- **[Hájek, 2005]** proposes to take MFL as the **underlying logic of FDLs** (mimicking the relation between DL and classical logic).
- In this framework FDL has been strictly related to **first order Fuzzy Logic**.
- We will consider FDLs from the point of view of Mathematical Fuzzy Logic.
- In this sense, **we are not considering** semantics that are not based on residuated structures.

Syntax

and

Semantics

Description Signature

A **description signature** is a tuple $\mathbf{D} = \langle N_I, N_A, N_R \rangle$, where

- N_I, a set of **individual names**;
  - Notation: $a, b, c, \ldots$
  - Examples: `John`, `Mary`, `Prague`, `MainSquare`,
- N_A a set of concept names (the **atomic concepts**);
  - Notation: $A, B, C, \ldots$
  - Examples: `Person`, `Female`, `Tall`, `Fat`, `Hight`,
- N_R a set of role names (the **atomic roles**)
  - Notation: $R_1, R_2, \ldots$
  - Examples: `hasChild`, `hasSister`, `hasNear`, `hasTemperature`,

Complex Concepts

$\bot$ |	(empty concept)	
$\top$ |	(universal concept)	
A |	(atomic concept)	
$\sim C$ |	(strong complementary concept)	$(\mathcal{C})$
$\sim A$ |	(restricted strong compl. concept)	
$\neg C$ |	(weak complementary concept)	$(\mathfrak{I})$
ΔC |	(delta operator)	$(\mathcal{D})$
$\dot{S}_i C$ |	(stressed concept)	$(\mathcal{M})$
$\dot{D}_j C$ |	(depressed concept)	$(\mathcal{M})$
$C \boxplus D$ |	(concept strong union)	$(\mathcal{U})$
$C \boxtimes D$ |	(concept strong intersection)	$(\mathcal{AL})$
$C \sqcup D$ |	(concept weak union)	$(\mathfrak{I})$
$C \sqcap D$ |	(concept weak intersection)	$(\mathfrak{I})$
$C \sqsupset D$ |	(concept implication)	$(\mathfrak{I})$
$\forall R.C$ |	(value restriction)	$(\mathcal{AL})$
$\exists R.C$ |	(existential quantification)	$(\mathcal{E})$
$\exists R.\top$ |	(restricted existential quantif.)	

Interpretations

A **T-interpretation** is a pair

$$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$$

where

- $\Delta^{\mathcal{I}}$ is a nonempty (crisp) set called **domain**,
- $\cdot^{\mathcal{I}}$ is a **fuzzy interpretation function** such that:
  1. $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow T$,
  2. $R^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \longrightarrow T$,
  3. $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$

Semantics of complex concepts

$$
\begin{array}{rcl}
\bot^{\mathcal{I}}(x) & := & 0 \\
\top^{\mathcal{I}}(x) & := & 1 \\
\overline{r}^{\mathcal{I}}(x) & := & r \\
(\sim C)^{\mathcal{I}}(x) & := & 1 - C^{\mathcal{I}}(x) \\
(\neg C)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) \to \bot^{\mathcal{I}}(x) \\
(\triangle C)^{\mathcal{I}}(x) & := & \triangle C^{\mathcal{I}}(x) \\
(\dot{S}_i C)^{\mathcal{I}}(x) & := & \dot{s}_i C^{\mathcal{I}}(x) \\
(\dot{D}_j C)^{\mathcal{I}}(x) & := & \dot{d}_j C^{\mathcal{I}}(x) \\
(C \boxtimes D)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) * D^{\mathcal{I}}(x) \\
(C \boxplus D)^{\mathcal{I}}(x) & := & 1 - ((1 - C^{\mathcal{I}}(x)) * (1 - D^{\mathcal{I}}(x))) \\
(C \sqcap D)^{\mathcal{I}}(x) & := & \min\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \\
(C \sqcup D)^{\mathcal{I}}(x) & := & \max\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \\
(C \sqsupset D)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \\
(\forall R.C)^{\mathcal{I}}(x) & := & \inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\} \\
(\exists R.C)^{\mathcal{I}}(x) & := & \sup_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(y)\}
\end{array}
$$

MFL based semantics

MTL-chains are used as algebras of truth values.

An **MTL-chain** is a structure:

$$\mathbf{T} = \langle T, *, \Rightarrow, \wedge, \vee, 0, 1 \rangle$$

where:

- T is a **linearly ordered** set,
- $\langle T, *, 1 \rangle$ is a **commutative monoid**,
- $\Rightarrow$ is the **residuum** of $*$,
- $\wedge$ and $\vee$ are the **minimum** and **maximum operations** w.r.t. the order of T,
- 0 and 1 are the **minimum** and **maximum elements** of T.

We are considering MTL-chains with **domain** either

$$[0,1] \qquad \text{or} \qquad \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}$$

and **operations**:

	Gödel	Product	Łukasiewicz
$x * y$	$\min(x, y)$	$x \cdot y$	$\max(0, x + y - 1)$
$x \Rightarrow y$	$\begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise} \end{cases}$	$\min(1, 1 - x + y)$
$\neg x$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$1 - x$

Expanding the algebraic language

Moreover, we will consider **expansions** of these algebras by means of the operations that are **the semantics of the following connectives**:

- a suitable set of **truth constants** $\bar{r}$,
- the **Monteiro-Baaz delta operator** $\triangle$,
- an **involutive negation** $\sim$,
- a set of **truth hedges**.

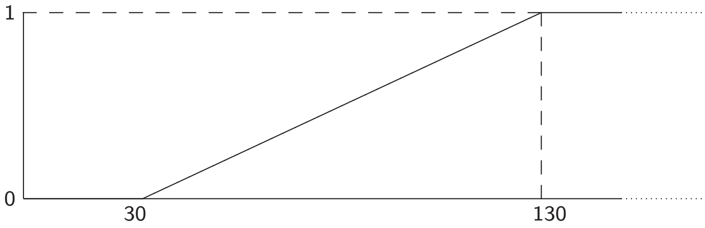
Challenges of the new framework

- With a Zadeh' style semantics, there was **no substantial difference** with the classical framework.
- However, in the new framework based on MFL there are **several differences** with the classical framework.
- Such differences include the following items:
  - **two kinds of conjunctions** can be considered in the many-valued framework, with different mathematical properties, and the same holds for disjunction,
  - **implication** is, in general, **not definable** from other connectives,
  - the **quantifiers are not definable from each other** by means of the equivalence $\exists R.C \equiv \neg\forall R.\neg C$,
  - the **strong disjunction is not definable** from the residuated negation $\neg C := C \rightarrow \bot$ and the conjunction $\sqcap$.

Two kinds of conjunctions: an example

Consider predicates `Weighty` and `Tall` that can be computed in the following way:

- the semantics of `Weighty` is
$$\max\{0, \min\{1, \frac{|W|-30}{100}\}\}.$$
- the semantics of `Tall` is
$$\max\{0, \min\{1, \frac{|H|-120}{100}\}\}.$$



The range of both predicates takes 100 values. Thus, if the chain of values is of type $Ł_n$, then we consider $Ł_{100}$.

Why *t*-norm based conjunction?

In order to give a degree to the concept Fat, we should rely on the rule:

$$\texttt{Fat} \equiv \texttt{Weighty} \boxtimes \sim\texttt{Tall} \geq 1$$

Suppose we have:

- Tall(John)=0.5,
- Tall(Jack)=0.5,
- Weighty(John)=0.6,
- Weighty(Jack)=0.9,

With **minimum**, we have that

$$\texttt{Fat(John)}=0.5=\texttt{Fat(Jack)}$$

With the **Łukasiewicz *t*-norm**, we have that

$$\texttt{Fat(John)} = 0.1 \quad \text{and} \quad \texttt{Fat(Jack)} = 0.4.$$

Why minimum based conjunction?

In order to give a degree to the **intersection of the respective fuzzy sets**, it is indeed more convenient to use a conjunction whose semantics is the minimum t-norm. Suppose we have

- $\texttt{Tall(John)} = 0.5$,
- $\texttt{Awesome(John)} = 0.4$,

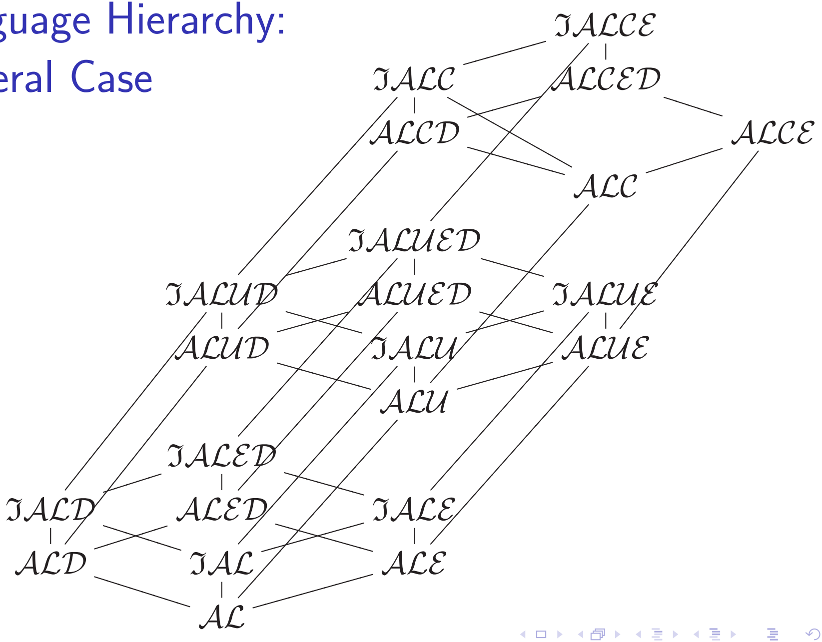
With **minimum**, we have that

$$\texttt{Tall} \sqcap \texttt{Awesome(John)} = 0.4,$$

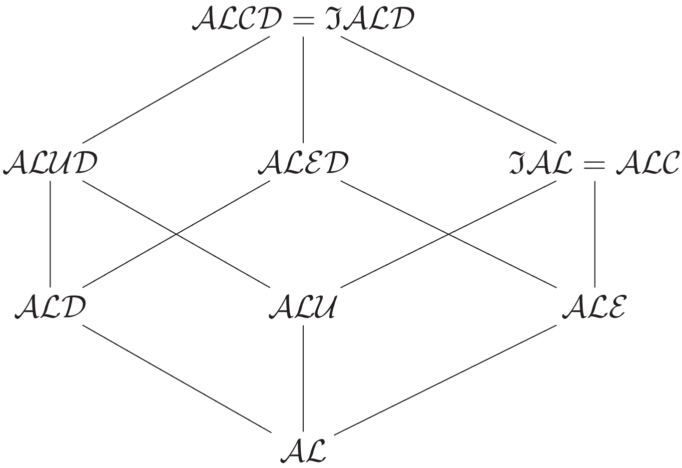
With the **Łukasiewicz t-norm**, we have that

$$\texttt{Tall} \boxtimes \texttt{Awesome(John)} = 0,$$

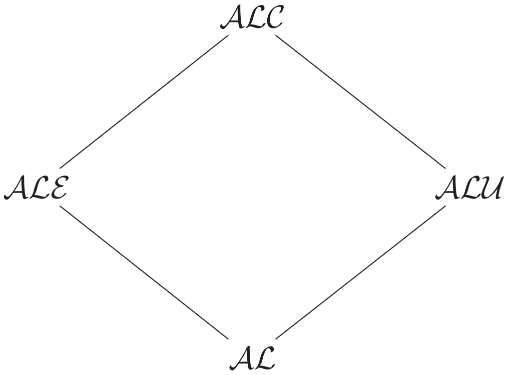
Language Hierarchy: General Case



Language Hierarchy: $[0,1]_Ł$ Case



Language Hierarchy: L_n Case



Languages with Higher Expressivity

$\geq n\, R$	unqualified number restriction	$\mathcal{N}$
$\leq n\, R$		
$= n\, R$		
$\geq n\, R.C$	qualified number restriction	$\mathcal{Q}$
$\leq n\, R.C$		
$= n\, R.C$		
$\{a\}$	nominals	$\mathcal{O}$
R^-	inverse roles	$\mathcal{I}$
U	universal role	$\mathcal{R}$
$\neg R$	role negation	$\mathcal{R}$
$R \sqcap S$	role intrsection	$\mathcal{R}$
$R \sqcup S$	role union	$\mathcal{R}$
$R \circ S$	role composition	$\mathcal{R}$
R^+	transitive roles	$\mathcal{S}$

Moreover, in the language $\mathcal{R}$ are allowed axioms involving roles:

Semantics of complex concepts

$$
\begin{aligned}
(\geq n\,R)^{\mathcal{I}}(x) &= \{x \in \Delta^{\mathcal{I}} \colon |\{y \in \Delta^{\mathcal{I}} \colon R^{\mathcal{I}}(x,y) > 0\}| \geq n\} \\
(\leq n\,R)^{\mathcal{I}}(x) &= \{x \in \Delta^{\mathcal{I}} \colon |\{y \in \Delta^{\mathcal{I}} \colon R^{\mathcal{I}}(x,y) > 0\}| \leq n\} \\
(= n\,R)^{\mathcal{I}}(x) &= \{x \in \Delta^{\mathcal{I}} \colon |\{y \in \Delta^{\mathcal{I}} \colon R^{\mathcal{I}}(x,y) > 0\}| = n\} \\
(\geq n\,R.C)^{\mathcal{I}}(x) &= \{x \in \Delta^{\mathcal{I}} \colon |\{y \in \Delta^{\mathcal{I}} \colon R^{\mathcal{I}}(x,y) \wedge C^{\mathcal{I}}(y) > 0\}| \geq n\} \\
(\leq n\,R.C)^{\mathcal{I}}(x) &= \{x \in \Delta^{\mathcal{I}} \colon |\{y \in \Delta^{\mathcal{I}} \colon R^{\mathcal{I}}(x,y) \wedge C^{\mathcal{I}}(y) > 0\}| \leq n\} \\
(= n\,R.C)^{\mathcal{I}}(x) &= \{x \in \Delta^{\mathcal{I}} \colon |\{y \in \Delta^{\mathcal{I}} \colon R^{\mathcal{I}}(x,y) \wedge C^{\mathcal{I}}(y) > 0\}| = n\} \\
\{a\}^{\mathcal{I}}(x) &= \{a^{\mathcal{I}}\} \subseteq \Delta^{\mathcal{I}} \\
(R^{-})^{\mathcal{I}}(x,y) &= R(y,x) \\
U^{\mathcal{I}}(x,y) &= 1 \text{ for all } x,y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \\
(\neg R)^{\mathcal{I}}(x,y) &= 1 - R^{\mathcal{I}}(x,y) \\
(R \sqcap S)^{\mathcal{I}}(x,y) &= \inf_{\langle x,y\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y), S^{\mathcal{I}}(x,y)\} \\
(R \sqcup S)^{\mathcal{I}} &= \sup_{\langle x,y\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y), S^{\mathcal{I}}(x,y)\} \\
(R \circ S)^{\mathcal{I}}(x,y) &= \sup_{x,y,z \in \Delta^{\mathcal{I}}}\{\min\{R^{\mathcal{I}}(x,z), S^{\mathcal{I}}(z,y)\}\}
\end{aligned}
$$

Fuzzy KBs

and

Reasoning Tasks

Fuzzy Knowledge Bases

- A TBox consists of **fuzzy concept inclusion** axioms:

  $$\langle C \sqsubseteq D \geq r \rangle \quad \longrightarrow \quad \inf_{x \in \Delta^{\mathcal{I}}} \{ C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \} \geq r$$

- An ABox consists of
  - **fuzzy concept assertion** axioms:

    $$\langle C(a) \geq r \rangle \quad \longrightarrow \quad C^{\mathcal{I}}(a^{\mathcal{I}}) \geq r$$
    $$\langle C(a) \leq r \rangle \quad \longrightarrow \quad C^{\mathcal{I}}(a^{\mathcal{I}}) \leq r$$
    $$\langle C(a) = r \rangle \quad \longrightarrow \quad C^{\mathcal{I}}(a^{\mathcal{I}}) = r$$

  - **fuzzy role assertion** axioms:

    $$\langle R(a, b) \geq r \rangle \quad \longrightarrow \quad R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq r$$

Set inclusion and implication

- The semantics for subsumption between concepts should be the **inclusion between fuzzy sets**, that is,

$$C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}().$$

- With **residuated implication** this is equivalent to the validity of concept

$$C \sqsupset D$$

- With **Kleene-Dienes implication**

$$\max\{1 - C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\}$$

the above relation between fuzzy set inclusion and implication does not hold anymore.

Set inclusion and implication: an example

- The conjunction $A \sqcap B$ is always subsumed by both concepts,

$$A \sqcap B \sqsubseteq A.$$

- In fact, for every interpretation $\mathcal{I}$ and every $x \in \Delta^{\mathcal{I}}$, it holds that

$$(A \sqcap B)^{\mathcal{I}}(x) = \min\{A^{\mathcal{I}}(x), B^{\mathcal{I}}(x)\} \leq A^{\mathcal{I}}(x).$$

- Let now

$$A^{\mathcal{I}}(x) = B^{\mathcal{I}}(x) = 0.5,$$

then,

$$\begin{aligned} &((A \sqcap B) \sqsupset A)^{\mathcal{I}}(x) = \\ = \; &\max\{1 - \min\{A^{\mathcal{I}}(x), B^{\mathcal{I}}(v)\}, A^{\mathcal{I}}\} = \\ = \; &\max\{1 - \min\{0.5, 0.5\}, 0.5\} = \\ = \; &0.5 \end{aligned}$$

Reasoning Tasks

- Fuzzy knowledge base consistency.
- Concept r-satisfiability w.r.t. (possibly empty) KB.
- Concept $(\geq r)$-satisfiability w.r.t. (possibly empty) KB.
- Positive satisfiability w.r.t. (possibly empty) KB.
- oncept $(\geq r)$-subsumption w.r.t. (possibly empty) KB,
- Entailment.
- Best satisfiability degree of a concept w.r.t. a KB.
- Best entailment degree.

Reduction among reasoning tasks

- As in the classical case, **polynomial reductions** among reasoning tasks to each other have been considered;
- in order to translate decidability results to other reasoning tasks, the reductions that are usually considered are those **to KB consistency**;
- due to the fact that in our framework some undecidability results have been proved, we have also to consider reductions **of KB consistency**;
- in the FDL framework, what kind of reduction are possible, **often depends on the algebra of truth values**, not only on the language considered.

Reduction to KB consistency

With a semantics defined on a **finite MTL chain**, the following statements hold for language $\mathfrak{I}\mathcal{F}\mathcal{L}_0$:

- **Concept** r**-satisfiability** with respect to a (possibly empty) KB can be polynomially reduced to KB consistency.
- **Concept** $(\geq r)$**-satisfiability** with respect to a (possibly empty) KB can be polynomially reduced to KB consistency.
- **Concept positive satisfiability** with respect to a (possibly empty) KB can be polynomially reduced to KB consistency.
- **Concept** $(\geq r)$**-subsumption** can be polynomially reduced to KB consistency.
- **Entailment** of an axiom by a KB can be polynomially reduced to KB consistency.

Reduction of KB consistency

With a semantics defined on the family $\{[0,1]_{Ł}, [0,1]_{\Pi}, [0,1]_{G}, Ł_n, G_n\}$ the following statements hold for language $\mathbf{T}$-$\mathcal{FL}_0$ with rational truth constants:

- KB consistency can be polynomially reduced to **concept** *r*-**satisfiability** with respect to a non-empty KB.
- KB consistency can be polynomially reduced to **concept 1-satisfiability** with respect to a non-empty KB.
- KB consistency can be polynomially reduced to **concept positive satisfiability** with respect to a non-empty KB.
- KB consistency can be polynomially reduced to the **entailment** of an axiom by a KB.

Decidability

and

Complexity

Finite Algebras

- For semantics based on **finite algebras** all reasoning problems are **decidable**.
- Moreover, usually the same complexity bounds as in classical DL are preserved.
- For all FDLs between $\mathcal{AL}$ and $\mathcal{SHI}$ all the decision problems with respect to a non-empty KB are EXPTIME-**complete**.
- Up to $\mathcal{SHOIQ}$ are in NEXPTIME.
- For all sublogic of $\mathfrak{I}\mathcal{ALCE}$ all the decision problems with respect to an empty KB are PSPACE-**complete**.

Infinite Algebras

- In $[0,1]_G$ all reasoning tasks are **decidable** up to $\mathcal{SHOIQ}$
- In $[0,1]_Ł$:
  - all tasks are **undecidable** for (sublogics of) $\mathcal{ALE}$ if a non-empty KB is involved,
  - all tasks are **decidable** for (sublogics of) $\mathfrak{I}\mathcal{ALCE}$ if a no KB is involved, but **complexity is unknown**.
- In $[0,1]_\Pi$:
  - KB consistency is **decidable** in $\mathcal{SHOIQ}$ if there are only lower bound axioms,
  - **undecidable** in (sublogics of) $\mathcal{ALC}, \mathcal{IAL}$, otherwise,
  - for reasoning tasks with no KB involved is actually **unknown**.

Reasoning Algorithms

Reasoning Algorithms

There exist essentially three methods for reasoning with TBoxes:

- crispification-based
- tableau-based
- automata-based

Without TBoxes, Hájek proposed a reduction to **propositional logic**

Crispification Approach

- This family of methods consists in reducing fuzzy reasoning to finitely many crisp DL reasoning tasks.
- for logics with strict negation and without upper bounds axioms it consists in **ignoring fuzziness** by means of the **double negation** of concepts.
- for logics based on finite or Gödel chains it consists in **simulating the values through crisply cutted concepts** (α-cuts).
- In the second case the transformation is **not polynomial**.

Tableau-based Approach

- Tableau rules decompose concepts into simpler restrictions.
- Adequate termination conditions are necessary (**blocking**).
- After termination, **check for contradictions**:
  - **syntactical contradiction**: assertion with two different degrees;
  - **non-solvable** system of constrains (typically **MILP**).
- Also in these cases, the transformation is **not polynomial**, but can be solved in PSPACE.

Automata-based Approach

- Tree automata can verify the **existence of tree-shaped models**.
- Non-determinism and infinite structures can be **handled easily**.
- Good for obtaining **tight complexity** bounds
- Good method for theoretical use, but **not easy to implement**.

Thank you for the attention !