Fuzzy Description Logics from a Mathematical Fuzzy Logic point of view

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Introduction and motivations

Previous framework

- Description Logics (DLs) are logic-based knowledge representation languages.
- They are an attempt to find a fair trade-off between expressivity and computational complexity in Knowledge Representation.
- In the early 90's it began an effort to generalize the classical version to the fuzzy case.
- The first works on Fuzzy Description Logic (FDL) considered a semantics based on Fuzzy Set Theory.

A framework for FDL based on MFL

- [Hájek, 1998] considers formal calculi of many-valued logic to be the kernel of fuzzy logic.
- This framework is nowadays called Mathematical Fuzzy Logic (MFL).
- [Hájek, 2005] proposes to take MFL as the underlying logics of FDLs (mimicking the relation between DL and classical logic).
- In this framework FDL has been strictly related to first order Fuzzy Logic.
- The work presented in our dissertation follows Hájek's approach and further develops it.

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  - Complexity results for FDL
  - Algorithmic research
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Semantics of early FDL

The first works on FDL considered a semantics based on fuzzy set-theoretical operations. The truth functions of this semantics are:

$\min\{x, y\}$	for the conjunction $\sqcap$
$\max\{x, y\}$	for the disjunction $\sqcup$
$1 - x$	for the negation $\neg$

The semantics of $C \sqsubseteq D$ was defined as the following (crisp) condition:

$$C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x), \text{ for every } x \text{ in the domain of the interpretation } \mathcal{I}.$$

MFL based semantics

MTL-chains are used as algebras of truth values.

An MTL-chain is a structure:

$$\mathbf{T} = \langle T, *, \Rightarrow, \wedge, \vee, 0, 1 \rangle$$

where:

- T is a linearly ordered set,
- $\langle T, *, 1 \rangle$ is a commutative monoid,
- $\Rightarrow$ is the residuum of $*$,
- $\wedge$ and $\vee$ are the minimum and maximum operations w.r.t. the order of T,
- 0 and 1 are the minimum and maximum elements of T.

We are considering MTL-chains with domain either

$$[0,1] \qquad \text{or} \qquad \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}$$

and operations:

	Gödel	Product	Łukasiewicz
$x * y$	$\min(x, y)$	$x \cdot y$	$\max(0, x + y - 1)$
$x \Rightarrow y$	$\begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise} \end{cases}$	$\min(1, 1 - x + y)$
$\neg x$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$1 - x$

PART I

A proposal of FDL based on MFL

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Syntax of complex concepts

$C, D \longrightarrow$	$\bot$	empty concept	$\mathcal{AL}$
	$\top$	universal concept	$\mathcal{AL}$
	A	atomic concept	$\mathcal{AL}$
	$C \boxtimes D$	strong conjunction	$\mathcal{AL}$
	$\forall R.C$	value restriction	$\mathcal{AL}$
	$\exists R.\top$	restricted existential quantif.	$\mathcal{AL}$
	$\sim A$	atomic complementation	$\mathcal{AL}$
	$\bar{r}$	a constant concept for every $r \in S$	$\mathcal{X}^S$
	$\triangle C$	delta operator	$\mathcal{D}$
	$C \sqsupset D$	implication	$\mathfrak{I}$
	$\neg C$	residuated negation	$\mathfrak{I}$
	$C \sqcap D$	weak conjunction	$\mathfrak{I}$
	$C \sqcup D$	weak disjunction	$\mathfrak{I}$
	$\sim C$	complementation	$\mathcal{C}$
	$C \boxplus D$	strong disjunction	$\mathcal{U}$
	$\exists R.C$	existential quantification	$\mathcal{E}$

Interpretations

A **T**-interpretation is a pair

$$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$$

where

- $\Delta^{\mathcal{I}}$ is a nonempty (crisp) set called domain,
- $\cdot^{\mathcal{I}}$ is a fuzzy interpretation function such that:
  1. $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow T$,
  2. $R^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \longrightarrow T$,
  3. $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$

Semantics of complex concepts

$$\begin{aligned}
\bot^{\mathcal{I}}(x) &:= 0\\
\top^{\mathcal{I}}(x) &:= 1\\
\overline{r}^{\mathcal{I}}(x) &:= r\\
(\sim C)^{\mathcal{I}}(x) &:= 1 - C^{\mathcal{I}}(x)\\
(\neg C)^{\mathcal{I}}(x) &:= C^{\mathcal{I}}(x) \to \bot^{\mathcal{I}}(x)\\
(\triangle C)^{\mathcal{I}}(x) &:= \triangle C^{\mathcal{I}}(x)\\
(C \boxtimes D)^{\mathcal{I}}(x) &:= C^{\mathcal{I}}(x) * D^{\mathcal{I}}(x)\\
(C \sqcap D)^{\mathcal{I}}(x) &:= \min\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\}\\
(C \boxplus D)^{\mathcal{I}}(x) &:= 1 - ((1 - C^{\mathcal{I}}(x)) * (1 - D^{\mathcal{I}}(x)))\\
(C \sqcup D)^{\mathcal{I}}(x) &:= \max\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\}\\
(C \sqsupset D)^{\mathcal{I}}(x) &:= C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\\
(\forall R.C)^{\mathcal{I}}(x) &:= \inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\}\\
(\exists R.C)^{\mathcal{I}}(x) &:= \sup_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(y)\}
\end{aligned}$$

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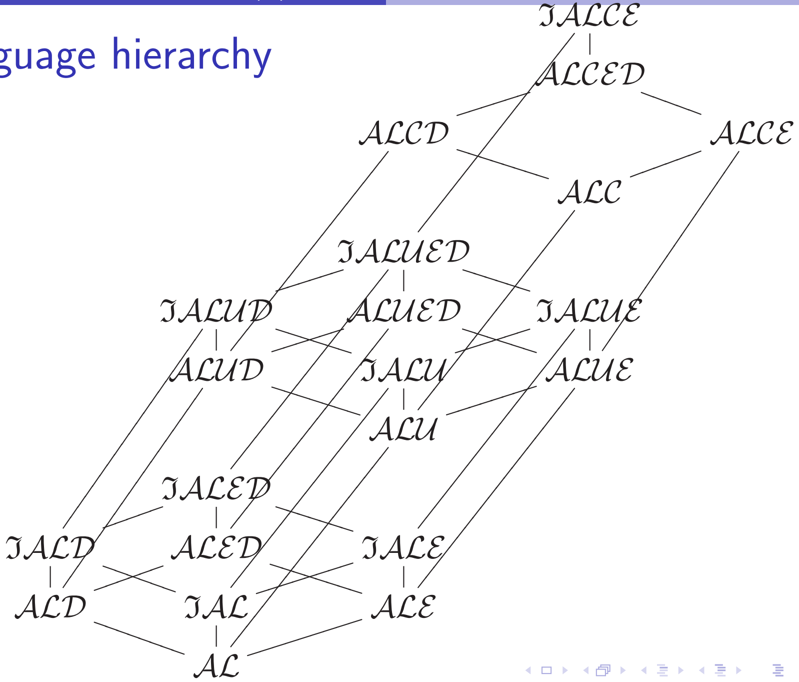
Definability of concept constructors

In general,

- The implication $\sqsupset$ is not definable from conjunction and negation.
- The existential quantifier $\exists R.$ is not definable from the universal quantifier $\forall R.$ and the negations $\neg$ or $\sim$.
- The Delta constructor $\triangle$ is not definable.

Language hierarchy



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Fuzzy knowledge bases

- A TBox consists of fuzzy concept inclusion axioms:

$$\langle C \sqsubseteq D \geq r \rangle \qquad \longrightarrow \qquad \inf_{x \in \Delta^{\mathcal{I}}} \{ C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \} \geq r$$

- An ABox consists of
  - fuzzy concept assertion axioms:

$$\langle C(a) \geq r \rangle \qquad \longrightarrow \qquad C^{\mathcal{I}}(a^{\mathcal{I}}) \geq r$$

  - fuzzy role assertion axioms:

$$\langle R(a,b) \geq r \rangle \qquad \longrightarrow \qquad R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq r$$

Reasoning tasks

- Fuzzy knowledge base consistency
- Graded concept satisfiability
- Positive satisfiability
- Graded concept subsumption
- Entailment
- Best satisfiability degree of a concept with respect to a KB
- Best entailment degree of an axiom with respect to a KB

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Translation of concepts to first order formulas

The translation is defined on concept, role and individual names

$$A \rightsquigarrow A(x)$$

$$R \rightsquigarrow R(x, y)$$

$$a \rightsquigarrow a$$

It is inductively extended to complex concepts as, for example:

$$\forall R.(A \boxtimes A) \rightsquigarrow (\forall y)\Big(R(x, y) \rightarrow (A(y) \otimes A(y))\Big)$$

The translation of complex concepts can be extended to fuzzy inclusions

$$\langle A \boxtimes B \sqsubseteq C \geq \overline{r}\rangle \quad \rightsquigarrow \quad \overline{r} \to (\forall x)(A(x) \otimes B(x) \to C(x)),$$

and fuzzy assertions.

$$\langle (A \boxtimes A)(a) \geq \overline{r}\rangle \quad \rightsquigarrow \quad (\overline{r} \to A(x) \otimes A(x))[a/x],$$

Result

The respective semantics are invariant under the translation.

This shows that our proposal of FDL indeed corresponds to a fragment of first order Fuzzy Logic.

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Translation of concepts to multi-modal formulas

The translation is defined on concept names

$$A \rightsquigarrow p_A$$

For every role name R a pair of modal operators is added to the propositional language

$$R \quad \rightsquigarrow \quad \Box_R, \Diamond_R$$

The translation is inductively extended to complex concepts, as for example:

$$\forall R.(A \boxtimes A) \rightsquigarrow \Box_R(p_A \otimes p_A)$$

Translation of multi-modal formulas to concepts

The translation is defined on propositional variables

$$p \quad \rightsquigarrow \quad A_p$$

For every pair of modal operators $\Box_i$ and $\Diamond_i$ a role name is added to the FDL signature

$$\{\, \Box_i, \Diamond_i \,\} \qquad \rightsquigarrow \qquad R_i$$

The translation is inductively extended to complex formulas as, for example:

$$\Box_i(p \otimes p) \quad \rightsquigarrow \quad \forall R_i.(A_p \boxtimes A_p)$$

- The translation of complex concepts can be extended to fuzzy inclusions.

$$\langle A \boxtimes B \sqsubseteq C \geq r \rangle \quad \rightsquigarrow \quad \Box_U(\bar{r} \rightarrow ((A \otimes B) \rightarrow C))$$

- Due to the lack of a translation of individual names, a translation of fuzzy assertions can not be obtained.

Result

The respective semantics are invariant under the translation.

PART II

Computability and complexity issues

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    - ⋆ Undecidability of KB satisfiability in $[0,1]_{Ł}$-$\mathcal{ALC}$
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Related previous results

- Standard tautologies are not recursively axiomatizable and not arithmetical for first order logic over $[0,1]_\Pi$ [Montagna, 2001].
- Concept satisfiability, validity and subsumption problems in the $\mathcal{ALC}$ description language over $[0,1]_{\text{Ł}}$ are decidable [Hájek, 2005].
- Language $\mathfrak{I}\mathcal{ALE}$ over $[0,1]_\Pi$ has not the finite model property [Hájek, 2005].

Concept satisfiability and subsumption in Π-$\mathfrak{I}\mathcal{ALE}$

Result

Concept validity and positive satisfiability problems for the language $\mathfrak{I}\mathcal{ALE}$ over $[0,1]_\Pi$ are decidable.

- We prove it by providing a recursive reduction of such problems to the logical consequence problem in propositional Product Logic.
- The reduction exploits the quasi-witnessed model property of first order Product Logic [Laskowski, Malekpour, 2007].
- The result then follows from the fact that logical consequence in propositional Product Logic is a decidable problem.
- Notice that we are not considering satisfiability with respect to a knowledge base.

Quasi-witnessed models and standard semantics

The quasi-witnessed model property can be proved in first order Product Logic with respect to the standard semantics.

Tautologies and positively satisfiable formulas in first order logic over $[0,1]_\Pi$ are the same of those in quasi-witnessed standard models.

φ is true in every model over $[0,1]_\Pi$	$\iff$	φ is true in every quasi-witnessed model over $[0,1]_\Pi$
φ is positively satisfiable in a model over $[0,1]_\Pi$	$\iff$	φ is positively satisfiable in a quasi-witnessed model over $[0,1]_\Pi$
φ is 1-satisfiable in a model over $[0,1]_\Pi$	?	φ is 1-satisfiable in a quasi-witnessed model over $[0,1]_\Pi$

Important remarks

- This shows the fact that restricting the first order language over $[0,1]_\Pi$ to the $\mathfrak{I}\mathcal{ALE}$ fragment we obtain decidability of the set of tautologies of the restricted language, in contrast to the fact that, without such restriction, the set of tautologies is not arithmetical.
- The given algorithm is able to cope with infinite interpretations by means of finite resources.

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Related previous results: concept satisfiability

- Concept satisfiability w.r.t. witnessed models coincides with concept satisfiability w.r.t. finite models in the language $\mathcal{ALC}$ [Hájek, 2005].

- Concept satisfiability w.r.t. witnessed models coincides with unrestricted concept satisfiability in the language $\mathcal{ALC}$ over $[0,1]_Ł$ [Hájek, 2005].

- Concept satisfiability is decidable in the language $\mathcal{ALC}$ over $[0,1]_Ł$ [Hájek, 2005].

Related previous results: knowledge base satisfiability

- KB satisfiability is decidable in very expressive languages over $[0,1]_G$ [Bobillo, Delgado, Gómez-Romero, Straccia, 2009].
- KB satisfiability is decidable in the language $\mathcal{ALCH}$ over finite t-norms [Bobillo, Straccia, 2010].
- KB satisfiability does not coincide with KB satisfiability w.r.t. finite models in the language $\mathcal{ALC}$ over $[0,1]_{Ł}$ and $[0,1]_{\Pi}$ [Bobillo, Bou, Straccia, 2011].
- KB satisfiability w.r.t. (strongly) witnessed models is undecidable in the language $\mathfrak{I}\mathcal{ALCE}$ over $[0,1]_{\Pi}$ [Baader, Peñaloza, 2011].
- KB satisfiability is undecidable in the language $\mathcal{ALCE}$ over lattices with t-norms [Borgwardt, Peñaloza, 2011].

KB satisfiability in $[0,1]_{Ł}$-$\mathcal{ALC}$

Result (C., Straccia)

- *KB satisfiability models is undecidable in the language $\mathcal{ALC}$ over $[0,1]_{Ł}$.*
- *KB satisfiability w.r.t. finite models is undecidable in the language $\mathcal{ALC}$ over $[0,1]_{Ł}$.*

- The results are obtained by means of a reduction of the Post Correspondence Problem to the corresponding KB satisfiability problem.
- The reduction is inspired in the one used in [Borgwardt, Peñaloza, 2011].

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Concept satisfiability: a uniform approach

The computational problem Satisf is the following one:

INPUT: $(\mathbf{T}, C, t)$ where $\mathbf{T}$ is a finite MTL-chain, C is a concept of $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCED}^S$ and $t \in T$.

OUTPUT: Yes/No depending whether there is/isn't an interpretation $\mathcal{I}$ and an individual $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = t$.

Moreover, for every finite MTL-chain $\mathbf{T}$, the computational problem $Satisf_{\mathbf{T}}$ is the one obtained by fixing the finite MTL-chain in the previous problem.

Related previous results

- Concept satisfiability is decidable in the language $\mathcal{ALC}$ over $[0,1]_{Ł}$, [Hájek, 2005].
- Concept satisfiability is decidable in the language $\mathcal{ALCH}$ over finite t-norms [Bobillo, Straccia, 2010].
- Concept satisfiability is PSPACE-complete in the language $\mathcal{ALCE}$ over a finite residuated lattice based semantics, [Borgwardt, Peñaloza, 2011].

PSPACE completeness of language $\mathfrak{I}\mathcal{ALCED}^T$ over any finite MTL-chain

Result

Concept satisfiability and subsumption are PSPACE-complete in the language $\mathfrak{I}\mathcal{ALCED}^T$ over any finite MTL-chain.

- The upper bound argument is based on a PSPACE implementation of the reduction to the corresponding propositional logic presented in [Hájek, 2005].
- The lower bound argument is inspired in the reduction from the problem of deciding the truth of Quantified Boolean Formulas (QBF) problem provided in [Ladner, 1977].

Important remarks

- We obtain a PSPACE implementation of the reduction provided in [Hájek, 2005].

  Note that this PSPACE implementation is not possible if the algebra of truth values **T** is infinite.

- With respect to [Bobillo, Straccia 2010] we obtain complexity bounds.

- The result from [Borgwardt, Peñaloza, 2011] is more general than ours, but here we use an alternative procedure.

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A reduction to propositional logic

The reduction to propositional logic proposed in [Hájek, 2005] has been studied under different points of view:

- A generalization to the case of quasi-witnessed models.
- The size of the propositional theory produced by the reduction.
- An implementation that runs in PSPACE.

Other algorithms

Other algorithms that have been studied during the writing of the dissertation are:

- A generalization to the case of the minimal finite-valued Łukasiewicz Modal Logic of the classical algorithm based on Hintikka sets proposed in [Blackburn, De Rijke, Venema, 2001].
- A reduction of the ABox consistency problem in the language $\mathcal{ALC}$ over $[0,1]_{Ł}$ to the Mixed Integer Linear Programming problem [C.,Straccia].

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Summary

- Expansion of the early description languages, by means of concept constructors that are used in Mathematical Fuzzy Logic.
- Study of the hierarchies of FDL languages.
- Analysis of the relations to two other related formalisms, i.e. first order Fuzzy Logic and Fuzzy Modal Logic.
- Decidability of the concept satisfiability and subsumption problems for language $\mathfrak{I}\mathcal{ALE}$ over $[0,1]_\Pi$.
- Undecidability of the knowledge base consistency problem for language $\mathcal{ALC}$ over $[0,1]_Ł$.
- PSPACE completeness of the concept satisfiability and subsumption problems for language $\mathfrak{I}\mathcal{ALCED}^T$ over any finite MTL-chain.
- A study of decision procedures based on a reduction to the corresponding propositional logic.

Directions for future research

- The study of the computational complexity of more expressive FDLs based on finite t-norms.
- A systematic study of the intractability sources for $\mathfrak{I}\mathcal{ALCED}^S$ FDL languages.
- A computational comparison between the algorithms for FDLs based on finite t-norms; in particular the ones based on Hintikka sets, on Hájek sets, on Automata Theory, on completion forests and on a reduction to classical DLs.
- The implementation of the algorithms provided in this dissertation.

Publications

Publications

- M. Cerami, F. Esteva; *First Order SMTL Logic and Quasi-Witnessed Models*. ESTYLF 2010, pp. 145–150.
- M. Cerami, F. Esteva; *Strict core fuzzy logics and quasi-witnessed models*. Archive for Math. Logic, 50(5), pp. 625–641, 2011.

- M. Cerami, À. García-Cerdaña, F. Esteva; *From Classical Description Logic to n-graded Fuzzy Description Logic*. Fuzz-IEEE 2010, pp. 1506–1513.
- M. Cerami, F. Esteva, À. García-Cerdaña; *On finitely valued Fuzzy Description Logics: The Łukasiewicz case*. IPMU 2012, pp. 235–244.

- M. Cerami, F. Esteva, F. Bou; *Decidability of a Description Logic over Infinite-Valued Product Logic*. KR-2010, pp. 203–213.
- M. Cerami, U. Straccia; *On the (Un)decidability of Fuzzy Description Logic under Łukasiewicz t-norm*. Submitted to *Information Sciences*.

- F. Bou, M. Cerami, F. Esteva; *Finite-valued Łukasiewicz Modal Logic is PSPACE-complete*. IJCAI 2011, pp.774–780.