Universitat de Barcelona (UB)
Facultat de Matemàtiques
Departament de Probabilitat, Lògica i Estadstica
Programa de doctorat en Lògica Pura i Aplicada

PhD Thesis

Fuzzy Description Logics from a Mathematical Fuzzy Logic point of view

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Consejo Superior de Investigaciones Cientificas

OCTOBER 2012

Ai miei genitori Renato ed Emanuela,
a mio fratello Marcello,
a mia sorella Marina.

Aknowledgments

L'argomento di questa tesi dottorale appartiene alla matematica e all'intelligenza artificiale, ma chi l'ha scritta é un essere umano che non avrebbe potuto portare a termine la realizzazione della stessa senza il supporto di molta gente e istituzioni. Prima di tutto voglio ricordare che senza l'educazione e il supporto che ho sempre ricevuto dalla mia famiglia non sarei neppure arrivato a poter iniziare gli studi che mi hanno portato a questo punto. Anche se fisicamente non sono stati vicini in questi quattro anni, in ogni caso senza mio padre Renato, mia madre Emanuela, mio fratello Marcello e mia sorella Marina non sarei la persona che ha potuto realizzare questa tesi.

La realitzaciò d'aquesta tesi tampoc huria estat possible sense el suport dels meus directors de tesi. Primer de tot el meu agraïment va per en Francesc Esteva, qui, no només va creure en mi quan ni tan sols jo hi creia, si no que també ha tingut la paciència de soportar-me i ensenyar-me durant aquests quatre anys. Sense negar l'alt valor cientific e intel·lectual d'aquest home, ha sigut un plaer compartir aquests quatre anys d'estudis amb una persona que també té un alt valor humà. També ha sigut un plaer i un honor treballar amb en Fèlix Bou, ja que no tothom pot dir que ha treballat amb un geni. Finalment en Lluis Godo qui, encara que no hagi treballat molt amb ell, sempre ha estat un punt de referència per a mi.

A part del suport intel·lectual rebut durant aquests anys, hauria estat impossible dur a terme aquesta dura tasca, sense un bon suport moral. Primer de tot aquest suport l'he trobat en el lloc de treball. Em considero una persona molt afortunada per haver pogut passar aquests quatre anys a l'Institut d'Investigaciò en Intel·ligencia Artificial de Bellaterra, perquè, sense negar l'altíssim nivel científic de les persones que hi treballen, també es un lloc d'altíssim nivel humà en el que he passat moments inoblidables. El meu agraïment va per a tothom, pero de forma especial he de recordar a Marc Pujol, Pere Pardo, Andrew Koster, Ferran Ollé, Jesus Cerquides, Juan Antonio Rodríguez, Manuel Atencia, Meritxell Vinyals, Marc Esteva, Norman Salazar, Tito Cruz, Daniel Villatoro i Daniel Polak.

A part de la sort de trobar un entorn de treball molt favorable, durant aquests anys, el suport m'acompnyava a la tornada cada dia cap a Barcelona encara que, de vegades, es traduïa mès en un impediment al estar treballant. Aquì he tingut la sort i el plaer de compartir molts moments inoblidables amb persones genials que em varen acompanyar també en els moments en que ho passava malament. Entre aquestes persones destacan Michele Lamartina, Marc Sallent, Daniel Palao i Enric Català.

También quiero recordar que este trabajo ha sido posible gracias a la financiación de varias instituciones bajo forma de becas i proyectos. En primer lugar quiero recordar la beca predoctoral JAEPredoc, n.074 del CSIC, gracias a la cual he podido trabajar durante estos cuatro años. También quiero agradecer el proyecto I143-G15 (LogICCC-LoMoReVI) de E.U. Eurocores-ESF/FWF, el proyecto 2009-SGR-1434 de la Generalitat de Catalunya y el proyecto ARINF TIN2009-14704-C03-03 del Gobierno Español. Quiero aprovechar aquí la ocasión para recordar que sin financiación no hay ciencia y sin ciencia el mundo podría volver a ser plano como antes.

Marco Cerami,
Barcelona,
6 de Setembre de 2012

Abstract

Description Logic is a formalism that is widely used in the framework of Knowledge Representation and Reasoning in Artificial Intelligence. They are based on Classical Logic in order to guarantee the correctness of the inferences on the required reasoning tasks. It is indeed a fragment of First Order Predicate Logic whose language is strictly related to the one of Modal Logic. Fuzzy Description Logic is the generalization of the classical Description Logic framework thought for reasoning with vague concepts that often arise in practical applications.

Fuzzy Description Logic has been investigated since the last decade of the 20^{th} century. During the first fifteen years of investigation their semantics has been based on Fuzzy Set Theory. A semantics based on Fuzzy Set Theory, however, has been shown to have some counter-intuitive behavior, due to the fact that the truth function for the implication used is not the residuum of the truth function for the conjunction. In the meanwhile, Fuzzy Logic has been given a formal framework based on Many-valued Logic. This framework, called Mathematical Fuzzy Logic, has been proposed has the kernel of a well mathematically founded Fuzzy Logic.

In this dissertation we propose a Fuzzy Description Logic whose semantics is based on Mathematical Fuzzy Logic as its mathematically well settled kernel. To this end we provide a novel notation that is strictly related to the notation that is used in Mathematical Fuzzy Logic. After having settled the notation, we investigate the hierarchies of description languages over different t-norm based semantics and the reductions that can be performed between reasoning tasks. The new framework that we establish gives us the possibility to systematically investigate the relation of Fuzzy Description Logic to Fuzzy First Order Logic and Fuzzy Modal Logic. Next we provide some (un)decidability results for the case of infinite t-norm based semantics with or without knowledge bases. Finally we investigate the complexity bounds of reasoning tasks without knowledge bases for basic Fuzzy Description Logics over finite t-norms.

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Preface

Fuzzy Description Logic (FDL) is a topic that has been studied since the beginning of the last decade of the 20^{th} century. In its earlier papers FDLs were defined as in the classical DLs but substituting the semantic of concepts and roles, that are crisp sets and crisp relations in DLs, by fuzzy sets and fuzzy relations valued on the real unit interval. Notice that during the first period of research on FDLs, no formal fuzzy logic was defined, simply some operations on the real unit interval were used to define functional operations on fuzzy sets (see, for instance [ATV83]). Nevertheless the first papers on FDLs use as truth functions the so-called Zadeh's operations on $[0, 1]$, that are $\min, \max$ and the negation $n(x) = 1 - x$ and the so called S-implication (also called Kleene-Dienes implication in many fuzzy papers), defined by $x \to y = \neg x \vee y$. From the seminal papers [Yen91] and [TM98], several papers on this topic, [Str98], [Str01], [SSP$^+$05a], [Str05b] [BS07] among others, have been published. They define quite rich languages (very similar to the ones studied and used in the classical DLs), study them and provide satisfiability algorithms.

In those first works, *Fuzzy Logic* is usually understood in the sense of Zadeh's fuzzy logic, both in the *wide sense* and in the *narrow sense*. We quote:

"In a narrow sense, fuzzy logic, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic. However, the agenda of FLn is quite different from that of traditional multivalued logics. In particular, such key concepts in FLn as the concept of a linguistic variable, canonical form, fuzzy if-then rule, fuzzy quantification and defuzzification, predicate modification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. In its wide sense, fuzzy logic, FLw, is fuzzily synonymous with the fuzzy set theory, FST, which is the theory of classes with unsharp boundaries. FST is much broader than FLn and includes the latter as one of its branches." ([Zad94])

Fuzzy logic underlying early works on FDL was understood in Zadeh's both narrow and wide sense. In a narrow sense because it was distinct from multivalued logic, both in its formalization and in its agenda. In fact, among the key concepts proposed in Zadeh's agenda of fuzzy logic, there are linguistic modifiers and fuzzy quantifiers that are of particular interest from the FDLs point of view. In a wide sense because, since the beginning, a semantics based on fuzzy set theory has been proposed for fuzzy concepts and roles.

In the last years of the 20^{th} century a group of researchers leaded by P. Hájek defined a formal framework called *Mathematical Fuzzy Logic* (MFL), based on residuated many-valued logics, unifying different research streams on fuzzy and many-valued logic. Taking into account some counter-intuitive behavior of the Kleene-Dienes implication (see page 3 for an explanation of this fact and Section 2.8 for the same argument in the case of FDL), P. Hájek proposes a multiple-valued logic having a t-norm as semantics for conjunction and its residuum as semantics for implication. As a motivation for this new framework, Hájek considered that despite the agenda of many-valued logic was narrower than that of fuzzy logic in the sense of Zadeh, it could be considered as a mathematically rigorous kernel around which building a fuzzy logic with a wider agenda. Moreover, that agenda can take great advantage from having a mathematically well-settled base. We quote:

"[...] even if I agree with Zadeh's distinction between many-valued logic and fuzzy logic in the narrow sense, I consider formal calculi of many-valued logic (including non-"traditional" ones, of course) to be the kernel or base of fuzzy logic in the narrow sense and the task of explaining things Zadeh mentions by means of these calculi to be a very promising task (not yet finished)." ([Háj98c])

In fact half of his book [Háj98c] deals with many-valued residuated systems (propositional and first order) and its properties and formal study, but the other half part deals with the task of explaining things Zadeh mention by means of these calculi, a task that is far from being finished. Nowadays Mathematical Fuzzy Logic is a well established field with quite nice and interesting results (See [CHN11] for a state of the art on the topic).

Mathematical Fuzzy logic, being a well formalized residuated many-valued system, allows us to define FDLs from first order fuzzy logics using the same relations existing between classical Logic and classical DLs. This is the proposal of P. Hájek in [Háj05]. In fact, this new way of seeing fuzzy logic allowed to think of a Fuzzy Description Logic based on many-valued logic as the logical and mathematical background behind it. The first step has been done in [Háj05], where this new framework for FDL has been proposed. This idea has been followed in [GCAE10], where FDLs based on different continuous t-norms and their residua have been considered. Since then, some papers on FDL based on a t-norm have been published and some important results on (un)decidability of FDL languages based on t-norms have been obtained.

From what we have said it becomes clear that we consider FDLs based on Mathematical Fuzzy Logic[1] (the only ones studied in this dissertation) as the kernel of the real FDLs in the same sense than MFL is considered the kernel of Fuzzy Logic by Hájek in the introduction of his book (the text quoted above). As a consequence, the agenda of FDLs has two different goals. On the one hand

[1]What some researchers like to call Many-valued Description Logics.

it is important to follow the study of the FDLs based on residuated many-valued logics in the sense proposed in this dissertation and to explore applications. On the other hand, it will be very interesting to incorporate new notions like the ones of fuzzy modifiers and fuzzy quantifiers, as well as to deal with uncertainty in this framework taking into account the work done in these subjects[2]. The aim of this dissertation is to settle down the theoretical foundations of the program, sketched by Hájek, of an FDL based on MFL. In this sense we do not strictly follow the tradition on FDL previous to [Háj05] and this fact becomes clear from the notation we propose for concept constructors, that is thought as a compromise between the notation used in classical DL and the one used in MFL.

0.1 Overview and structure of the dissertation

This dissertation is structured in five chapters and two appendixes:

Chapter 1 In the first chapter we introduce the preliminaries of the work presented in this dissertation and it has two parts. The first part is devoted to introduce the framework of Mathematical Fuzzy Logic. We begin with a little bit of history about the process that brought to the general framework of MFL from the fields of many-valued logics and fuzzy set theory. Then we formally define what fuzzy propositional logic, fuzzy modal logic and fuzzy first order logic are. Our definitions of these formalisms are made from a semantical point of view. This is due to the fact that the semantic approach is more suited to be then used in FDL. Finally, we mention some properties, for each of the three formalisms, that are useful in the overall development of the dissertation. The second part of this chapter is devoted to introduce classical Description Logic. Also in this case we report a little bit of history. Subsequently we define the formalism as is usually done in the literature and report those results on complexity that are generalized in the dissertation to the many-valued case.

Chapter 2 In the second chapter we introduce our proposal of Fuzzy Description Logic. The syntax of this proposal is based, as usual, on concept constructors. Nevertheless, the reader aware of the literature in FDL will find the syntactical notation for concept constructors rather non-standard. This is due to the fact that, in order to generalize the framework of DL to the fuzzy case, novel symbols are needed. Throughout the chapter the reasons of these changes with respect to the literature are deeply explained. Then a t-norm based semantics is introduced. Once introduced the semantics, various kinds of consequences are explained. Among them let us point out: (i) the changes in the hierarchy of inclusions between languages, (ii) the simplifications that can be performed on the kinds of axioms that have been introduced in the literature, (iii) the reduction between reasoning tasks that can be considered in a many-valued framework. In

[2]See for example [DRSV] for a state of the art about evaluation of fuzzy quantifiers, [H0́1] and [EGN11] for a treatment of fuzzy modifiers (hedges) in the setting of MFL and finally [LS08] for a proposal to cope with uncertainty in FDLs.

this chapter relations to other formalisms are also considered. The formalisms considered are fuzzy first order logic and fuzzy multi-modal logic. For both a translation from FDL concepts to formulas is provided. The translation is then proved to be meaningful through a corresponding translation between the respective semantics. In the case of multi modal logic a translation from formulas to FDL concepts is provided as well. In the last section on related work, besides a brief (and rather non-exhaustive) history of FDL there are also deeply motivated the practical reasons for which we prefer a residuated t-norm based semantics, than the semantics that was adopted before [Háj05].

Chapter 3 The third chapter deals with decidability issues. First, a definition of the reduction to fuzzy propositional logic used in [Háj05] to prove decidability of concept witnessed satisfiability for language $\mathcal{ALC}$ is reported. We report this reduction in this chapter because it is analyzed under several points of view throughout the dissertation. Second, the decidability of concept quasi-witnessed satisfiability with respect to empty knowledge bases for language $\mathfrak{I}\mathcal{ALE}$ over infinite-valued product semantics is proved. The proof uses a generalization to quasi-witnessed interpretations of the previous defined reduction to fuzzy propositional logic with respect to witnessed interpretations. Third, the undecidability of knowledge base satisfiability with respect to general knowledge bases for language $\mathcal{ALC}$ over infinite-valued Łukasiewicz semantics is proved. The proof uses a recursive reduction to Post Correspondence Problem that was inspired by [BP11f]. In the last section the related work on decidability of FDLs over residuated semantics is reported.

Chapter 4 The fourth chapter deals with computational complexity issues. First, it is showed that the reduction from [Háj05] reported at the beginning of Chapter 4 is not polynomial. Second, the PSPACE-completeness of the satisfiability and validity problems of formulas in the minimal modal logic of Kripke frames over finite Łukasiewicz chains is proved. The upper bound argument uses a generalization of a procedure based on Hintikka sets. The procedure that has been generalized is used in [BdRV01] to prove a PSPACE upper bound for the same problem in classical Modal Logic. The lower bound argument is just sketched, since it is the same proof provided in [Lad77]. Third, the PSPACE-completeness of the concept satisfiability problem in language $\mathfrak{I}\mathcal{ALCED}^T$ language over a finite MTL-chain is proved. This time the upper bound argument uses a PSPACE procedure based on the reduction reported at the beginning of Chapter 4. The lower bound argument is the same as for the minimal modal logic of Kripke frames over finite Łukasiewicz chains and so the details are not explicitly given. In the last section the related work on computational complexity of FDLs over residuated semantics is reported.

Chapter 5 In the fifth chapter we make an overview on the contributions of this dissertation, resume the problems that have been left open and provide some future research lines.

Appendix A The results of this appendix, even though they are on first order product logic with standard semantics, are crucial for the result about decidability of some FDLs over product logic given in the first part of Chapter 4. In this appendix it is proved that tautologies and positive satisfiable formulas of first order Product logic coincide with tautologies and positive satisfiable formulas over the one-generated subalgebra and thus they coincide with tautologies and positive satisfiable formulas with respect to quasi-witnessed models. Finally a remark explains that the problem of whether the formulas that are 1-satisfiable for all models over $[0, 1]_\Pi$ and those that are 1-satisfiable over quasi-witnessed models coincide is an open problem.

Appendix B This appendix contains the paper *Strict core fuzzy logics and quasi-witnessed models* by M. Cerami and F. Esteva published in Archive of Mathematical Logic [CE11]. This paper deals with the relation between first order strict core fuzzy logics and quasi-witnessed models with general semantics. It provides two axioms that added to the axiomatic definition of any strict core fuzzy logic define a logic complete with respect to quasi-witnessed models. Finally it is proved that the only first order fuzzy logics of a continuous t-norm where these new axioms are already provable are Łukasiewicz and Product logics. The paper deals with general semantics and thus, if the reader is only interested on FDLs (related to standard semantics), this appendix can be skipped. However, we have decided to include it since we think its content could help the interested reader to better understand quasi-witnessed models. Notice that we report the contents of the published version despite the notation does not coincide with the one used in this dissertation, nevertheless the notation is explained in the same appendix.

0.2 Publications related to the dissertation

The redaction of this dissertation is based, although not only, on the work developed in the following publications of the author. Note that the list of publications follows the order of the topics as developed in this dissertation, rather than a chronologic order.

1. M. Cerami, A. García-Cerdaña, F. Esteva. *From classical description logic to n-graded fuzzy description logics.* In P. Sobrevilla, editor, *Proceedings of the FUZZ-IEEE 2010 Conference*, pages 1506–1513, Barcelona, 2010.

   In this publication a preliminary investigation on the task of generalizing classical $\mathcal{ALC}$ to finite-valued $\mathfrak{I}\mathcal{ALCE}$ is provided. A novel notation for concept constructors and a new hierarchy of fuzzy languages, adapted to the framework of FDL are introduced. It is also began the study of the relations to fuzzy first order logics. The results of this publication are generalized in Chapter 3.

2. M. Cerami, F. Esteva, A. Garcìa–Cerdaña. *On finitely valued fuzzy description logics: The Łukasiewicz case.* In *Proceedings of the 14th Interna-*

tional Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2012), pages 235–244 Catania, 2012.

In this publication some improvements of the case of $Ł_n$-$\mathcal{ALC}$ with respect to the general finite-valued case are studied. It is also deepened the study of the relations to finite-valued First Order Łukasiewicz Logic. Moreover, it is begun the study of the relations of $Ł_n$-$\mathcal{ALC}$ to many-valued Łukasiewicz Modal Logic. The results of this publication are generalized in Chapter 3.

3. M. Cerami, F. Esteva, F. Bou. *Decidability of a description logic over infinite-valued product logic.* In Lin, F., Sattler, U., and Truszczynsky, M., editors, *Proceedings of the Twelfth International Conference on Principles of Knowledge Representation and Reasoning* (KR 2010), pages 203–213. AAAI Press. Toronto, 2010.

   In this publication the decidability of the concept positive satisfiability and 1-subsumption problems for language $[0,1]_\Pi$-$\mathfrak{I}\mathcal{ALE}$ is proved. In order to achieve such result, an algorithm that reduces the positive satisfiability problem for language $[0,1]_\Pi$-$\mathfrak{I}\mathcal{ALE}$ to the semantic consequence problem in Propositional Product Logic is provided. The details of this work are given in Section 3.2 and in Appendix A.

4. M. Cerami, U. Straccia. *On the undecidability of fuzzy description logics with GCIs with Łukasiewicz t-norm.* Technical report, Computing Research Repository. Available as CoRR technical report at http://arxiv.org/abs/1107.4212.

   In this manuscript the undecidability of knowledge consistency for language $[0,1]_Ł$-$\mathcal{ALC}$ is proved. In order to achieve this result, a recursive reduction from the reverse of the Post Correspondence Problem to knowledge base consistency is provided. The details of this work are given in Section 3.4. This preprint has been submitted for publication and a preliminary version has been presented at the WL4AI workshop, embedded in the 20^{th} European Conference on Artificial Intelligence (ECAI 2012), Montpellier, August 27-31, 2012.

5. F. Bou, M. Cerami, F. Esteva. *Finite-valued Łukasiewicz modal logic is PSPACE-complete.* In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence* (IJCAI 2011), pages 774–779. Barcelona, 2011.

   In this publication the PSPACE-completeness for the satisfiability and validity problems for the logic of all Kripke $Ł_n$-frames is proved. In order to achieve this result, a generalization to the n-valued Łukasiewicz case, of a procedure based on Hintikka sets is provided. The details of this work are given in Section 4.2.

6. F. Bou, M. Cerami, F. Esteva. *Concept satisfiability in finite-valued fuzzy description logics is PSPACE-complete* (extended abstract). In *Proceed-*

ings of the Conference on Logic, Algebras and Truth Degrees 2012 (LATD 2012). Kanazawa, 2012.

In this extended abstract a PSPACE procedure for concept satisfiability for $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCE}$, where $\mathbf{T}$ is a finite MTL-chain, is proposed, but no proof is given in this abstract. The same procedure is described in Section 4.3 of this dissertation where a full proof of PSPACE-completeness for $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCE}$ is provided.

7. M. Cerami, F. Esteva. *First order SMTL logic and quasi-witnessed models.* In *Proceedings of the XV Congreso Español de Tecnología y Lógica Fuzzy (ESTYLF 2010)*, pages 145–150, Huelva, 2010.

   In this publication the relation between extensions of SMTL first order logic and quasi-witnessed models is studied. In particular, a couple of axioms are proposed that identify the axiomatic extensions of an SMTL logic that are complete with respect to quasi-witnessed models. On the contents of this publication rely the results on quasi-witnessed models given in Section 1.1.3.

8. M. Cerami, F. Esteva. *Strict core fuzzy logics and quasi-witnessed models. Archive for Mathematical Logic*, 50(5-6):625–641.

   This publication is an improvement of the results in *First order SMTL logic and quasi-witnessed models.* The results are generalized to the case of Strict-core fuzzy logics (that are here defined). On the contents this publication rely the results on quasi-witnessed models given in Section 1.1.3.

Chapter 1

Preliminaries

In this dissertation we apply results of Mathematical Fuzzy Logic (MFL) in order to generalize the framework of Classical Description Logics (DL) to the fuzzy and many-valued cases. This chapter on preliminaries contains basic results on MFL and DL that will be needed for the overall development and understanding of the work presented in the rest of the chapters of this document. In Section 1.1 we provide an overview on Mathematical Fuzzy Logic and, in Section 1.2, an overview on Classical Description Logics.

1.1 Mathematical Fuzzy Logic

What we call *Mathematical Fuzzy Logic* (MFL) is a recent paradigm that aims at treating vague reasoning by means of formally defined many-valued and fuzzy logic systems. This nowadays paradigm is the result of a process that began in ancient times with the discovery of fuzzy predicates.

The suspicion that vague sentences and predicates can lead to an unusual behavior of the reasoning process has been already present since the IV century b.C. when, according to the tradition, the greek philosopher Eubulides from Mileto proposed what is known as the *sorites paradox* (or, in modern english, the *heap paradox*). There exist several formulations of such paradox, we report here a version that is quite close to the original one:

10.000 sand grains are a heap.

If we take a sand grain away from a sand heap,
the result keeps being a heap.

---

0 sand grain is a heap.

At first sight it can seem a linguistic trick. Nevertheless, this paradox can be easily formalized. If we consider, for every natural n such that $0 \leq n \leq 10.000$, the proposition:

$$p_n = \text{“}n \text{ sand grains are a heap”},$$

we can formalize the paradox in the following form:

$$\begin{array}{l} p_{10.000} \\ p_{10.000} \to p_{9.999} \\ p_{9.999} \to p_{9.998} \\ \vdots \\ p_1 \to p_0 \\ \hline p_0 \end{array}$$

As we can easily see, when we try to express it formally, if we remain in the classical framework, it keeps being a paradox. As happened with other ancient paradoxes, also for sorites paradox one had to wait a long time before a solution could be found and this solution needed a widening of the classical two-valued framework.[1]

In modern times, the first who thought in terms of three-valued logic has been C. S. Peirce in a manuscript of 1909, but he did not make it public. The birth of many-valued logics is, indeed, attributed to J. Łukasiewicz, who, in 1920 starting from philosophical considerations about the problem of *contingent future events*, defined the first three-valued logic and published it in [Łuk20]. Subsequently and jointly with A. Tarski, in [ŁT30] he defined a logic whose propositions are valued in the real unit interval $[0, 1]$. A couple of years later, in order to prove that Intuitionistic Propositional Logic is not complete with respect to any finite linear model, K. Gödel defined in [Göd32] a class of finite linear algebras one for every finite cardinality. Considering the class of algebras defined by Gödel, M. Dummet in [Dum59] defined axiomatically a logical calculus and proved its completeness with respect to that class of semantics. Another relevant many-valued logic has been introduced 1996 to be defined. In that year, in fact, P. Hájek, L. Godo and F. Esteva proposed in [HGE96] an axiomatic system where the truth function for the conjunction is the product between real numbers in the real unit interval $[0, 1]$ and called it Product Logic.

Beyond the context of many-valued logics, L. A. Zadeh defined, in [Zad65], the notion of *fuzzy set*. Zadeh's definition of fuzzy set is based on a generalization of the range of the set characteristic function to the real unit interval $[0, 1]$. Set operations are also generalized to the operations of $\min\{x, y\}$, $\max\{x, y\}$ and $1-x$ for intersection, union and complementation respectively, where $x, y \in [0, 1]$ are the images of the generalized characteristic functions of two different sets over the same object element. Following the intuition, the subset relation has been defined as true between two fuzzy sets when the instantiation of the subset by means of every domain element is less or equal than the instantiation of the superset by means of the same element.

[1] See, for example, the solution given in [Gog69] or the one stated in Section 1.1.4.

In the logical system behind Zadeh's fuzzy set theory the truth function chosen for the implication operator was the so-called *Kleene-Dienes implication*. This operation is a straightforward generalization of the semantics of the classical material implication, i.e.

$$x \Rightarrow y \quad := \quad \max\{1 - x, y\}, \tag{1.1}$$

where $x, y \in [0, 1]$ are thought as the truth values of the antecedent and the consequent of the implication respectively. Nevertheless, this semantics for the implication gives rise to the lack of the classical correspondence between implication and subset/superset relationship. Indeed, given two fuzzy sets C and D, with such semantics, the fact that $C \subseteq D$ is no more equivalent to the fact that, for every object x belonging to the domain, $C(x) \Rightarrow D(x) = 1$, as it would be desirable. A way to overcome this shortcoming is, as it began to become clear during the 80's (see, for example [TV85]), the use of the residuum of a class of operations over the real unit interval [0,1] coming from the theory of probabilistic metric spaces, called *t-norms*. The use a t-norm, as a semantics for the conjunction, and its residuum, as a semantics for the implication, in fact, solves the above mentioned problem and provides a mathematically well-founded background for fuzzy set theory.

At the end of the 90's P. Hájek, considering all these results, in [Háj98c], defined what nowadays is known as Mathematical Fuzzy Logic. Thanks to this new framework, great advances on the subject have been done until the present day. The interested reader can find in [CHN11] an exhaustive survey on the subject.

In the rest of this Section we will provide the definitions of fuzzy logics under the propositional, modal and first-order points of view as well as the results that we need in order to easily develop the central subject of this dissertation.

Despite the fact that sometimes logics are defined by means of a set of axioms and inference rules, here we will not follow this pattern. We are, indeed, rather interested in the semantical definition of a logic in terms of validity and logical consequence that, in the case of the logics here considered, can be either a variety or a single algebra.

1.1.1 Propositional logic

In this section we introduce the fuzzy propositional logics we are going to use from their underlying semantics. Even though propositional logics are not a central matter in this dissertation it is important to introduce them for three reasons. The first is that the language of multi-modal logic, which we will introduce in Section 1.1.2 and which is a notational variant of $\mathcal{ALC}$-like languages, is defined as an expansion of the language of propositional logic. The second reason is that the algebraic semantics of the fuzzy propositional logics here considered, called *MTL-chains* and specially the standard ones (defined on the real unit interval), are the algebras of truth values of the FDLs considered in our framework and, therefore, they deserve special attention in order to understand the central part of this dissertation. The third reason is that many decidability and complexity results presented in this dissertation are based on a reduction to propositional logic.

Language

A *propositional language* $\mathbf{l} = \{\star_1, \dots \star_i\}$ is a finite set of *propositional connectives*. The *arity* $\mathbf{a}(\star) \in \mathbb{N}$ of the propositional connective $\star$ is the number of formulas that $\star$ takes as arguments. The *type* $\mathbf{t}(\mathbf{l}) \in \mathbb{N}^{|\mathbf{l}|}$ of language $\mathbf{l}$ is the tuple given by the arities of the propositional connectives in $\mathbf{l}$.

Given a language $\mathbf{l} = \{\star_1, \dots \star_i\}$ and a denumerable set of *propositional variables* $At = \{p_1, p_2, \dots\}$, the set of $\mathbf{l}$-*formulas*, denoted by $Fm_{\mathbf{l}}$ is built inductively in the following way:

1. each propositional variable is a formula,
2. each 0-ary connective is a formula,
3. if $\varphi_1, \dots, \varphi_j$ are formulas and $\star \in \mathbf{l}$ is a j-ary connective, then $\star(\varphi_1, \dots, \varphi_j)$ is a formula.

Notice that, since in our context we will make use of propositional connectives that are at most binary, we will, in general, use the notation $\varphi \star \psi$, instead of the above prenex notation, in which the same formula should be denoted by $\star(\varphi, \psi)$.

Given two formulas φ, ψ and a propositional variable p (not necessarily occurring in φ) the *substitution* of p by ψ in φ, denoted by $\varphi[\psi/p]$ is the formula obtained by replacing every occurrence of p in φ by the formula ψ.

Throughout this dissertation we will use the propositional language that contains the symbols $\otimes$, $\oplus$, $\wedge$, $\vee$, $\rightarrow$, $\bot$ and $\top$ standing for the connectives of *strong conjunction*, *strong disjunction*, *weak conjunction*, *weak disjunction*, *implication*, *bottom constant* and *top constant* respectively.

Semantics

Within the framework of propositional fuzzy logic two kinds of semantics have been considered, namely the *general* and the *standard* semantics. Considering the general semantics means working with an equational class of algebras, while considering the standard semantics means working either with a chain (or a class of isomorphic algebras) or with a class of chains whose domain is the real unit interval $[0, 1]$.

In this section we introduce the classes of algebras that we need in order to define the logics we work with and to report some interesting results. As we will see later on, the structures that we are going to define are fundamental for the understanding of the present dissertation because they turn out to be the algebras of truth values in which description concepts will take their values.

General semantics: the variety of MTL-algebras and its subvarieties An *algebra* $\mathbf{T}$ is a structure composed by a nonempty set T, called the *domain* or *universe* of $\mathbf{T}$ and a set of operations $\mathbf{l_T}$ called (as in the case of propositional logic) language, such that, for every $a_1, \ldots, a_j \in T$ and each j-ary operation $\hat{\star} \in \mathbf{l_T}$, it holds that $\hat{\star}(a_1, \ldots, a_j) \in T$. Equational classes of algebras $\mathbb{K}$ are usually defined through a finite set of equations that are supposed to be true in every algebra belonging to $\mathbb{K}$ and only in them. Definitional equations are expressions of the form:

$$(\forall \vec{x})(\forall \vec{y})(t_1(\vec{x}) = t_2(\vec{y})) \tag{1.2}$$

where t_1 and t_2 are terms. Following the tradition in MFL, we will use the expression:

$$t_1(\vec{x}) \approx t_2(\vec{x})$$

in order to abbreviate expression (1.2). Being all these classes of algebras equationally defined, each one form a *variety*.

Equational classes $\mathbb{K}$ satisfy that, for every two algebras $\mathbf{T}, \mathbf{T}' \in \mathbb{K}$, it holds that $\mathbf{t}(\mathbf{l_T}) = \mathbf{t}(\mathbf{l_{T'}})$. So, it makes sense to speak about the type $\mathbf{t}_\mathbb{K}$ of a variety $\mathbb{K}$.

A *lattice* is an algebra[2] $\mathbf{T} = \langle T, \vee, \wedge \rangle$ with two binary operations $\vee$ and $\wedge$, called *join* and *meet*, which satisfies the following equations:

$(E1)$	$x \vee y \approx y \vee x$	
$(E2)$	$x \wedge y \approx y \wedge x$	(commutativity)
$(E3)$	$x \vee (y \vee z) \approx (x \vee y) \vee z$	
$(E4)$	$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$	(associativity)
$(E5)$	$x \vee x \approx x$	
$(E6)$	$x \wedge x \approx x$	(idempotence)
$(E7)$	$x \approx x \vee (x \wedge y)$	
$(E8)$	$x \approx x \wedge (x \vee y)$	(absorption)

In a lattice $\mathbf{T}$ an *order relation* can be defined between every two elements $a, b \in T$, in the following way:

$$a \leq b \iff a \wedge b = a \iff a \vee b = b.$$

For every subset $X \subseteq T$, we have that

- an *upper bound* of X is an element $a \in T$ such that, for every element $b \in X$, it holds that $b \leq a$,

[2] For further information about lattices, the interested reader can look at [BS81].

- a *lower bound* of X is an element $a \in T$ such that, for every element $b \in X$, it holds that $a \leq b$,
- the *least upper bound* or *supremum* of X (denoted $\sup(X)$) is an upper bound $a \in T$ of X such that $a \leq b$ for every upper bound b of X. If, moreover, $\sup(X) \in X$, we call it the *maximum* of X (denoted $\max(X)$),
- the *greatest lower bound* or *infimum* of X (denoted $\inf(X)$) is a lower bound $a \in T$ of X such that $a \geq b$ for every lower bound b of X. If, moreover, $\inf(X) \in X$, we call it the *minimum* of X (denoted $\min(X)$).

A lattice $\mathbf{T}$ is *bounded* if $\min(T)$ and $\max(T)$ always exist; it is *complete* if, for every subset $X \subseteq T$, $\inf(X)$ and $\sup(X)$ always exist. Clearly, if a lattice is complete, it is bounded as well.

A *monoid* is an algebra $\mathbf{T} = \langle T, *, 1\rangle$, where:

- $*$ is an associative binary operation,
- $1 \in T$ is the neutral element of operation $*$, in the sense that,

$$x * 1 \approx 1 * x \approx x.$$

A monoid $\mathbf{T}$ is commutative if the operation $*$ is so.

We say that an algebra $\mathbf{T} = \langle T, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a *bounded commutative integral residuated lattice* if:

- $\langle T, \wedge, \vee, 0, 1\rangle$ is a bounded lattice where $0 = \inf(T)$ and $1 = \sup(T)$,
- $\langle T, *, 1\rangle$ is a commutative monoid,
- there exists a unique binary operation $\Rightarrow$ satisfying for all $a, b, c \in T$ the following condition (called *residuation*):

$$a * b \leq c \iff a \leq b \Rightarrow c. \tag{1.3}$$

  The operation $\Rightarrow$ is called the *residuum* of the operation $*$ and it is defined as:

$$a \Rightarrow b \quad := \quad \max\{c \in T \colon a * c \leq b\}. \tag{1.4}$$

Further operations that can be defined are:

$$\neg a \quad := \quad a \Rightarrow 0 \tag{1.5}$$

$$a \Leftrightarrow b \quad := \quad (a \Rightarrow b) * (a \Rightarrow b) \tag{1.6}$$

$$a^n \quad := \quad \overbrace{a * \ldots * a}^{n \text{ times}} \tag{1.7}$$

called *residuated negation*, *biconditional* and *n-ary conjunction* respectively.

An *MTL-algebra* $\mathbf{T} = \langle T, \wedge, *, \Rightarrow, 0, 1\rangle$ is a bounded commutative integral residuated lattice which satisfies the equation:

(PL) $(x \Rightarrow y) \vee (y \Rightarrow x) \approx 1$ (pre-linearity)

Notice that, in residuated lattices where equation (PL) is valid, the operation $\vee$ does not need to be present in the algebraic language, since it is definable as:

$$a \vee b \quad := \quad ((a \Rightarrow b) \Rightarrow b) \wedge ((b \Rightarrow a) \Rightarrow a) \tag{1.8}$$

so, in what follows, we will omit it from the languages of the algebras where it is a definable operation.

An *IMTL-algebra* $\mathbf{T} = \langle T, \wedge, *, \Rightarrow, 0, 1\rangle$ is a MTL-algebra which satisfies the equation:

(Inv) $x \approx (x \Rightarrow 0) \Rightarrow 0$ (involutive negation)

An *SMTL-algebra* $\mathbf{T} = \langle T, \wedge, *, \Rightarrow, 0, 1\rangle$ is a MTL-algebra which satisfies the equation:

(S) $x \wedge (x \Rightarrow 0) \approx 0$ (strictness)

A Π*MTL-algebra* $\mathbf{T} = \langle T, \wedge, *, \Rightarrow, 0, 1\rangle$ is an SMTL-algebra which satisfies the equation:

(Π) $((z \Rightarrow 0) \Rightarrow 0) \Rightarrow (((x * z) \Rightarrow (y * z)) \Rightarrow (x \Rightarrow y)) \approx 1$ (simplification)
or, equivalently, an MTL-algebra that satisfies (Π) and (S).

A *BL-algebra* $\mathbf{T} = \langle T, \wedge, *, \Rightarrow, 0, 1\rangle$ is an MTL-algebra which satisfies the equation:

(D) $x \wedge y \approx x * (x \Rightarrow y)$ (divisibility)

Notice that, in presence of divisibility, $\wedge$ becomes a definable operation. In subvarieties of $\mathbb{BL}$, it can, in fact be defined by the divisibility equation and, therefore, we will omit it from the algebraic language of subvarieties of $\mathbb{BL}$.

A Π-*algebra* $\mathbf{T} = \langle T, *, \Rightarrow, 0, 1\rangle$ is an SMTL-algebra which satisfies the equations (D) and (Π).

A *Gödel-algebra* $\mathbf{T} = \langle T, *, \Rightarrow, 0, 1\rangle$ is a BL-algebra (or an MTL algebra) which satisfies the equation:

(Id) $x \approx x * x$ (idempotence)

An *MV-algebra* $\mathbf{T} = \langle T, *, \Rightarrow, 0, 1\rangle$ is a BL-algebra which satisfies (Inv) or, equivalently, an IMTL-algebra which satisfies (D).

If any of these algebras is linearly ordered, we say that it is a *chain*. If the domain T is a finite set, we will speak about *finite algebras* and *finite chains*.

In Figure 1.1 we show the lattice of inclusions between these varieties, where a variety $\mathbb{A}$ is included in $\mathbb{A}'$ if they are connected by a sequence of upward

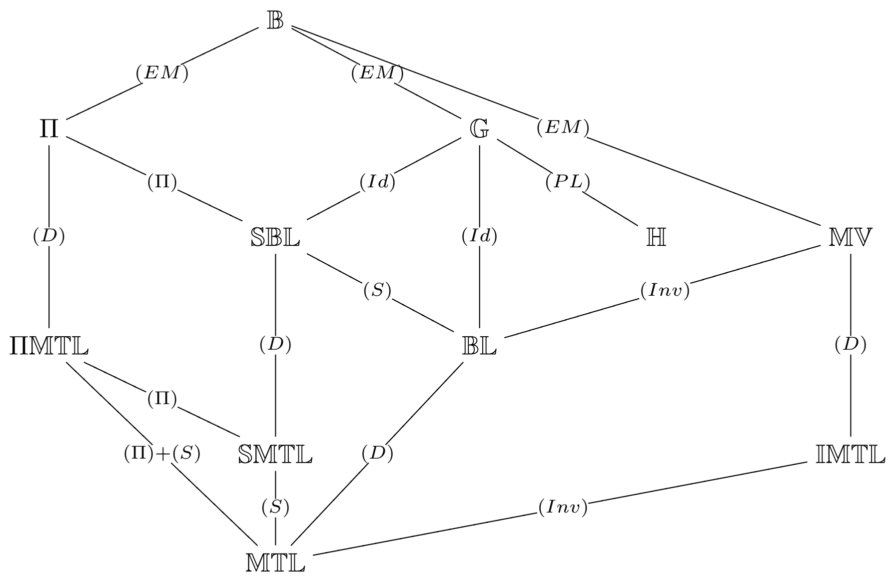

Figure 1.1: The hierarchy of subvarieties of $\mathbb{MTL}$

edges. Note that in Figure 1.1 $\mathbb{B}$ and $\mathbb{H}$ denote the variety of *Boolean algebras* and *Heyting algebras*, respectively and (EM) denotes the *excluded middle law*:

(EM) $\quad x \vee (x \Rightarrow 0) \approx 1 \quad$ (excluded middle)

Standard semantics Particular attention has been paid in the literature to the MTL-chains whose lattice reduct is $[0, 1]$ with the usual order. These chains, called "standard chains", are related to a special kind of operation called "*t*-norms".

Definition 1.1.1. A *t-norm* is a binary operation $*$ on the real unit interval $[0, 1]$ that is associative, commutative, non-decreasing in both arguments and having 1 as neutral (unit) element.

Left continuity of $*$, i.e.

$$a * \bigvee X = \bigvee_{b \in X} \{a * b\}$$

is a sufficient and necessary condition for the existence of the residuum of the *t*-norm $*$. Using this residuum, the following result characterizes standard chains.

Proposition 1.1.2. *A structure $\langle [0, 1], \wedge, *, \Rightarrow, 0, 1 \rangle$ is a standard MTL-chain if and only if $*$ is a left-continuous t-norm and $\Rightarrow$ is its residuum. This structure* *will be denoted from now on by* $[0,1]_*$. *Moreover a standard chain satisfies divisibility (hence it is a BL-chain) if and only if the* t*-norm is continuous.*

The most used representative of the standard chains (unique up to isomorphisms), are the ones defined by the so-called Łukasiewicz, product and minimum t-norms and their residua (collected in Table 1.1).

$\hat{\star}$	Minimum (Gödel)	Product (of real numbers)	Łukasiewicz
$x * y$	$\min(x,y)$	$x \cdot y$	$\max(0, x+y-1)$
$x \Rightarrow y$	$\begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise} \end{cases}$	$\min(1, 1-x+y)$
$x \Rightarrow 0$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$1-x$

Table 1.1: The three main continuous t-norms.

In the case of Łukasiewicz and Gödel t-norms $*$, the operations in Table 1.1 can be defined on the domain of a finite subalgebra of $[0,1]_*$. In these cases, we can talk about *finite t-norms.* So, for every natural number n, $\mathbf{Ł_n}$ and $\mathbf{G_n}$ will denote the restriction of Łuksiewicz and Gödel t-norms, respectively, on the subalgebra of cardinality $n+1$ over the domain $T = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ in the $n+1$-valued case. Notice that, there not exist finite product subalgebras of $[0,1]_\Pi$ (of cardinality > 2), so a finite product t-norm can not be defined.

Ordinal sums Let I be a bounded countable set and, for every $i \in I$, let $*_i \colon [0,1]^2 \longrightarrow [0,1]$ be a (continuous) t-norm. Let $[l_i, u_i]$ be a family of non-empty, pairwise disjoint closed intervals of [0,1] and let $f_i \colon [0,1] \longrightarrow [l_i, u_i]$ be a family of bijections which respects operation $*_i$. Then the *ordinal sum* of the family $\{*_i \colon i \in I\}$ of t-norms is the operation $* \colon [0,1]^2 \longrightarrow [0,1]$ defined as

$$x * y = \begin{cases} f_i(f_i^{-1}(x) *_i f_i^{-1}(y)), & \text{if } x, y \in [l_i, u_i] \\ \min\{x, y\}, & \text{otherwise} \end{cases}$$

Once defined what an ordinal sum of t-norm is, we can report an important result about continuous t-norms from [Lin65]. It is the corresponding result for t-norms of the one that in [MS57] has been proved for families of abelian semi-groups.

Theorem 1.1.3. *Every continuous* t*-norm is an ordinal sum of isomorphic copies of Łukasiewicz, product and Gödel* t*-norms.*

This result has been generalized to BL-chains in [Háj98a]. We report an useful consequence of that result:

Theorem 1.1.4. *Every standard BL-chain is an ordinal sum of isomorphic copies of Łukasiewicz, product and Gödel chains.*

As a consequence we have the following corollary.

Corollary 1.1.5. *Every finite standard BL-chain is an ordinal sum of isomorphic copies of Łukasiewicz and Gödel finite chains.*

Defining logics from varieties Given an algebra $\mathbf{T}$ and a propositional language $\mathbf{l}$ such that $\mathbf{t}(\mathbf{l}) = \mathbf{t}(\mathbf{l_T})$, a *propositional* $\mathbf{T}$*-evaluation* (we will say *propositional evaluation*, when the algebra $\mathbf{T}$ is clear from the context) is a mapping $e\colon At \longrightarrow T$.

The propositional evaluation e can be inductively extended to the set of formulas $Fm_\mathbf{l}$ in such a way that for every pair of formulas $\varphi, \psi \in Fm_\mathbf{l}$, every logical connective $\star \in \mathbf{l}$ and its respective algebraic operation $\hat{\star} \in \mathbf{l_T}$, it holds that:

- $e(\bot) = 0$,
- $e(\top) = 1$,
- $e(\varphi \star \psi) = e(\varphi)\hat{\star}e(\psi)$

It is possible to define, over the set $Fm_\mathbf{l}$ of formulas on the language $\mathbf{l}$, the following notions based on the general semantics given by a class $\mathbb{K}$ of algebras. Given a set of formulas Γ and a formula φ, we say that:

- φ is 1-*satisfiable* if there exists an algebra $\mathbf{T} \in \mathbb{K}$ and a propositional $\mathbf{T}$-evaluation such that $e(\varphi) = 1$. In this case we say that $\mathbf{T}$ *satisfies* φ, in symbols $\mathbf{T}, e \models \varphi$.
- φ is *positively satisfiable* if there exists an algebra $\mathbf{T} \in \mathbb{K}$ and a propositional $\mathbf{T}$-evaluation such that $e(\varphi) > 0$. In this case we say that $\mathbf{T}$ *positively satisfies* φ, in symbols $\mathbf{T}, e \models^{pos} \varphi$.
- φ is a $\mathbb{K}$-*tautology* (denoted $\models_\mathbb{K} \varphi$) if, for every $\mathbf{T} \in \mathbb{K}$ and every propositional $\mathbf{T}$-evaluation, it holds that $e(\varphi) = 1$.
- φ is a *logical consequence* of Γ (denoted $\Gamma \models_\mathbb{K} \varphi$) if, for every $\mathbf{T} \in \mathbb{K}$ and every propositional $\mathbf{T}$-evaluation such that $e(\psi) = 1$, for every $\psi \in \Gamma$, it holds that $e(\varphi) = 1$.

Once defined these notions it makes sense the notion of *logic associated with a variety* $\mathbb{K}$.

Definition 1.1.6. Let $\mathbb{K}$ be a variety and $\mathbf{l}$ a propositional language such that $\mathbf{t}(\mathbf{l}) = \mathbf{t}(\mathbf{l}_\mathbb{K})$. We define the logic of $\mathbb{K}$ (called $\mathcal{L}(\mathbb{K})$) as the set of its tautologies.

Some basic results in the literature are the following:

- For every subclass $\mathbb{K}$ of $\mathbb{MTL}$, let $\mathbb{V}(\mathbb{K})$ be the minimal variety that contains $\mathbb{K}$, then the tautologies of $\mathcal{L}(\mathbb{K})$ are the same as the tautologies of $\mathcal{L}(\mathbb{V}(\mathbb{K}))$.
- For every subvariety $\mathbb{K}$ of $\mathbb{MTL}$, the logic $\mathcal{L}(\mathbb{K})$ is *chain complete*, that is:

$$\mathcal{L}(\mathbb{K}) = \mathcal{L}(\mathbb{KC})$$

  where $\mathbb{KC}$ is the class of $\mathbb{K}$-chains.

Defining logics from standard algebras Given a standard algebra $[0,1]_*$ it is possible to define, over the set $Fm_{\mathbf{l}}$ of formulas on the language $\mathbf{l}$, the notions of 1-satisfiability, positive satisfiability, tautologicity and logical consequence based on the standard semantics. Given a t-norm $*$, by setting $\mathbb{K} = \{[0,1]_*\}$ in the definitions at page 10, we obtain what is called the *logic of the t-norm* $*$, denoted $\mathcal{L}(*)$.

Note that it is well known that, for each continuous t-norm $*$, the logic of a t-norm $\mathcal{L}(*)$ enjoys the so-called *Finite strong standard completeness*. That is, if $\mathbb{V}([0,1]_*)$ is the minimal variety that contains $[0,1]_*$, then, for every finite set of formulas $\Gamma \cup \varphi \subseteq Fm_{\mathbf{l}}$, it holds that

$$\Gamma \models_{\mathbb{V}([0,1]_*)} \varphi \iff \Gamma \models_{[0,1]_*} \varphi$$

The same is not true for infinite sets of formulas Γ, except when $[0,1]_* = [0,1]_G$.

It is also well known (see [Háj98c]) that, if there is an evaluation e from the set of propositional formulas to either $[0,1]_G$ or $[0,1]_\Pi$ such that, for a formula φ, it holds that $e(\varphi) = r \in (0,1)$, then, for any $r' \in (0,1)$ it can be found an evaluation e' such that $e'(\varphi) = r'$. So, in what follows, we will speak only about 1-satisfiability and positive satisfiability in the case of Gödel and product standard chains.

If $\mathbf{T}$ is either $[0,1]_Ł$ or $Ł_n$ it is possible to define the notion of r-satisfiability, for $r \in \mathbf{T}$, as well. Indeed, φ is *r-satisfiable* if there exists a propositional $\mathbf{T}$-evaluation e such that $e(\varphi) = r$. In this case we say that $\mathbf{T}$ *r-satisfies* φ, in symbols $\mathbf{T}, e \models^r \varphi$.

Expanding the language In this dissertation we are going to consider some *language expansions* of the logic of a t-norm. Language expansions of the language of a given logic $\mathcal{L}(*)$ are obtained by adding new propositional connectives to its language $\mathbf{l}_{\mathcal{L}(*)}$.

Truth constants Defining the logic of a t-norm gives the possibility to consider a language expansion that is particularly interesting in relation with Fuzzy Description Logics: the one obtained by adding truth constants to the language of a given logic $\mathcal{L}(*)$. Given a subalgebra S of $[0,1]_*$ we consider the logic $\mathcal{L}_S(*)$, defined on the language $\mathbf{l}_S := \mathbf{l}_{\mathcal{L}(*)} \cup \{\overline{r} \colon r \in S\}$. The evaluations of $\mathcal{L}_S(*)$ are obtained by adding the conditions for truth constants:

- $e(\overline{r}) = r$, for any $r \in S$.

Involutive negation In the case that the residuated negation is not involutive, that is, when the t-norm $*$ considered is not the Łukasiewicz one, an interesting expansion is the one obtained by adding an *involutive negation* $\sim$ as an extra unary connective. An involutive negation of $\mathcal{L}(*)$ is any unary connective $\sim$ such that

$$\begin{aligned} \text{if } e(\varphi) \leq e(\psi) \quad \text{then} \quad & e(\sim \psi) \leq e(\sim \varphi) && (1.9) \\ e(\varphi) \quad = \quad & e(\sim\sim \varphi) && (1.10) \end{aligned}$$

for every propositional evaluation e and formula φ, that is, the truth function associated to $\sim$, is an order reversing involution on $[0, 1]$. In this dissertation we will only consider evaluations e such that:

$$e(\sim \varphi) \quad := \quad 1 - e(\varphi) \tag{1.11}$$

for every formula φ.

The logic obtained, which we will denote by $\mathcal{L}_{\sim}(*)$, is defined on the language $\mathbf{l}_{\sim} := \mathbf{l}_{\mathcal{L}(*)} \cup \{\sim\}$.

Strong disjunction A binary connective that we will often use in the context of FDL, is the one of strong disjunction $\oplus$. Its truth function defined in a standard algebra is:

$$a \veebar b \quad := \quad 1 - ((1 - a) * (1 - b)) \tag{1.12}$$

where $a, b \in T$, with either $T = [0, 1]$ or the domain of a finite standard algebra. Once defined the strong disjunction, we define the following abbreviation of the n-ary strong disjunction (for $n \in \mathbb{N}$):

$$n \cdot \varphi \quad := \quad \overbrace{\varphi \oplus \ldots \oplus \varphi}^{n \text{ times}} \tag{1.13}$$

where $\varphi \in Fm_{\mathbf{l}}$. Notice that, having the involutive negation defined as in (1.11), the connective of *strong disjunction* can be define in the language of the logic $\mathcal{L}_{\sim}(*)$ as:

$$\varphi \oplus \psi \quad := \quad \sim(\sim \varphi \otimes \sim \psi) \tag{1.14}$$

Monteiro-Baaz Delta operator Another interesting connective that can be added to the language of a given logic $\mathcal{L}(*)$ is the unary connective $\triangle$, whose associated truth function is:

$$\delta(a) = \begin{cases} 0, & \text{if } a \neq 1, \\ 1, & \text{otherwise} \end{cases} \tag{1.15}$$

for every $a \in [0,1]$.

The resulting propositional calculus, which we will denote by $\mathcal{L}_\triangle(*)$, is defined on the language $\mathbf{l}_\triangle := \mathbf{l}_{\mathcal{L}(*)} \cup \{\triangle\}$. The following formulas, introduced in [Baa96] in the framework of Gödel Logic:

($A_\triangle 1$) $\triangle\varphi \vee \neg \triangle \varphi$,

($A_\triangle 2$) $\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$,

($A_\triangle 3$) $\triangle\varphi \to \varphi$,

($A_\triangle 4$) $\triangle\varphi \to \triangle \triangle \varphi$,

($A_\triangle 5$) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$.

become then tautologies of the logic $\mathcal{L}_\triangle(*)$.

Further properties Most of the logics considered so far enjoy the following properties.

Definition 1.1.7.
1. We say that a logic $\mathcal{L}$ enjoys the *Local Deduction Theorem* (*LDT*, for short) if for each finite theory Γ and formulas φ, ψ, it holds that $\Gamma, \varphi \models \psi$ iff there exists a natural number n such that $\Gamma \models \varphi^n \to \psi$.
2. We say that a logic $\mathcal{L}$ enjoys *Delta Deduction Theorem* ($\triangle DT$, for short) if, for each finite theory Γ and formulas φ, ψ, it holds that $\Gamma, \varphi \models \psi$ iff $\Gamma \models \triangle\varphi \to \psi$.
3. We say that a logic $\mathcal{L}$ enjoys *Invariance under Substitution* (*Sub*, for short) if, for every formulas φ, ψ, χ and every formula ζ occurring in χ, it holds that $\varphi \leftrightarrow \psi \models \chi[\varphi/\zeta] \leftrightarrow \chi[\psi/\zeta]$.

Next we recall the definition of *core fuzzy logic* and of $\triangle$-*core fuzzy logic* given in [HC06] and that of *strict core fuzzy logic* given in [CE11].

Definition 1.1.8.
1. We say that a logic $\mathcal{L}$ is a *core fuzzy logic* if it is finitary, enjoys *LDT*, *Sub* and expands $\mathcal{L}(\mathbb{MTL})$.
2. We say that a logic $\mathcal{L}$ is a *strict core fuzzy logic* if it is finitary, enjoys *LDT*, *Sub* and expands $\mathcal{L}(\mathbb{SMTL})$.
3. We say that a logic $\mathcal{L}_\triangle$ is a $\triangle$-*core fuzzy logic* if it enjoys $\triangle DT$, *Sub* and expands $\mathcal{L}_\triangle(\mathbb{MTL})$.

1.1.2 Fuzzy (multi-)modal logic

In this section we introduce the framework of multi-modal logic. It is important because, as we will see, multi-modal language is a notational variant of the description language mainly considered in this work. For this reason, we will consider the general framework of *multi-modal language* of which the uni-modal language is a particular case.

Modal Logic was already known and studied in ancient times by the Aristotele's school. In its modern version it has been defined by C. I. Lewis and C. H. Langford, who, in [LL32] established a modern notation, gave sets of axioms for some logical system and provided a matrix-like truth-functional semantics for those systems. Further studies, due, above all, to E. J. Lemmon (see [Lem57, Lem66a, Lem66b]) introduced an algebraic semantics for Lewis and Langford Systems.

Nevertheless, the real cornerstone in the study of Modal Logic has been the work of S. Kripke, who, in [Kri63, Kri65], defined what is nowadays known as *Kripke semantics*, based on a particular kind of relational structures, called *Kripke frames*. This kind of structures gave a clear and well-defined semantics to modal systems, allowing great advancements in the study of Modal Logic, also under the syntactic and computational points of view.

Syntax

Given a propositional language $\mathbf{l}$, a multi-modal language $\mathbf{l}_\Box$ is obtained by adding a non-empty finite subset of the set of unary modal connectives $\{\Box_n : n \in \mathbb{N}\} \cup \{\Diamond_n : n \in \mathbb{N}\}$. From this new language, the set of modal formulas $Fm_{\mathbf{l}_\Box}$ is built by recursively applying the same rules as for propositional formulas, but in the new multi-modal language.

So, a multi-modal language $\mathbf{l}_\Box$ is defined as an expansion of a given propositional language $\mathbf{l}$ by means of a set of modal connectives.

An exhaustive study of Uni-modal Logic in the classical framework can be found in [BdRV01]. Within the framework of many valued logics, the study of modal expansions is much more recent and some advances have been done in [Fit92a, Fit92b, HH96, Háj98c, MO09, Háj10, BEGR11, CR11]. We adopt the definitions given in [BEGR11] in the next pages.

Semantics

As was said, a notion that has become fundamental for the study of Modal Logic, under a semantical point of view, is that of Kripke frames and Kripke models. In this work we consider a many-valued generalization of the classical notion of Kripke model following the one provided in [BEGR11].

Definition 1.1.9 (Kripke frames and models)**.** Given an algebra $\mathbf{T}$ and $m \in \mathbb{N}$, a $\mathbf{T}$-*valued Kripke frame* is a tuple $\mathfrak{F} = \langle W, R_1, \dots, R_m \rangle$, where

- W is a non-empty crisp set, called *domain* or *set of possible worlds*,

- for every $1 \leq i \leq m$, R_i is a binary relation (called *accessibility relation*) valued in $\mathbf{T}$; i.e., it is a mapping $R_i\colon W \times W \longrightarrow T$.

A Kripke frame is said to be *crisp* if, for every $1 \leq i \leq m$, the range of R_i is included in $\{0, 1\}$. The class of all $\mathbf{T}$-valued frames will be denoted by Fr and the class of crisp frames by CFr.

A Kripke $\mathbf{T}$-*model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where $\mathfrak{F}$ is a $\mathbf{T}$-valued Kripke frame and V is a mapping $V\colon At \times W \longrightarrow T$ assigning to each propositional variable and each world in W a value in T. The map V can be uniquely extended to a map, which we also denote by V, assigning to each pair formed by a formula $\varphi \in Fm_{\mathbf{l}_\Box}$ and a world $w \in W$ an element of T in such a way that:

- $V(\top, w) = 1$;
- $V(\bot, w) = 0$;
- $V(\overline{r}, w) = r \in T$ for each truth constant $\overline{r} \in \mathbf{l}$;
- $V(\star(\varphi_1, \ldots, \varphi_n), w) = \hat{\star}(V(\varphi_1, w), \ldots, V(\varphi_n, w))$, for every n-ary propositional connective $\star \in \mathbf{l}$ and its truth function $\hat{\star} \in \mathbf{l_T}$;
- for each $1 \leq i \leq m$, $V(\Box_i \varphi, w) = \inf_{w' \in W}\{R_i(w, w') \Rightarrow V(\varphi, w')\}$;
- for each $1 \leq i \leq m$, $V(\Diamond_i \varphi, w) = \sup_{w' \in W}\{R_i(w, w') * V(\varphi, w')\}$.

The *universal modality*, denoted $\Box_U$ is a modality whose associated accessibility relation is the total relation, i.e., the fuzzy relation $U\colon W \times W \to T$ such that, for every $v, w \in W$, it holds that $U(v, w) = 1$.

Logic

In the following definition we define a list of notions that can be considered about a many-valued multi-modal logic, some of them will be equivalent to reasoning tasks in Fuzzy Description Logic.

Definition 1.1.10. Consider a formula $\varphi \in Fm_{\mathbf{l}_\Box}$ an algebra $\mathbf{T}$, $r \in T$ and a Kripke $\mathbf{T}$-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, then:

- we say that $w \in W$ *satisfies* φ, denoted $\mathfrak{M}, w \Vdash \varphi$, if $V(\varphi, w) = 1$;
- we say that $w \in W$ *positively satisfies* φ, denoted $\mathfrak{M}, w \Vdash^{pos} \varphi$, if $V(\varphi, w) > 0$;
- we say that $w \in W$ *r-satisfies* φ, denoted $\mathfrak{M}, w \Vdash^{r} \varphi$, if $V(\varphi, w) = r$;
- we say that $\mathfrak{M}$ *locally satisfies* (resp. positively, r-) φ, denoted $\mathfrak{M} \models_l \varphi$ (resp. $\mathfrak{M} \models_l^{pos} \varphi$, $\mathfrak{M} \models_l^{r} \varphi$), if there exists $w \in W$ such that $\mathfrak{M}, w \Vdash \varphi$ (resp. $\mathfrak{M}, w \Vdash^{pos} \varphi$, $\mathfrak{M}, w \Vdash^{r} \varphi$); in this sense we say that φ is *locally satisfiable* (resp. positively, r-) if there is a Kripke $\mathbf{T}$-model $\mathfrak{M}$ which locally satisfies (resp. positively, r-) it;

- we say that $\mathfrak{M}$ *globally satisfies* (resp. positively, r-) φ, denoted $\mathfrak{M} \models_g \varphi$ (resp. $\mathfrak{M} \models_g^{pos} \varphi$, $\mathfrak{M} \models_g^r \varphi$), if $\inf_{w \in W}\{V(\varphi, w)\} \geq 1$ (resp. for every $w \in W$ it holds that $V(\varphi, w) > 0$, $\inf_{w \in W}\{V(\varphi, w)\} \geq r$); in this sense we say that φ is *globally satisfiable* (resp. positively, r-) if there is a Kripke $\mathbf{T}$-model $\mathfrak{M}$ which globally satisfies (resp. positively, r-) it;

- we say that φ is a *local consequence* of a set of formulas $\Gamma \subseteq Fm_{\mathbf{l}_\Box}$, denoted $\Gamma \models_l \varphi$ if, for every Kripke $\mathbf{T}$-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $w \in W$, if w 1-satisfies every formula in Γ, then w 1-satisfies φ;

- we say that φ is a *global consequence* of a set of formulas $\Gamma \subseteq Fm_{\mathbf{l}_\Box}$, denoted $\Gamma \models_g \varphi$, if, every Kripke $\mathbf{T}$-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ which globally 1-satisfies every formula in Γ, globally 1-satisfies φ as well;

- we say that φ is *valid in the frame* $\mathfrak{F}$, denoted $\mathfrak{F} \models \varphi$, if it is globally 1-satisfied in every Kripke $\mathbf{T}$-model based on $\mathfrak{F}$; in this sense, given a class K of frames, we write $\mathsf{K} \models \varphi$ to mean that φ is valid in every frame in that class.

Following the notation used in [BEGR11], we will denote the set of formulas that are valid in every frame of a class K under the algebra of truth values $\mathbf{T}$, by $\mathbf{\Lambda}(\mathsf{K}, \mathbf{T})$.

Besides these sets of valid formulas, we will consider the sets $\mathbf{Sat}(\mathsf{K}, \mathbf{T})$, $\mathbf{Sat_{pos}}(\mathsf{K}, \mathbf{T})$ and $\mathbf{Sat_r}(\mathsf{K}, \mathbf{T})$ of satisfiable, positively satisfiable and r-satisfiable formulas respectively, in a class of frames K and under the algebra of truth values $\mathbf{T}$.

1.1.3 First order predicate logic

Syntax

In order to define what a *predicate language* is, we need the notion of *signature.*

Definition 1.1.11. A *predicate signature* $\mathbf{s}$ consists of a countable set of relation symbols (also called *predicates*) $P_1, \ldots, P_n, \ldots$, each one with arity ≥ 1, a countable set of function symbols $f_1, \ldots, f_n, \ldots$, each one with its arity, a countable set of constant symbols $c_1, \ldots, c_n, \ldots$, that are 0-ary function symbols.

Given a countable set Var of individual variables, the set of *Terms* over a predicate signature is defined inductively as follows:

- every variable $x \in Var$ is a term,

- every constant $c \in \mathbf{s}$ is a term,

- if $t_1, \ldots, t_n$ are terms and $f \in \mathbf{s}$ is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term.

Now, let $\mathbf{l}$ be a propositional language, as defined in Section 1.1.1, then the set of symbols $\mathbf{l}\forall := \mathbf{l} \cup \{\forall, \exists\}$ is a first order language. The set $Fm_{\mathbf{l}\forall,\mathbf{s}}$ of *Formulas* over a first order language $\mathbf{l}\forall$ and predicate signature $\mathbf{s}$ is defined inductively as follows:

- $\bot$ and $\top$ are formulas,
- if $t_1, \ldots, t_n$ are terms and $P \in \mathbf{s}$ is an n-ary predicate, then $P(t_1, \ldots, t_n)$ is a formula (called *atomic formula*),
- if $\varphi_1, \ldots, \varphi_n$ are formulas and $\star \in \mathbf{l}$ is an n-ary logical connective, then $\star(\varphi_1, \ldots, \varphi_n)$ is a formula,
- if $\varphi(x)$ is a formula and x a variable, then $(\forall x)\varphi(x)$ and $(\exists x)\varphi(x)$ are formulas.

As usual a variable that does not fall within the scope of a quantifier is said to be *free*, otherwise, it is said to be *bound*. The notation $\varphi(x_1, \ldots, x_n)$ means that the variables that are free in φ are among $x_1, \ldots, x_n$. We say that a formula that has no free variable is *closed*, otherwise it is *open*. Given a term t and a formula $\varphi(x_1, \ldots, x_n)$, we denote by $\varphi(x_1, \ldots, x_n)[t/x_1]$ the result of substituting every occurrence of variable x_1 for t in $\varphi(x_1, \ldots, x_n)$.

Semantics

From a semantical point of view, first order models consist of a domain, an algebra of truth values and an assignment function.

Definition 1.1.12. A *first order structure* for a given signature $\mathbf{s}$ and an MTL-chain $\mathbf{T}$ is a pair $(\mathbf{M}, \mathbf{T})$, where $\mathbf{M}$=$(M, (P_{\mathbf{M}})_{P\in\mathbf{s}}, (f_{\mathbf{M}})_{f\in\mathbf{s}}, (c_{\mathbf{M}})_{c\in\mathbf{s}})$, is such that:

1. The set M, called *domain*, is a non-empty set,
2. for each predicate symbol $P \in \mathbf{s}$ of arity n, $P_{\mathbf{M}}$ is an n-ary $\mathbf{T}$-fuzzy relation on M, i.e. an n-ary function $P_{\mathbf{M}} \colon M^n \longrightarrow T$,
3. for each function symbol $f \in \mathbf{s}$ of arity n, $f_{\mathbf{M}}$ is an n-ary (crisp) function on M and
4. for each constant symbol $c \in \mathbf{s}$, $c_{\mathbf{M}}$ is an element of M.

The truth value $\|\varphi\|_v^{(\mathbf{M},\mathbf{T})}$ of a predicate formula φ under a given assignment v is defined as follows.

Definition 1.1.13. Let $\mathbf{s}$ be a first order signature, $\mathbf{T}$ an MTL-chain and $(\mathbf{M}, \mathbf{T})$ a first order structure. Then an *assignment* v is a mapping $v : Var \longrightarrow M$. As usual each assignment, defined on the set of individual variables, extends univocally to an assignment (that we will denote by v as well) satisfying, for every terms $t_1, \ldots, t_n$ and each n-ary function $f \in \mathbf{s}$, that

$$v(f(t_1, \ldots, t_n)) = f_{\mathbf{M}}(v(t_1), \ldots, v(t_n)).$$

To denote that assignment v assigns objects $a_1, \ldots, a_n$ to variables $x_1, \ldots, x_n$, we will write $v([a_1/x_1], \ldots, [a_n/x_n])$.
Moreover, each assignment v, defined on the set of individual variables yields a first order model $\|\cdot\|_v^{(\mathbf{M},\mathbf{T})} : Fm_{\mathbf{l}\forall,\mathbf{s}} \longrightarrow T$ such that:

1. for each n-tuple of terms $t_1, \ldots, t_n$ and each n-ary relation $P \in \mathbf{s}$, it holds that

$$\|P(t_1, \ldots, t_n)\|_v^{(\mathbf{M},\mathbf{T})} = P_{\mathbf{M}}(v(t_1), \ldots, v(t_n)) \tag{1.16}$$

2. if $\varphi_1, \ldots, \varphi_n$ are formulas, $\star \in \mathbf{l}$ an n-ary logical connective and $\hat{\star} \in \mathbf{l_T}$ its truth function, then

$$\|\star(\varphi_1, \ldots, \varphi_n)\|_v^{(\mathbf{M},\mathbf{T})} = \hat{\star}(\|\varphi_1\|_v^{(\mathbf{M},\mathbf{T})}, \ldots, \|\varphi_n\|_v^{(\mathbf{M},\mathbf{T})}) \tag{1.17}$$

3. if $\varphi(x_1, \ldots, x_n)$ is a formula and v is a first order assignment such that $v(x_i) = a_i$ and $a_i \in M$, for $1 < i \leq n$, then we have that

$$\|(\forall x_1)\varphi(x_1, x_2, \ldots, x_n)\|_v^{(\mathbf{M},\mathbf{T})} = \inf_{a \in M}\{\|\varphi(a, a_2, \ldots, a_n)\|^{(\mathbf{M},\mathbf{T})}\} \tag{1.18}$$

4. if $\varphi(x_1, \ldots, x_n)$ is a formula and v is a first order assignment such that $v(x_i) = a_i$ and $a_i \in M$, for $1 < i \leq n$, then we have that

$$\|(\exists x_1)\varphi(x_1, x_2, \ldots, x_n)\|_v^{(\mathbf{M},\mathbf{T})} = \sup_{a \in M}\{\|\varphi(a, a_2, \ldots, a_n)\|^{(\mathbf{M},\mathbf{T})}\} \tag{1.19}$$

Clearly, depending on the model, the infimum and supremum of a set of values of formulas do not necessarily exist and, in this case we will say that a given quantified formula has an undefined truth value. Following [Háj98c], we will say that if, for a given model $\|\cdot\|_v^{(\mathbf{M},\mathbf{T})}$, both infima and suprema of sets of values are defined for every formula, then $\|\cdot\|_v^{(\mathbf{M},\mathbf{T})}$ is a *safe* model. Moreover, if, for a given first order structure $(\mathbf{M}, \mathbf{T})$, each assignment v, defined in it, is safe, we will say that $(\mathbf{M}, \mathbf{T})$ is a safe structure.

From now on and for simplicity, we will omit the name "safe" before the first order structures, i.e., *when we speak about a first order structure* $(\mathbf{M}, \mathbf{T})$*, we implicitly mean a* **safe** *first order structure* $(\mathbf{M}, \mathbf{T})$.

Logic The notions of *tautologies*, *logical consequence* and *satisfiability* are defined as in the case of propositional logic, but, this time o closed formulas of a predicate language and with respect to first order structures. The same holds for the notions of *deduction theorems* and *core fuzzy logics*.

As in the case of propositional logics, in fact, it makes sense the notions of *first order logic of a variety* $\mathbb{K}$ and *first order logic of a (continuous) t-norm* $*$.

Definition 1.1.14. Let $\mathbb{K}$ be a class of chains and $\mathbf{l}$ a propositional language such that $\mathbf{t}(\mathbf{l}) = \mathbf{t}(\mathbf{l}_{\mathbb{K}})$. We define the first order logic of the class $\mathbb{K}$ (called $\mathcal{L}\forall(\mathbb{K})$) as the set of tautologies over $\mathbb{K}$.

Consider a t-norm $*$ and the minimal variety $\mathbb{V}([0,1]_*)$ containing $[0,1]_*$, then we will denote the logic $\mathcal{L}\forall(\mathbb{V}([0,1]_*))$ by *the first order logic of the t-norm* $*$ (sometimes we will specify that it is the *the first order logic of the t-norm* $*$ *with general semantics*).

Definition 1.1.15. Let $[0,1]_*$ be a standard algebras and $\mathbf{l}$ a propositional language such that $\mathbf{t}(\mathbf{l}) = \mathbf{t}(\mathbf{l}_{[0,1]_*})$. We define the first order logic of a t-norm $*$ (called $\mathcal{L}\forall(*)$) as the set of tautologies over $[0,1]_*$.

When $*$ is the Łukasiewicz (product, Gödel, respectively) t-norm, we will denote the logic $\mathcal{L}\forall(*)$ by *Łukasiewicz (product, Gödel, respectively) first order logic with standard semantics.* Notice that the Łukasiewicz and product first order logic with general semantics do not coincide with the Łukasiewicz and product first order logic with standard semantics (see [Háj98c]).

As for propositional logic, if there is a model $\|\cdot\|_v^{(\mathbf{M},\mathbf{T})}$, with either $\mathbf{T} = [0,1]_G$ or $\mathbf{T} = [0,1]_\Pi$ such that, for a closed formula φ, it holds that model $\|\varphi\|_v^{(\mathbf{M},\mathbf{T})} = r \in (0,1)$, then, for any $r' \in (0,1)$ it can be found a model $\|\cdot\|_v^{(\mathbf{M}',\mathbf{T})}$ such that model $\|\varphi\|_v^{(\mathbf{M}',\mathbf{T})} = r'$. So, in what follows, we will speak only about 1-satisfiability and positive satisfiability in the case of Gödel and product standard chains.

If $\mathbf{T}$ is either $[0,1]_Ł$ or $Ł_n$ it is possible to define the notion of r-satisfiability as well. Indeed, φ is *r-satisfiable*, for $r \in T$, if there exists a first order structure $(\mathbf{M},\mathbf{T})$ and an assignation v such that $\|\varphi\|_v^{(\mathbf{M},\mathbf{T})} = r$. In this case we say that $(\mathbf{M},\mathbf{T})_v$ *r-satisfies* φ, in symbols $(\mathbf{M},\mathbf{T})_v \models^r \varphi$.

The witnessed model property Generalizing the classical case, the value of a universally (existentially) quantified formula is defined, like in (1.18) ((1.19) respectively) as the infimum (supremum) of the corresponding set of values. In the context of Classical Logic, as well as every finitely valued logic, infima and suprema turn out to be minima and maxima, respectively. However, when we move to infinitely valued logics, this is not the case. The infimum or supremum of a set of values X may be an element $r \notin X$, i.e., a quantified formula may have no *witness*. Following these ideas, Hájek introduced in [Háj05] the notion of *witnessed model*, i.e., a model in which each quantified formula has a witness and proved that this is an important property because it implies a limited form of finite model property for certain fragments of predicate fuzzy logic (see [Háj05] and Section 3.1 of this dissertation).

Witnessed models have been firstly defined in [Háj05] in the following way:

Definition 1.1.16. For any structure $(\mathbf{M}, \mathbf{T})$,

- a formula $(\forall x)\varphi(x, x_1, \ldots, x_n)$ is $\mathbf{T}$-witnessed in $\mathbf{M}$ if, for each tuple $c_1, \ldots, c_n \in M$, there is an assignment $v\colon Var \longrightarrow M$ and an element $c \in M$ such that $v(x_i) = c_i$, for $1 \leq i \leq n$, $v(x) = c$ and

$$\|(\forall x)\varphi(x, x_1, \ldots, x_n)\|_v^{(\mathbf{M},\mathbf{T})} = \|\varphi(x, x_1, \ldots, x_n)\|_v^{(\mathbf{M},\mathbf{T})},$$

- a formula $(\exists x)\varphi(x, x_1, \ldots, x_n)$ is $\mathbf{T}$-witnessed in $\mathbf{M}$ if, for each tuple $c_1, \ldots, c_n \in M$, there is an assignment $v\colon Var \longrightarrow M$ and an element $c \in M$ such that $v(x_i) = c_i$, for $1 \leq i \leq n$, $v(x) = c$ and

$$\|(\exists x)\varphi(x, x_1, \ldots, x_n)\|_v^{(\mathbf{M},\mathbf{T})} = \|\varphi(x, x_1, \ldots, c_n)\|_v^{(\mathbf{M},\mathbf{T})}.$$

$\mathbf{M}$ is $\mathbf{T}$-witnessed if all quantified formulas are $\mathbf{T}$-witnessed in $\mathbf{M}$.

Later on, in [HC06], Hájek and Cintula consider the following couple of formulas already given by Baaz in [Baa96]:

(C∃) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,

(C∀) $((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

They proved that formulas (C∃) and (C∀) identify a first order core fuzzy logic associated to the class $\mathbb{K}$ restricted to witnessed models (hence denoted $\mathcal{L}\forall^w(\mathbb{K})$) in the sense that these two formulas are tautologies of $\mathcal{L}\forall^w(\mathbb{K})$. Moreover, in [Háj07a] it is proved that, Łukasiewicz first order logic $\mathcal{L}\forall(\mathbb{V}([0,1]_*))$ is the only logic of a t-norm equivalent to its restriction to witnessed models $\mathcal{L}\forall^w(\mathbb{V}([0,1]_*))$, i.e., (C∃) and (C∀) are tautologies of $\mathcal{L}\forall(\mathbb{V}([0,1]_{\text{Ł}}))$. Thus Łukasiewicz is the only infinite-valued logic of a t-norm with general semantics which is complete with respect to witnessed models. We will refer to this property as the *witnessed model property*, which Łukasiewicz first order logic with general semantics has.

Moreover,

- In [Háj98c, Theorem 5.4.30] it is proven that, if a formula φ is not true in a $[0,1]_{\text{Ł}}$-model, then there exists an integer n such that φ is not true in an Ł_n-model.
- In [Háj05, Lemma 3] Hájek proves that, if a formula φ is 1-satisfiable in a $[0,1]_{\text{Ł}}$-model, then it is 1-satisfiable in a witnessed $[0,1]_{\text{Ł}}$-model.

So, tautologies and 1-satisfiable formulas in $\mathcal{L}\forall(*_{\text{Ł}})$ coincide with tautologies and 1-satisfiable formulas in $\mathcal{L}\forall^w(*_{\text{Ł}})$ and, therefore, $\mathcal{L}\forall(*_{\text{Ł}})$ enjoys the witnessed model property as well.

The quasi-witnessed model property Neither Gödel, nor Product first-order Logic (neither defined by the respective varieties nor by the t-norms) have the witnessed model property because formulas (C∀) are not a tautologies of these logics. Nevertheless, in [LM07] it is proved that Product Predicate Logic enjoys a weaker property, what we call *quasi-witnessed model property.* Quasi-witnessed models[3] are models in which, whenever the value of a universally quantified formula is strictly greater than 0, then it has a witness, while existentially quantified formulas have always a witness.

Definition 1.1.17. For any first-order structure $(\mathbf{M}, \mathbf{T})$, a formula $(\forall x)\varphi(x, x_1, \ldots, x_n)$ is $\mathbf{T}$-*quasi-witnessed* in $\mathbf{M}$ if for each tuple $c_1, \ldots, c_n$ of elements in M,

1. either there exists an assignment $v\colon Var \longrightarrow M$ and an element $c \in M$ such that $v(x_i) = c_i$, for $1 \leq i \leq n$, $v(x) = c$ and

$$\|(\forall x)\varphi(x, x_1, \ldots, x_n)\|_v^{(\mathbf{M},\mathbf{T})} = \|\varphi(x, x_1, \ldots, x_n)\|_v^{(\mathbf{M},\mathbf{T})},$$

2. or there exists an assignment $v\colon Var \longrightarrow M$ such that $v(x_i) = c_i$, for $1 \leq i \leq n$ and

$$\|(\forall x)\varphi(x, x_1, \ldots, x_n)\|_v^{(\mathbf{M},\mathbf{T})} = 0.$$

A formula $(\exists x)\varphi(x, x_1, \ldots, x_n)$ is $\mathbf{T}$-*quasi-witnessed* in $\mathbf{M}$ if it is witnessed.

We say that a first-order structure $(\mathbf{M}, \mathbf{T})$ is quasi-witnessed if for every assignment v of the variables on $\mathbf{M}$ every formula is quasi-witnessed.

In [CE11][4] we introduced both the so-called *strict core fuzzy logics* and the following couple of formulas (generalizations of formulas (C∃) and (C∀) of Hájek-Cintula to cope with quasi-witnessed models). If $\mathcal{L}\forall$ is any strict core fuzzy first-order logic, we denote by $\mathcal{L}\forall^{qw}$ the restriction of $\mathcal{L}\forall$ to quasi-witnessed models.

Consider the couple of formulas:

(C∃) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,

(ΠC∀) $\neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

The first one, (C∃), is the same as in the case of witnessed models and the second one says that formula (C∀) is true in a structure $(\mathbf{M}, \mathbf{T})$ only when the truth value of $(\forall x)\varphi(x)$ is different from 0, i.e., when $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{M},\mathbf{T})} = 1$.

[3] These models are called "closed models" in [LM07] but we decided, after some discussions with colleagues, to use the more informative name of "quasi-witnessed models". We take into account the fact that the name "closed" is used in mathematics and logic in different contexts with different meanings and could induce some confusion.

[4] The full paper is reported in the Appendix B.

In the same paper, we proved, following the approach of [HC06] that formulas (C∃) and (ΠC∀) are tautologies of the restriction of any first-order strict core fuzzy logic to quasi-witnessed models. From this result, the one in [LM07] about the completeness of Product first-order Logic with general semantics with respect to quasi-witnessed models, follows as a corollary. Moreover, we proved that formulas (C∃) and (ΠC∀) are tautologies in no logic of a continuous t-norm, but Product and Łukasiewicz predicate logics with general semantics.

In the case of the Product first order logic with standard semantics, the landscape is not the same. In Appendix A it is reported a result from [CEB10], where it is proved that 1-validity and positive satisfiability restricted to quasi-witnessed models and unrestricted positive satisfiability indeed coincide under the standard product semantics $[0,1]_\Pi$ and the same holds for tautologies. For the 1-satisfiability problem under standard product semantics, completeness with respect to quasi-witnessed models is still an open problem.

1.1.4 A solution to sorites paradox

After having introduced the framework of MFL, we can explain a formal solution to the sorites paradox (see [Nog08]). Consider the propositional logic $\mathcal{L}(*)$, where $*$ is the Łukasiewicz t-norm and the propositional evaluation $e\colon \{p_n : n \in \mathbb{N}\} \to [0,1]$ such that:

- $e(p_{10.000}) = 1$,
- for every $n \in \mathbb{N}$ such that $0 < n \leq 10.000$, it holds that $e(p_n \to p_{n-1}) = 0.9999$.

which, intuitively, means that it is totally true that 10.000 sand grains are a heap, but, when we took a sand grain away from a heap, what we obtain is *a bit less* a heap than the original one. Then, from the semantics of Łukasiewicz implication, we have that:

- $e(p_n) = \frac{n}{10.000}$,
- in particular, $e(p_0) = 0$.

which, intuitively says that, while 10.000 sand grains are a heap, 0 sand grain are not, and every amount of sand grain that falls in between is a sand heap in a degree that is neither fully true, nor totally false, but proportional to the number of sand grain. Hence, Łukasoewicz Logic, as defined before, does not have to preserve the value of the assumption in sorites paradox; and so, there is no paradox now.

1.2 (Classical) Description Logic

This dissertation proposes a generalization of Classical Description Logic (DL) to the many-valued and fuzzy case. In order to make a confrontation of the new generalized framework with the classical one, we briefly introduce in this section the classical framework on DL. For an exhaustive presentation of the subject, the reader is invited to look at the general *Handbook of Description Logic* [BCM+03] and the more recent paper [BHS08].

1.2.1 A little bit of history

Description Logics are usually considered as an evolution of *frame-based systems*. The main examples of frame-based systems are Quillian's *Semantic networks* (see [Qui67]) and Minsky's *Frame systems* (see [Min81]). Frame-based systems were formalisms based on researches about human cognitive behavior. In this sense, given a memory model, their goal was to obtain a program that imitates human mental skills, e.g. natural language understanding. For this reason these systems were thought in a way that they could support language ambiguity and this fact made them far from based on formal logic, when their authors were not explicitly against the use of logic.

During the second half of 70's began to be clear the limits of frame-based systems. Among those limits we can find the following ones:

- it was not so clear what the systems had to compute (see [Woo75]),
- there was not a simple way to give these system a clear formal semantics,
- most aspects of these systems can be formalized by means of first order logic and it seems that the contributions of frame-based systems is not so novel (see [Hay77]).

Despite the first version of KL-ONE, developed by R. J. Brachman in [Bra79], was not based on formal logic, nevertheless this new representation system brought some significative novelty to the old framework of frame-based systems. We report some of them:

- it considers the tasks of extracting implicit conclusions from existing knowledge,
- it gives the user the possibility of defining new complex concepts and roles,
- it introduces the difference between *individual concepts* and *generic concepts*,
- the difference between the concept definitions with sufficient and necessary condition and those with just necessary ones is studied,
- *classification* (computation of the hierarchy of subsumptions) and *realization* (computation of the more specific atomic concept) are added to the reasoning tasks,

Besides these novelties, KL-ONE had some weaknesses that became evident quite early. Among those weaknesses we can find the lack of a clear formal semantics and the fact that the algorithms for deciding classification and realization were incomplete. In order to overcome the weaknesses of KL-ONE it has been proposed, as guidelines for new systems:

1. the fact of thinking the system under the point of view of *functionality*, i.e. focussing on the reasoning services provided to the user, more than under the point of view of the mere concept representation,

2. a clearer distinction between the knowledge representing relations among concepts and that representing assertions about individuals.

Besides the weaknesses that KL-ONE-like systems had, they brought a new way to see knowledge representation systems. On the one hand, in fact it has been adopted the so-called *functional approach*, that consisted in putting the attention on the services provided by the KR systems, more than on the way it represents knowledge. This change of perspective can be seen at the origin of the growing interest that, since the 80's, researchers put on decision algorithms and their complexity. On the other hand, the need of a clear semantics can be seen at the origin of the fact that systems began to be more and more logic-based and an unambiguous Tarsky-style semantics was adopted.

The fact of putting attention on the reasoning tasks and on the logical language of the systems allowed to think about those systems in a more abstract way as clearly defined *description languages*. This means, as well, that the languages are now quantitatively comparable, mainly under two points of view: the computational complexity of reasoning, on the one side, and the expressivity of the language, on the other. Since the 80's, the history of proper DL systems is, indeed, characterized by the tradeoff between complexity and expressivity of the language and the search of a fair equilibrium between these two features has been the main fuel of the great advancements that researches in DL have seen since then.

The DL systems of the 80's, like BACK and LOOM, used so-called *structural subsumption algorithms*. These kinds of algorithm perform a comparison in the syntactic structure of two given concepts after having transformed them in a suitable normal form. Structural subsumption algorithms are relatively efficient when applied to very inexpressive languages, as proven in [BL84]. Nevertheless, in more expressive languages these algorithms turn out to be incomplete. Further researches of the same period, like [BL85], allowed by the use of abstract languages, revealed that expressivity improvements increase intractability of the reasoning tasks. In particular, [Neb90] revealed that reasoning in presence of a Terminological Box is a computationally intractable problem in itself.

The 90's saw the introduction of a new kind of algorithms: the *tableau based algorithms* (see, for example, [HNSS90]). These kind of algorithms revealed to be complete also for quite expressive DLs and allowed a systematic study of complexity of reasoning in various DLs, in particular, those related with logical languages (see [DLN$^+$92, SSS91]). Moreover, they are suitable to be highly optimized in such a way that they can lead to a good practical behavior of the system. In the same period the relationships between DLs and classical modal logics ([Sch91]) and with fragments of classical first order logic ([Bor96]) are investigated.

Nowadays very expressive DL systems are used as the reasoning engines of the Semantic Web and for knowledge representation in medical and bio-informatic data bases.

1.2.2 Syntax

Knowledge is represented in DL systems through the construction of *concepts* by means of a set of symbols that consists of:

- a set N_A of *concepts names*,
- a set N_R of *role names*,
- a set N_I of *individual names*,
- and a family of *concept* and *role constructors*.

The difference between description languages consists in the family of concept and role constructors utilized to build up concept descriptions and each of these sets is denoted by a sequence of letters. In what follows we briefly introduce, by means of syntactic rules, the symbology used to denote each constructor, the name of the constructor and the letter used to denote the language that the constructor utilizes. Since, however, we are interested in those languages that are related to modal and first order logic, we will give an account of DL languages up to the one that is known as $\mathcal{ALC}$ (see also [BHS08]). For this reason, besides not considering most existing concept constructors, we will not consider any role constructors as neither, since they fall outside $\mathcal{ALC}$.

Concept constructors

Given a variable $A \in N_A$ for an atomic concept, a variable $R \in N_R$ for a role name and variables for complex concepts C, D, an $\mathcal{ALC}$ *concept* is inductively built in accordance with the following syntactic rules:

C, D	$\longrightarrow$	$\bot$	empty concept	$\mathcal{FL}_0$
		$\top$	universal concept	$\mathcal{FL}_0$
		A	atomic concept	$\mathcal{FL}_0$
		$C \sqcap D$	conjunction	$\mathcal{FL}_0$
		$\forall R.C$	value restriction	$\mathcal{FL}_0$
		$\exists R.\top$	restricted existential quantif.	$\mathcal{FL}^-$
		$\neg A$	atomic complementation	$\mathcal{AL}$
		$\neg C$	complementation	$\mathcal{C}$
		$C \sqcup D$	disjunction	$\mathcal{U}$
		$\exists R.C$	existential quantification	$\mathcal{E}$

Here the name $\mathcal{FL}$ stands for "frame language" because it has more or less the same expressive power of frame-based systems. On the other hand, the name $\mathcal{AL}$ stands for "attributive language", since it marks the difference between frame-based systems and the new systems based on a description of attributes and predicates.

1.2.3 Semantics

An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a nonempty set $\Delta^{\mathcal{I}}$ (called *domain*) and of an *interpretation function* $\cdot^{\mathcal{I}}$ that assigns:

1. to each individual name $a \in N_I$ an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ such that $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ if $a \neq b$ (*Unique Name Assumption*, different individuals denote different objects of the domain),

2. to each atomic concept A a subset $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ of the domain set,

3. to each role name R a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ on the domain set.

Moreover, the interpretation function is inductively extended to complex concepts as follows:

$$
\begin{array}{rcl}
\bot^{\mathcal{I}} & = & \emptyset \\
\top^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\exists R.\top)^{\mathcal{I}} & = & \{a \in \Delta^{\mathcal{I}} \colon \text{exists } b \in \Delta^{\mathcal{I}} \text{ such that } R^{\mathcal{I}}(a,b)\} \\
(\forall R.C)^{\mathcal{I}} & = & \{a \in \Delta^{\mathcal{I}} \colon \text{for every } b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,b) \to C^{\mathcal{I}}(b)\} \\
(\exists R.C)^{\mathcal{I}} & = & \{a \in \Delta^{\mathcal{I}} \colon \text{exists } b \in \Delta^{\mathcal{I}} \text{ such that } R^{\mathcal{I}}(a,b) \wedge C^{\mathcal{I}}(b)\}
\end{array}
$$

1.2.4 Inclusions between languages: the $\mathcal{ALC}$ hierarchy

A straightforward consequence of the semantics of constructors is that every $\mathcal{ALE}$ and every $\mathcal{ALU}$ concepts are $\mathcal{ALC}$ concepts, but there are $\mathcal{ALE}$ concepts that are not $\mathcal{ALU}$ concepts and vice-versa. So, the hierarchy of languages between $\mathcal{AL}$ and $\mathcal{ALC}$ appears as in Figure 1.2.

1.2.5 Reasoning

As said before, besides the description of the world, a fundamental service provided by DL systems is that of inferring hidden conclusions from known premises. In this section we give an account of the syntax and semantics of the premises and the types of conclusions that can be inferred from those premises.

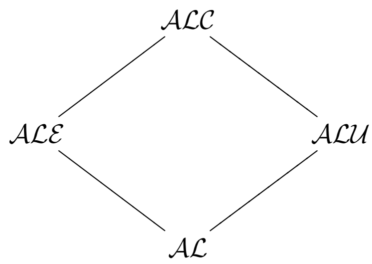

Figure 1.2: The hierarchy of sub-$\mathcal{ALC}$ languages

Knowledge bases

A *general concept inclusion* (GCI) (or *inclusion axiom*) is an expression of the form:

$$C \sqsubseteq D$$

where C, D are $\mathcal{ALC}$ concepts. An interpretation $\mathcal{I}$ satisfies an inclusion axiom $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

An *equivalence axiom* is an expression of the form:

$$C \equiv D$$

which, in the classical case, is an abbreviation for the pair of axioms $C \sqsubseteq D$ and $D \sqsubseteq C$. An interpretation $\mathcal{I}$ satisfies an equivalence axiom $C \equiv D$ if $C^{\mathcal{I}} = D^{\mathcal{I}}$.

A finite set $\mathcal{T}$ of GCIs is called a *terminology* or *TBox*. An axiom of the form $A \equiv C$, where A is a concept name, is called a *definition*. It is said that a concept name A *directly uses* a concept name B in a TBox $\mathcal{T}$ if there is a definition $A \sqsubseteq C$ in $\mathcal{T}$ such that B occurs in C. Furthermore, it is said that a concept name A *uses* a concept name B if B is in the transitive closure of the relation of directly using with respect to A. A TBox $\mathcal{T}$ is called *definitorial* or *acyclic* if:

- it contains only definitions,
- it contains at most one definition for each concept name occurring in it,
- no concept name occurring in it uses itself.

A *concept assertion axiom* (or *assertion*) is an expression of the form:

$$C(a)$$

where C is a concept and $a \in N_I$. An interpretation $\mathcal{I}$ satisfies an assertion $C(a)$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

A *role assertion axiom* is an expression of the form:

$$R(a, b)$$

where $R \in N_R$ and $a, b \in N_I$. An interpretation $\mathcal{I}$ satisfies a role assertion $R(a, b)$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}$.

A finite set of concept and role assertion axioms is called *ABox*. An ABox is said to be *local* if the same individual name a appears in each assertion.

Finally a *knowledge base* (KB for short) $\mathcal{K}$ consists of a TBox and an ABox, each one possibly empty.

Main inference problems

Consider a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A},)$, a pair of concepts C, D, a pair of roles R, S and a pair of individuals a, b, then we can define the main reasoning tasks considered in the literature.

- Checking whether C is *satisfiable* means checkin whether there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$. In this case we say that interpretation $\mathcal{I}$ satisfies concept C, in symbols $\mathcal{I} \models C$.
- Checking whether $\mathcal{K}$ is *consistent* means checkin whether there is an interpretation $\mathcal{I}$ that satisfies every assertion axiom in $\mathcal{A}$ and every inclusion axiom in $\mathcal{T}$. In this case we say that $\mathcal{I}$ is a *model* of $\mathcal{K}$, in symbols $\mathcal{I} \models \mathcal{K}$.
- Checking whether C is *satisfiable with respect to* $\mathcal{K}$ means checkin whether there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^{\mathcal{I}} \neq \emptyset$.
- Checking whether concept D *subsumes* concept C with respect to $\mathcal{K}$ (in symbols $\mathcal{K} \models C \sqsubseteq D$) means checkin whether, in every model $\mathcal{I}$ of $\mathcal{K}$, it holds that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- Checking whether two concepts C, D are *equivalent* with respect to $\mathcal{K}$ (in symbols $\mathcal{K} \models C \equiv D$) means checkin whether, in every model $\mathcal{I}$ of $\mathcal{K}$, it holds that $C^{\mathcal{I}} = D^{\mathcal{I}}$.
- Checking whether an individual a is an *instance* of C with respect to $\mathcal{K}$ (in symbols $\mathcal{K} \models C(a)$) means checkin whether, in every model $\mathcal{I}$ of $\mathcal{K}$, it holds that $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

Reduction to knowledge base consistency

Due to the classical semantics, in DL languages where all the boolean operators are present, each one of the above reasoning problems can be reduced to knowledge base (in)consistency in the following way:

- Concept C is satisfiable if and only if the knowledge base $\mathcal{K} = \{C(a)\}$ is consistent, where a is a new individual name.
- Concept C is satisfiable with respect to the knowledge base $\mathcal{K}$ if and only if the new knowledge base $\mathcal{K} \cup \{C(a)\}$ is consistent, where a is an individual name not occurring in $\mathcal{K}$.

- Concept D subsumes concept C with respect to the knowledge base $\mathcal{K}$ if and only if the new knowledge base $\mathcal{K} \cup \{(C \sqcap \neg D)(a)\}$ is inconsistent, for a new individual name a.

- Two concepts C, D are equivalent with respect to the knowledge base $\mathcal{K}$ if and only if the new knowledge base $\mathcal{K} \cup \{(C \sqcap \neg D)(a), (\neg C \sqcap D)(a)\}$ is inconsistent, for a new individual name a.

- An individual a is an instance of concept C with respect to the knowledge base $\mathcal{K}$ if and only if the new knowledge base $\mathcal{K} \cup \{(\neg C(a)\}$ is inconsistent.

Complexity

The study of the computational complexity of the reasoning tasks is fundamental in Description Logics and it has worked, since the beginning of the research on DL, as an engine for the improvements made on this subject. This is due to the fact that DLs have always been characterized by a tradeoff between expressivity and tractability of their languages, with the aim of searching for a fragment of first order logic that gives a good compromise between them. For many languages, the complexity classes they belong to, have been identified and often a systematic study of what causes the increment of complexity has been undertaken. Some examples of those systematic studies are:

- subsumption in language $\mathcal{FL}^-$ jumps from PTIME to co-NP ([Neb90]) when a terminology is considered,

- concept satisfiability in language $\mathcal{FL}^-$ jumps from PTIME to co-NP when disjunction and atomic complementation are added ([SSS91]),

- concept satisfiability in language $\mathcal{FL}^-$ jumps from PTIME to PSPACE when unrestricted complementation is added ([SSS91]),

- concept satisfiability in language $\mathcal{FL}^-$ jumps from PTIME to NP when unrestricted existential quantification is added ([DLN$^+$92]).

The classical complexity results we are more interested in are the following:

- Concept satisfiability for language $\mathcal{ALC}$ is PSPACE-complete ([SSS91]).

- Knowledge base consistency for language $\mathcal{ALC}$ is in EXPTIME ([DM00]).

This is so because they are generalized to the many-valued framework in the present dissertation.

Chapter 2

Fuzzy Description Logic

In this chapter we introduce our proposal of Fuzzy Description Logic following the guidelines provided by Hájek in [Háj05]. We introduce our proposal of a syntax and semantics for Fuzzy Description Logics up to the language that, in the classical case, would correspond to $\mathcal{ALC}$. We discuss the consequences that these choices have on the hierarchy of basic FDL languages. Moreover, we provide a translation from $\mathcal{ALC}$-like concepts to fuzzy first order formulas and prove that it preserves the meaning of the involved concepts. We also provide a translation from $\mathcal{ALC}$-like concepts to fuzzy multi-modal formulas and vice-versa and, again, prove that it preserves the meaning of the expressions involved. Finally, in a section dedicated to related works, we will report other proposals existing in the literature and discuss how our proposal fits among them.

The definitions provided in this chapter are thought over BL chains as algebras of truth values $\mathbf{T}$. Nevertheless, except for the definability of weak conjunction $\sqcap$ and weak disjunction $\sqcup$ they are true in general for MTL chains. The results are limited to $\mathbf{T}$ being either the Łukasiewicz, the Gödel or the Product standard chains (i.e. with domain $T = [0,1]$) in the infinite-valued case and $\mathbf{T}$ being a finite BL chain (over domain $T = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ in the $n+1$-valued case).

2.1 Syntax

In this section we introduce the syntax of concepts and fuzzy axioms.

2.1.1 Concepts

The signature in our proposal of FDLs is the same as for classical DLs since the difference is related to the semantics of concepts and roles. In fact here the semantics of concepts and roles will be fuzzy sets and fuzzy relations. A *description signature* is a tuple $\mathcal{D} = \langle N_I, N_A, N_R \rangle$, where:

- $N_I = \{a, b, \ldots\}$ is a countable set of *individual names*,

- $N_A = \{A, B, \dots\}$ is a countable set of *atomic concepts* or *concept names*,
- $N_R = \{R, S, \dots\}$ is a countable set of *atomic roles* or *role names*.

Complex concepts in the FDL languages considered in the present dissertation are built inductively from atomic concepts and roles by means of the corresponding subset of the following concept constructors:

C, D	$\longrightarrow$	$\bot$	empty concept	$\mathcal{FL}_0$
		$\top$	universal concept	$\mathcal{FL}_0$
		A	atomic concept	$\mathcal{FL}_0$
		$C \boxtimes D$	strong conjunction	$\mathcal{FL}_0$
		$\forall R.C$	value restriction	$\mathcal{FL}_0$
		$\exists R.\top$	restricted existential quantif.	$\mathcal{FL}^-$
		$\sim A$	atomic complementation	$\mathcal{AL}$
		$\overline{r} : r \in S$	constant concept	$\mathcal{X}^S$
		$\triangle C$	delta operator	$\mathcal{D}$
		$C \sqsupset D$	implication	$\mathfrak{I}$
		$C \sqcap D$	weak conjunction	$\mathfrak{I}$
		$C \sqcup D$	weak disjunction	$\mathfrak{I}$
		$\sim C$	complementation	$\mathcal{C}$
		$C \boxplus D$	strong disjunction	$\mathcal{U}$
		$\exists R.C$	existential quantification	$\mathcal{E}$

where $A \in N_A$, and $R \in N_R$.

The notation proposed here is thought in order to maintain, as much as possible, the similarity with classical DL notation while, at the same time, introducing the notation used in the framework of MFL. So:

- The language $\mathcal{FL}_0$, as in the classical case, contains
  - the empty concept $\bot$,
  - the universal concept $\top$,
  - the strong conjunction $\boxtimes$,
  - the value restriction $\forall R$,

  as concept constructors.
- The language $\mathcal{FL}^-$ is built, as in the classical case, by adding the restricted existential quantification $\exists R.\top$ to $\mathcal{FL}_0$.
- The language $\mathcal{AL}$ is built, again, as in the classical case, by adding the atomic complementation $\sim A$ to language $\mathcal{FL}^-$. In this case, we will use the symbol $\sim$ for complementation in languages that include $\mathcal{ALC}$, because it is traditionally used in MFL to denote the involutive negation.

- The language $\mathcal{X}^S$, where $\mathcal{X}$ stands for a generic FDL language and the superindex S denotes the domain of a suitable subalgebra of $\mathbf{T}$, contains a constant concept constructor $\overline{r}$ for each $r \in S$. For "the domain S of a suitable subalgebra" of $\mathbf{T}$ we mean, if not explicitly stated, the set $[0,1] \cap \mathbb{Q}$ in the infinite-valued case, and the full T in the finite-valued case.
- We introduce the symbol $\mathcal{D}$ for languages that have Delta operator $\triangle$.
- We prefix the symbol $\mathfrak{I}$ in those languages that have implication $\sqsupset$.
- We use symbol $\mathcal{U}$ in those languages that contain strong disjunction $\boxplus$.
- The name for languages that include the unrestricted existential quantification $\exists R.C$ will be denoted with $\mathcal{E}$ as in the classical case.

For the sake of clarity, sometimes it will be necessary to specify the algebra of truth values $\mathbf{T}$ considered. In those cases we will prepose the algebra name before the one of the FDL language, like $\mathbf{T}$-$\mathcal{FDL}$, otherwise, we will write $\mathcal{FDL}$.

2.1.2 Knowledge bases

Knowledge bases are defined as in the classical case, but in our framework inclusion and assertion axioms are graded, as usually done in the literature on FDL (see, for example [Str04a], [CGCE10]).

A *fuzzy concept inclusion axiom* (or *fuzzy inclusion*) is an expression of one of the following four forms:

$$\langle C \sqsubseteq D \geq r \rangle \quad \text{non-strict lower bound inclusion axioms} \quad (2.1)$$
$$\langle C \sqsubseteq D \leq r \rangle \quad \text{non-strict upper bound inclusion axioms} \quad (2.2)$$
$$\langle C \sqsubseteq D > r \rangle \quad \text{strict lower bound inclusion axioms} \quad (2.3)$$
$$\langle C \sqsubseteq D < r \rangle \quad \text{strict upper bound inclusion axioms} \quad (2.4)$$

where C, D are concepts and $r \in T \cap \mathbb{Q}$.

A *fuzzy concept assertion axiom* (or *fuzzy assertion*) is an expression of one of the following four forms:

$$\langle C(a) \geq r \rangle \quad \text{non-strict lower bound assertion axioms} \quad (2.5)$$
$$\langle C(a) \leq r \rangle \quad \text{non-strict upper bound assertion axioms} \quad (2.6)$$
$$\langle C(a) > r \rangle \quad \text{strict lower bound assertion axioms} \quad (2.7)$$
$$\langle C(a) < r \rangle \quad \text{strict upper bound assertion axioms} \quad (2.8)$$

where C is a concept, a is an individual name and $r \in T \cap \mathbb{Q}$.

Finally, a *fuzzy role assertion axioms* (or *fuzzy role assertion*) is an expression of the form:

$$\langle R(a,b) \geq r \rangle \quad \text{role assertion axioms} \quad (2.9)$$

where R is an atomic role, a, b are individual constants and $r \in T \cap \mathbb{Q}$. Note that for roles we only consider non-strict lower bound role assertions $\langle R(a,b) \geq r \rangle$. This is due to the fact that in the literature only fuzzy role assertions like (2.9) are considered.

A *knowledge base* (KB) for the languages considered in this dissertation has two components: a *fuzzy terminological box* or *fuzzy ontology* (TBox) and an *fuzzy assertional box* (ABox).

A fuzzy TBox for an FDL language is a finite set of fuzzy inclusions. A fuzzy ABox is a finite set of fuzzy assertions and role assertions. A fuzzy KB is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where the first component is a fuzzy TBox and the second one is a fuzzy ABox.

2.2 Semantics

2.2.1 Concepts

Given a BL-chain $\mathbf{T} = \langle T, *, \Rightarrow, \wedge, \vee, 0, 1 \rangle$[1], a $\mathbf{T}$-*interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a nonempty (crisp) set $\Delta^{\mathcal{I}}$ (called *domain*) and of a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns:

1. to each concept name $A \in N_C$ a fuzzy set, that is, a function $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow T$,

2. to each role name $R \in N_R$ a fuzzy relation, that is, a function $R^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \longrightarrow T$,

3. to each individual name $a \in N_I$ an object $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ such that $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ if $a \neq b$ (*Unique Name Assumption*, different individuals denote different objects of the domain).

The semantics of complex concepts is a function $C^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$ inductively defined as follows:

---

[1]Note that we are indeed considering algebras with domain either [0,1] or $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ and sometimes expanded with either operation δ, a set of 0-ary operations (truth constants) $T \cap \mathbb{Q}$ or the involutive negation $1 - x$. We remind that maximum $\veebar$ is defined by

$$x \veebar y := 1 - ((1-x) * (1-y)).$$

$$
\begin{array}{rcl}
\bot^{\mathcal{I}}(x) & := & 0 \\
\top^{\mathcal{I}}(x) & := & 1 \\
\overline{r}^{\mathcal{I}}(x) & := & r \\
(\sim C)^{\mathcal{I}}(x) & := & 1 - C^{\mathcal{I}}(x) \\
(\triangle C)^{\mathcal{I}}(x) & := & \triangle C^{\mathcal{I}}(x) \\
(C \boxtimes D)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) * D^{\mathcal{I}}(x) \\
(C \sqcap D)^{\mathcal{I}}(x) & := & \min\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \\
(C \boxplus D)^{\mathcal{I}}(x) & := & 1 - ((1 - C^{\mathcal{I}}(x)) * (1 - D^{\mathcal{I}}(x))) \\
(C \sqcup D)^{\mathcal{I}}(x) & := & \max\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \\
(C \sqsupset D)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \\
(\forall R.C)^{\mathcal{I}}(x) & := & \inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\} \\
(\exists R.C)^{\mathcal{I}}(x) & := & \sup_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(y)\}
\end{array}
$$

Note that, from the above semantics, residuated negation $\neg$, weak conjunction $\sqcap$ and weak disjunction $\sqcup$ are present in languages that include $\mathfrak{IFL}_0$ because, if the algebra of truth values $\mathbf{T}$ is a BL chain, these operators are definable from the implication $\sqsupset$, the strong conjunction $\boxtimes$ and the empty concept $\bot$. In fact:

- The constructor of weak conjunction $\sqcap$ is definable from the implication and the strong conjunction in the following way:

$$C \sqcap D := C \boxtimes (C \sqsupset D).$$

- The constructor of weak disjunction $\sqcup$ is definable from the implication and the weak conjunction in the following way:

$$C \sqcup D := ((C \sqsupset D) \sqsupset D) \sqcap ((D \sqsupset C) \sqsupset C).$$

- The constructor of residuated negation $\neg$ is definable from the implication and the empty concept in the following way:

$$\neg C := C \sqsupset \bot.$$

Hence, these operators can be considered as abbreviations in the language $\mathfrak{IFL}_0$ and in every language expanding it.

2.2.2 Fuzzy axioms

From the semantics of concepts we can define the semantics of fuzzy axioms. We say that a **T**-interpretation $\mathcal{I}$ *satisfies* axioms of type (2.1), (2.2), (2.3) and (2.4) respectively, if

$$\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \geq r \tag{2.10}$$

$$\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \leq r \tag{2.11}$$

$$\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} > r \tag{2.12}$$

$$\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} < r \tag{2.13}$$

We say that a **T**-interpretation $\mathcal{I}$ *satisfies* axioms of type (2.5), (2.6), (2.7) and (2.8) respectively if

$$C^{\mathcal{I}}(a^{\mathcal{I}}) \geq r \tag{2.14}$$

$$C^{\mathcal{I}}(a^{\mathcal{I}}) \leq r \tag{2.15}$$

$$C^{\mathcal{I}}(a^{\mathcal{I}}) > r \tag{2.16}$$

$$C^{\mathcal{I}}(a^{\mathcal{I}}) < r \tag{2.17}$$

We say that a **T**-interpretation $\mathcal{I}$ *satisfies* axioms of type (2.9), if

$$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq r \tag{2.18}$$

As we do for languages, when the algebra **T** is clear from the context or it does not matter which algebra is considered (e.g. the result explained does not depend on a particular algebra **T**), we will simply speak about *interpretations*.

2.2.3 Witnessed, quasi-witnessed and strongly witnessed interpretations

As we have already mentioned in Section 1.1.3 for the case of first order fuzzy logic, interpretations of concepts of type $\forall R.C$ and $\exists R.C$, need not have a witness when we deal with an infinite set of truth values. Taking these considerations into account, in [Háj05] the notion of *witnessed interpretation* has been introduced. Since then most researchers preferred to restrict the reasoning tasks to witnessed interpretations because it seems a quite natural restriction and reasoning tasks so restricted have a very good computational behavior.

Definition 2.2.1 (*Witnessed interpretation* [Háj05])**.** An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is *witnessed* in case that

(wit∃) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ there is $b \in \Delta^{\mathcal{I}}$ such that
$$(\exists R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a, b) * C^{\mathcal{I}}(b),$$

(wit∀) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ there is $b \in \Delta^{\mathcal{I}}$ such that
$$(\forall R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a, b) \Rightarrow C^{\mathcal{I}}(b).$$

Following the definition of *closed model* from [LM07], in [CEB10] the notion of *quasi-witnessed interpretation* has been introduced.

Definition 2.2.2 (*Quasi-witnessed interpretation* [CEB10])**.** An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is *quasi-witnessed* when it satisfies condition (**wit**∃) and

(qwit∀) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ either $(\forall R.C)^{\mathcal{I}}(a) = 0$ or there is $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) \ = \ R^{\mathcal{I}}(a, b) \Rightarrow C^{\mathcal{I}}(b).$$

In [BP11b] the notion of *strongly witnessed interpretation* has been introduced.

Definition 2.2.3 (*Strongly witnessed interpretation* [BP11b])**.** An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is *strongly witnessed* when it satisfies conditions (**wit**∃), (**wit**∀) and moreover

(swit∀) for every pair of concepts C, D, there is some $b \in \Delta^{\mathcal{I}}$ such that

$$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} = C^{\mathcal{I}}(b) \Rightarrow D^{\mathcal{I}}(b)$$

This further restriction to the notion of witnessed interpretation is not so much used because, if $\mathbf{T}$ is a continuous chain, it imposes a too strict constraint to the interpretations considered. Indeed, as mentioned in [BP11b], "it does not capture the spirit of fuzzy concept inclusions", since "it is not really necessary that the infimum of the values for the residuum is indeed reached". On the other hand, if $\mathbf{T}$ is a finite chain, it is straightforward that every interpretation $\mathcal{I}$ is strongly witnessed.

2.3 The Hierarchy of basic FDL languages

Due to the above defined semantics, in our framework under strict standard semantics the languages $\mathcal{ALE}$, $\mathcal{ALD}$ and $\mathfrak{IAL}$ are not strictly contained in $\mathcal{ALC}$. This is due to the fact that, in the logic of a strict t-norm, implication is not definable from conjunction and negation (neither the residuated negation, nor the involutive one). The existential quantifier is not definable from the universal one by means of the negation (neither the residuated negation, nor the involutive one) and the same definition as in classical DL. Moreover, the Delta constructor $\triangle$ needs the residuated negation $\neg$ (defined from the implication $\sqsupset$ and the constant $\bot$) and the involutive negation $\sim$ to become definable.

Since in our framework we do not have the same possibility of reducing languages like in the classical case, the hierarchy of basic languages obtained is more cumbersome. The new hierarchy of FDL languages over the logic of a strict t-norm is represented in Figure 2.1, that shows the partially ordered set of

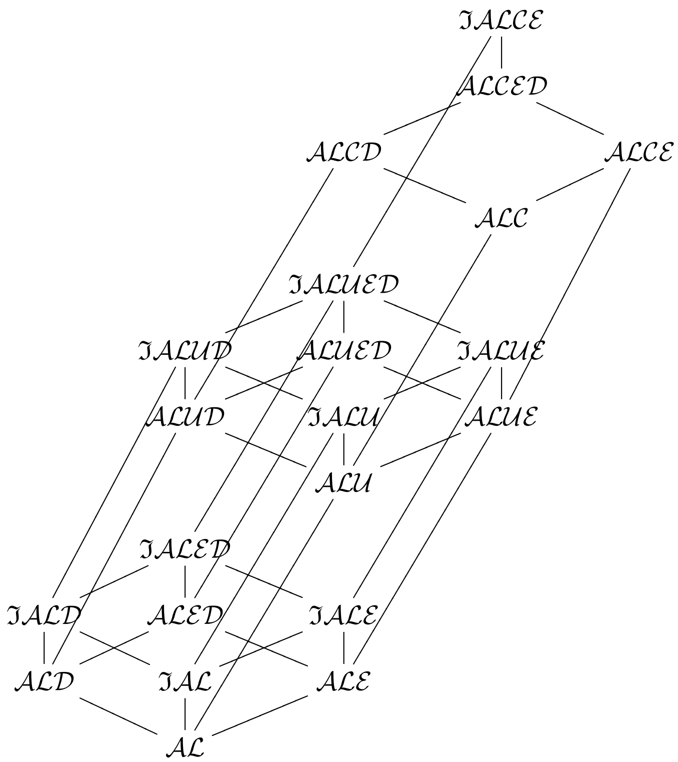

Figure 2.1: Hierarchy of basic languages under strict standard semantics

inclusions among the languages obtained by successively adding a basic operator or another.

In languages with complementation $\sim$, the strong disjunction $\boxplus$ is definable from the strong conjunction $\boxtimes$ and $\sim$ by the following De Morgan law, i.e., as

$$C \boxplus D := \sim(\sim C \boxtimes \sim D)$$

hence, the language $\mathcal{ALU}$ is strictly contained in the language $\mathcal{ALC}$.

Notice that the top of the poset in Figure 2.1 will be called in our framework $\mathfrak{I}\mathcal{ALCE}$, instead of $\mathcal{ALC}$, as in the classical case.

Figure 2.1 represent the worst scenario. Indeed, hierarchy in Figure 2.1 can be simplified when we deal with infinite Łukasiewicz Logic. In this case, indeed, the fact that the residuated negation is involutive implies that $\mathcal{E}$ and $\mathcal{U}$ are definable by duality from value restriction and the strong conjunction respectively, by means of the involutive negation as is usually done in classical DL and the same holds for the constructor of implication. Thus, in the case of FDLs based on Łukasiewicz Logic, the languages $\mathcal{ALCE}$, $\mathfrak{I}\mathcal{ALC}$, $\mathfrak{I}\mathcal{ALCE}$ coincide with $\mathcal{ALC}$. Nevertheless, due to the fact that in $[0,1]_{\text{Ł}}$ the constructor $\triangle$ is not

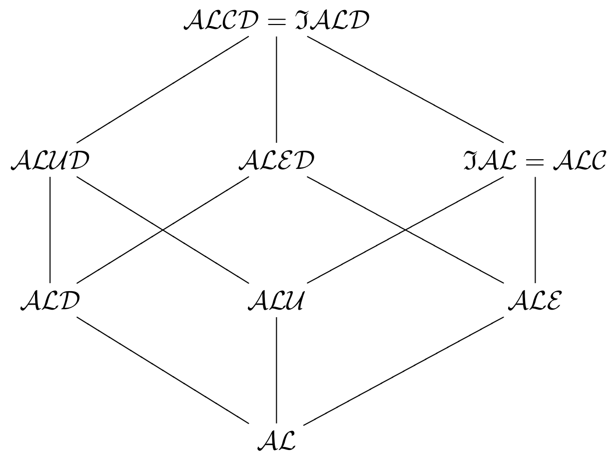

Figure 2.2: Hierarchy of basic languages under $[0,1]_{\text{Ł}}$

definable at all, the hierarchy of basic languages for Łukasiewicz is still more cumbersome than in the classical case. The new hierarchy of basic languages based on infinite Łukasiewicz standard semantics is given in Figure 2.2, that shows the partially ordered set of inclusions among the languages obtained by successively adding a basic operator or another.

It is worth remarking that the presence of truth constants modifies neither the hierarchy in Figure 2.1 nor the one in Figure 2.2.

Finally, in the case of finite Łukasiawicz t-norm, Ł_n, since the constructor $\triangle$ is definable already in language $\mathcal{FL}_0$ as

$$\triangle C := C^n$$

where C^n stands for $C \boxtimes \overset{n}{\ldots} \boxtimes C$, we have that the hierarchy of basic languages is the same as in the classical case, i.e. the one represented in Figure 1.2.

2.4 Simplifying knowledge bases

Since in FDL there is the possibility of considering a graded notion of subsumption, equivalence and assertion, there are obviously more types of fuzzy axioms in FDL than crisp axioms in classical DL, as we have seen in Section 2.1.2. A question that naturally arises is, then, whether those types can be simplified, that is, whether there are axioms that can be defined in terms of other axioms, as it is done in classical DL for the case e.g. of the equivalence axioms, that can be expressed as a conjunction of inclusion axioms. So, for simplification of knowledge bases we mean that, given a knowledge base $\mathcal{K}$, we can obtain a knowledge base $\mathcal{K}'$ that is satisfied by the same interpretations as $\mathcal{K}$, but is expressed in a simpler syntax, that is, by means of a smaller number of axiom types.

Besides the fact that, in general, a richer logical language allows to simplify knowledge bases, we have to consider the cases of different standard algebras of truth values $\mathbf{T}$ separately, because there are simplifications that can be performed under certain standard algebras but not under others. Moreover, depending on the standard algebra $\mathbf{T}$ considered, there are simplifications that can be performed in more or less rich FDL languages.

2.4.1 The case of infinite-valued Łukasiewicz Logic

We begin with the case when $\mathbf{T} = [0,1]_{Ł}$. In this case the presence of an involutive negation $\sim$ or delta operator $\triangle$ and truth constants from $\mathbb{Q} \cap [0,1]$ make a substantial difference in the simplifications that can be performed on the set of axioms.

Language $\mathcal{FL}_0$

There are kinds of simplifications that can be performed already in the more basic FDL languages. Exact value axioms $\langle C \sqsubseteq D = r\rangle$ and $\langle C(a) = r\rangle$ as well as equivalence axioms $\langle C \equiv D \rhd r\rangle$ are often considered in the literature. Here we explain why we are not considering them in our framework. Note that the simplification performed on equivalence axioms is the same that is usually considered in classical DL.

Exact value axioms Axioms of types:

$$\langle C \sqsubseteq D = r\rangle, \qquad \langle C(a) = r\rangle$$

are abbreviations for the simultaneous presence of non-strict lower and upper bound axioms, i.e. axioms (2.1) and (2.2) in the first case, and axioms (2.5) and (2.6) in the second case. In other words, the knowledge base $\mathcal{K} \cup \{\langle C \sqsubseteq D = r\rangle\}$ can be substituted by the knowledge base $\mathcal{K} \cup \{\langle C \sqsubseteq D \geq r\rangle, \langle C \sqsubseteq D \leq r\rangle\}$. The same holds for axioms $\langle C(a) = r\rangle$.

Fuzzy equivalence axioms Differently from the classical case, a fuzzy equivalence axiom can not always be equivalently substituted by a couple of fuzzy inclusions. Consider, for example the fuzzy equivalence

$$\langle C \equiv D \leq r\rangle \tag{2.19}$$

Clearly axiom (2.19) is not equivalent to the couple of fuzzy inclusions

$$\langle C \sqsubseteq D \leq r\rangle \tag{2.20}$$
$$\langle D \sqsubseteq C \leq r\rangle \tag{2.21}$$

because there are interpretations that satisfy e.g. (2.20) (and, hence, (2.19)), but not (2.21) (and, hence, not both). Notice that the same holds when, instead of a non-strict upper bound $\leq$, the equivalence has either a strict upper bound $<$ or an equality $=$.

On the contrary, as in the classical framework, the constraints expressed by means of (either strict or not) lower bound equivalence axioms can be expressed by means of the simultaneous presence of the corresponding two fuzzy inclusion axioms. This means that the knowledge base

$$\mathcal{K} \cup \{\langle C \equiv D \geq r \rangle\}$$

can be substituted by the knowledge base

$$\mathcal{K} \cup \{\langle C \sqsubseteq D \geq r \rangle, \langle D \sqsubseteq C \geq r \rangle\},$$

with $r \in T$. Of course, the same holds true with $>$, instead of $\geq$.

Summarizing:

- Axioms of type $\langle C \equiv D = r \rangle$ can be substituted by the couple of axioms $\langle C \equiv D \geq r \rangle$ and $\langle C \equiv D \leq r \rangle$.
- Axioms of type $\langle C \equiv D \geq r \rangle$ can be substituted by the couple of axioms $\langle C \sqsubseteq D \geq r \rangle$ and $\langle D \sqsubseteq C \geq r \rangle$.
- Axioms of type $\langle C \equiv D > r \rangle$ can be substituted by the couple of axioms $\langle C \sqsubseteq D > r \rangle$ and $\langle D \sqsubseteq C > r \rangle$.

Each knowledge base without exact value equivalences can be equivalently reduced to another one containing only (either strict or not) upper bound equivalence axioms.

Language $\mathcal{FL}_0\mathcal{C}$

Since in our framework we are considering the involutive negation $\sim$ whose truth function is $1 - x$, with $x \in [0, 1]$, we have the possibility of expressing axioms of types (2.6) and (2.8) in terms of axioms of types (2.5) and (2.7) in the following way:

- Axioms of type $\langle C(a) \leq r \rangle$ can be rewritten as $\langle \sim C(a) \geq 1 - r \rangle$.
- Axioms of type $\langle C(a) < r \rangle$ can be rewritten as $\langle \sim C(a) > 1 - r \rangle$.

Language $\mathfrak{I}\mathcal{FL}_0\mathcal{D}^{\mathbb{Q}\cap[0,1]}$

When the operator $\triangle$ and truth constants from $\mathbb{Q} \cap [0, 1]$ are present in the language as well as the implication $\sqsupset$, we have the possibility of expressing axioms of types (2.7) and (2.8) in terms of axioms of type (2.5) in the following way:

- Axioms of type $\langle C(a) > r \rangle$ can be rewritten as $\langle \neg \triangle (C \sqsupset \overline{r})(a) \geq 1 \rangle$.
- Axioms of type $\langle C(a) < r \rangle$ can be rewritten as $\langle \neg \triangle (\overline{r} \sqsupset C)(a) \geq 1 \rangle$.

Summary

In Table 2.1 we summarize which types of axioms are necessary in the presence of some constructors in any language extending $\mathcal{FL}_0$.

	Axioms								
Language	(2.1)	(2.2)	(2.3)	(2.4)	(2.5)	(2.6)	(2.7)	(2.8)	(2.9)
$\mathcal{FL}_0$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$+\ \mathcal{C}/\mathfrak{I}$	$\times$	$\times$	$\times$	$\times$	$\times$		$\times$		$\times$
$+\ \mathcal{C}/\mathfrak{I}, \mathcal{D}, ^{\mathbb{Q}\cap[0,1]}$	$\times$	$\times$	$\times$	$\times$	$\times$				$\times$

Table 2.1: Minimal types of axioms under Łukasiewicz standard semantics

2.4.2 The cases of infinite-valued Product and Gödel logics

When either $\mathbf{T} = [0,1]_\Pi$ or $\mathbf{T} = [0,1]_G$, more or less the same simplifications on the types of axioms can be performed as in the case when $\mathbf{T} = [0,1]_{\text{Ł}}$. However, there are differences that we will report in what follows. Notice that having the implication constructor $\sqsupset$ in the language does not imply the definability of complementation, and conversely. Nevertheless the constructor $\triangle$ is definable as soon as the implication $\sqsupset$ and the complementation $\sim$ are both in the language just considering

$$\triangle C \quad := \quad \neg \sim C \tag{2.22}$$

Language $\mathfrak{I}\mathcal{FL}_0\mathcal{C}^{\mathbb{Q}\cap[0,1]}$

From the definability of the Delta operator $\triangle$ we obtain that, when either $\mathbf{T} = [0,1]_\Pi$ or $\mathbf{T} = [0,1]_G$, a rewriting of axioms of type (2.7) in terms of axioms of type (2.5) can be performed already in the language $\mathfrak{I}\mathcal{FL}_0\mathcal{C}^{\mathbb{Q}\cap[0,1]}$ in the following way:

Axioms of type $\langle C(a) > r\rangle$ can be rewritten as $\langle \neg \triangle (C \sqsupset \overline{r})(a) \geq 1\rangle$.

Summary

In Table 2.2 we summarize which types of axioms are necessary in the presence of some constructors in any language extending $\mathcal{FL}_0$.

2.4.3 The case of finite t-norms

When the algebra of truth values $\mathbf{T}$ is a finite t-norm, that is, either G_n, Ł_n or any ordinal sum of copies of them, more simplifications than in the infinite-valued case can be performed in simpler languages.

Language	Axioms (2.1)	(2.2)	(2.3)	(2.4)	(2.5)	(2.6)	(2.7)	(2.8)	(2.9)
$\mathcal{FL}_0$	×	×	×	×	×	×	×	×	×
$+\ \mathcal{C}$	×	×	×	×	×		×		×
$+\ \mathfrak{I}, \mathcal{C}, ^{\mathbb{Q}\cap[0,1]}$	×	×	×	×	×				×

Table 2.2: Minimal types of axioms under either product or Gödel standard semantics

Language $\mathcal{FL}_0$

In the case of finite t-norms, besides the simplifications concerning exact value axioms and equivalence axioms, already in the language $\mathcal{FL}_0$ strict bound axioms (2.3), (2.4), (2.7) and (2.8) can be rewritten in terms of non-strict axioms (2.1), (2.2), (2.5) and (2.6), respectively. This is due to the fact that, with a finite set of truth values, if a truth value is strictly greater than a given value $r \in T$, then it is greater or equal to the lower truth value greater or equal than r. Note that, when $\mathbf{T}$ is a finite BL chain, there is always such a value. So, let $\mathcal{I}$ be an FDL interpretation and let $T = \{r_0, r_1, \ldots, r_n\}$ be such that its domain is ordered by $0 = r_0 < r_1 < \ldots < r_n = 1$, then:

- Axiom $\langle C \sqsubseteq D > r_i \rangle$, with $i < 1$, can be rewritten in terms of axiom $\langle C \sqsubseteq D \geq r_{i+1} \rangle$,
- Axiom $\langle C \sqsubseteq D < r_i \rangle$, with $i > 0$, can be rewritten in terms of axiom $\langle C \sqsubseteq D \leq r_{i-1} \rangle$,
- Axiom $\langle C(a) > r_i \rangle$, with $i < 1$, can be rewritten in terms of axiom $\langle C(a) \geq r_{i-1} \rangle$,
- Axiom $\langle C(a) < r_i \rangle$, with $i > 0$, can be rewritten in terms of axiom $\langle C(a) \leq r_{i-1} \rangle$.

Language $\mathcal{FL}_0\mathcal{C}$

In the same way as for the infinite-valued case, in presence of an involutive negation $\sim$, we have the possibility of expressing axioms of types (2.6) and (2.8) in terms of axioms of types (2.5) and (2.7).

Summary

In Table 2.3 we summarize which types of axioms are necessary in the presence of some constructors in any language extending $\mathcal{FL}_0$.

	Axioms				
Language	(2.1)	(2.2)	(2.5)	(2.6)	(2.9)
$\mathcal{FL}_0$	×	×	×	×	×
$\mathcal{FL}_0\mathcal{C}$	×	×	×		×

Table 2.3: Simplfying KBs under finite t-norms semantics

2.5 Reasoning tasks

Among the reasoning tasks that can be defined in a multi-valued framework we find the generalization of the ones that are usual in a classical framework, as reported in Section 1.2.5. Being the logic many-valued, these tasks can be considered in their graded versions. In addition to these reasoning tasks, more tasks which are proper of a many-valued framework have been introduced in the literature. In what follows, let $r \in T$.

- **Fuzzy knowledge base consistency** is the problem of checking whether, for a given fuzzy KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ there is a $\mathbf{T}$-interpretation $\mathcal{I}$ that satisfies every axiom in $\mathcal{K}$; in this case we say that the KB is *consistent* and that $\mathcal{I}$ *satisfies* $\mathcal{K}$, in symbols $\mathcal{I} \models \mathcal{K}$.

- Different notions of the **concept satisfiability** with respect to a (possibly empty) knowledge base $\mathcal{K}$ have been considered in the literature.

  - *Lower bound* r-satisfiability is the problem of checking whether, for concept C, there exists a $\mathbf{T}$-interpretation $\mathcal{I}$ which satisfies $\mathcal{K}$ and an object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) \geq r$. In this case we say that concept C is $(\geq r)$-*satisfiable.*
  - *Exact value* r-satisfiability is the problem of checking whether, for concept C, there exists a $\mathbf{T}$-interpretation $\mathcal{I}$ which satisfies $\mathcal{K}$ and an object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = r$. In this case we say that concept C is r-*satisfiable.* In the particular case when $r = 1$, we will simply say that C is *satisfiable.*
  - *Positive* satisfiability is the problem of checking whether, for concept C, there exists a $\mathbf{T}$-interpretation $\mathcal{I}$ which satisfies $\mathcal{K}$, an object $a \in \Delta^{\mathcal{I}}$ and a truth value $s \in T$, with $s > 0$, such that $C^{\mathcal{I}}(a) = s$. In this case we say that concept C is *positively satisfiable* or *consistent.*

  Notice that, as we have seen in Section 1.1.1, when the knowledge base $\mathcal{K}$ is indeed empty and the algebra $\mathbf{T}$ considered is either $[0,1]_\Pi$ or $[0,1]_G$, the notions of $(\geq r)$-satisfiability and r-satisfiability may not make sense when $r < 1$. The notions of $(\geq r)$- and r-satisfiability will be used only when $\mathbf{T}$ is $[0,1]_{\text{Ł}}$.

- **Concept r-subsumption** is the problem of checking whether, given concepts C, D, for every $\mathbf{T}$-interpretation $\mathcal{I}$ and every $a \in \Delta^{\mathcal{I}}$, it holds that $C^{\mathcal{I}}(a) \Rightarrow D^{\mathcal{I}}(a) \geq r$, in this case we say that concept D *subsumes* concept C *in a degree greater or equal to* r (or that D *r-subsumes* C).

- **Entailment of an axiom by a knowledge base** is the problem of checking whether, for a given fuzzy axiom φ and a fuzzy KB $\mathcal{K}$, every $\mathbf{T}$-interpretation $\mathcal{I}$ which satisfies $\mathcal{K}$, also satisfies φ; in this case we say that $\mathcal{K}$ *entails* φ, in symbols $\mathcal{K} \models \varphi$.

- The **best satisfiability degree of a concept with respect to a KB** (defined in [SB07]) is the problem of determining, for a given fuzzy concept C and a fuzzy knowledge base $\mathcal{K}$, the supremum of the satisfaction degree of C by interpretations satisfying $\mathcal{K}$; that is, $bsd(\mathcal{K}, C) = \sup_{\mathcal{I} \models \mathcal{K}}\{\sup_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x)\}\}$.

- The **best entailment degree of an axiom with respect to a KB** (defined in [Str01]) is the problem of determining, for a given (non-fuzzy) axiom $\varphi = C \sqsubseteq D$ or $\varphi = C(a)$ and a fuzzy knowledge base $\mathcal{K}$, the supremum of $r \in T$ with respect to which $\langle \varphi \geq r \rangle$ is entailed by $\mathcal{K}$; that is, $bed(\mathcal{K}, \varphi) = \sup\{r \colon \mathcal{K} \models \langle \varphi \geq r \rangle\}$.

2.5.1 Reductions among reasoning tasks

In the classical framework it is usual to consider reductions between reasoning tasks in order to apply procedures, that have been designed for a given task, to other tasks that are reducible to the given one. In particular, within the classical framework, every reasoning task can be polynomially reduced to knowledge base (in)consistency (see [BHS08, pag. 142]). In this section we will see which reductions can be performed in FDLs.

Proposition 2.5.1 (Reduction to KB consistency)**.** *Let* $\mathbf{T}$ *be in the family* $\{[0,1]_{Ł}, [0,1]_{\Pi}, [0,1]_{G}, Ł_n, G_n\}$*, then the following statements hold for language* $\mathbf{T}$*-*$\mathcal{FL}_0$*:*

1. *Concept r-satisfiability with respect to a (possibly empty) KB can be polynomially reduced to KB consistency.*

2. *Concept $(\geq r)$-satisfiability with respect to a (possibly empty) KB can be polynomially reduced to KB consistency.*

3. *Concept positive satisfiability with respect to a (possibly empty) KB can be polynomially reduced to KB consistency.*

4. *Concept r-subsumption can be polynomially reduced to KB consistency.*

5. *Entailment of an axiom by a KB can be polynomially reduced to KB consistency.*

Proof. 1. A concept C is r-satisfiable with respect to knowledge base $\mathcal{K}$ if and only if $\mathcal{K} \cup \{\langle C(a) \geq r \rangle, \langle C(a) \leq r \rangle\}$ is consistent, where $a \in N_I$ does not occur in $\mathcal{K}$.

2. A concept C is $(\geq r)$-satisfiable with respect to knowledge base $\mathcal{K}$ if and only if $\mathcal{K} \cup \{\langle C(a) \geq r \rangle\}$ is consistent, where $a \in N_I$ does not occur in $\mathcal{K}$.

3. A concept C is positively satisfiable with respect to knowledge base $\mathcal{K}$ if and only if $\mathcal{K} \cup \{\langle C(a) > 0 \rangle\}$ is consistent, where $a \in N_I$ does not occur in $\mathcal{K}$.

4. A concept C is r-subsumed by concept C if and only if the knowledge base $\mathcal{K} = \{\langle C \sqsubseteq D < r \rangle\}$ is inconsistent.

5. In the case of entailment, the way the reduction is performed, depends on the type of axiom entailed.

   - A knowledge base $\mathcal{K}$ entails an axiom $\langle C \sqsubseteq D \geq r \rangle$ if and only if $\mathcal{K} \cup \{\langle C \sqsubseteq D < r \rangle\}$ is inconsistent.
   - A knowledge base $\mathcal{K}$ entails an axiom $\langle C \sqsubseteq D \leq r \rangle$ if and only if $\mathcal{K} \cup \{\langle C \sqsubseteq D > r \rangle\}$ is inconsistent.
   - A knowledge base $\mathcal{K}$ entails an axiom $\langle C \sqsubseteq D > r \rangle$ if and only if $\mathcal{K} \cup \{\langle C \sqsubseteq D \leq r \rangle\}$ is inconsistent.
   - A knowledge base $\mathcal{K}$ entails an axiom $\langle C \sqsubseteq D < r \rangle$ if and only if $\mathcal{K} \cup \{\langle C \sqsubseteq D \geq r \rangle\}$ is inconsistent.
   - A knowledge base $\mathcal{K}$ entails an axiom $\langle C(a) \geq r \rangle$ if and only if $\mathcal{K} \cup \{\langle C(a) < r \rangle\}$ is inconsistent.
   - A knowledge base $\mathcal{K}$ entails an axiom $\langle C(a) \leq r \rangle$ if and only if $\mathcal{K} \cup \{\langle C(a) > r \rangle\}$ is inconsistent.
   - A knowledge base $\mathcal{K}$ entails an axiom $\langle C(a) > r \rangle$ if and only if $\mathcal{K} \cup \{\langle C(a) \leq r \rangle\}$ is inconsistent.
   - A knowledge base $\mathcal{K}$ entails an axiom $\langle C(a) < r \rangle$ if and only if $\mathcal{K} \cup \{\langle C(a) \geq r \rangle\}$ is inconsistent. □

Since in the languages considered in this dissertation there is only one type of role assertion axioms and the role negation constructor is lacking, entailment of role axioms (2.9) cannot be reduced to KB consistency.

Moreover, we will consider reductions to reasoning tasks other than KB consistency that will be useful in the following chapters. First of all we will see how to reduce KB consistency to other reasoning tasks.

Proposition 2.5.2 (Reduction of KB consistency)**.** *Let* $\mathbf{T}$ *be in the family* $\{[0,1]_Ł, [0,1]_\Pi, [0,1]_G, L_n, G_n\}$*, then the following statements hold for language* $\mathbf{T}$-$\mathcal{FL}_0^T$*:*

1. *KB consistency can be polynomially reduced to concept* r*-satisfiability with respect to a non-empty KB.*

2. *KB consistency can be polynomially reduced to concept 1-satisfiability with respect to a non-empty KB.*

3. *KB consistency can be polynomially reduced to concept positive satisfiability with respect to a non-empty KB.*

4. *KB consistency can be polynomially reduced to the entailment of an axiom by a KB.*

Proof. 1. Knowledge base $\mathcal{K}$ is consistent if and only if concept $\overline{r}$ is r-satisfiable w.r.t. $\mathcal{K}$.

2. Knowledge base $\mathcal{K}$ is consistent if and only if concept $\top$ is 1-satisfiable w.r.t. $\mathcal{K}$.

3. Knowledge base $\mathcal{K}$ is consistent if and only if concept $\top$ is positively satisfiable w.r.t. $\mathcal{K}$.

4. Knowledge base $\mathcal{K}$ is consistent if and only if axiom $\langle \bot \geq 1 \rangle$ is not entailed by $\mathcal{K}$ if and only if axiom $\langle \top \sqsubseteq \bot \geq 1 \rangle$ is not entailed by $\mathcal{K}$. □

Another interesting item is the reduction of reasoning tasks without knowledge bases to concept r-satisfiability and r-subsumption. in this case, however, we have to distinguish the cases depending on the standard algebra $\mathbf{T}$ considered.

Proposition 2.5.3. *Let* $\mathbf{T}$ *be in the family* $\{[0,1]_Ł, [0,1]_\Pi, [0,1]_G, Ł_n, G_n\}$*, then the following statements hold for language* $\mathbf{T}$-$\Im\mathcal{FL}_0^T$*:*

1. *Concept* $(\geq r)$*-satisfiability can be polynomially reduced to concept 1-satisfiability.*

2. *Concept positive satisfiability can be polynomially reduced to concept 1-subsumption.*

Proof. 1. Concept C is $(\geq r)$-satisfiable if and only if concept concept $\overline{r} \sqsupset C$ is 1-satisfiable.

2. Concept C is positively satisfiable if and only if concept concept C is not 1-subsumed by $\bot$. □

Proposition 2.5.4. *Let* $\mathbf{T} \in \{[0,1]_\Pi, [0,1]_G\}$ *and* $r \in (0,1]$*, then the following statements hold for language* $\mathcal{FL}_0$*:*

1. *Concept* $(\geq r)$*-satisfiability and positive satisfiability are equivalent problems.*

2. *Concept positive satisfiability can be polynomially reduced to concept 1-subsumption.*

Proof. 1. For every $r \in (0,1]$ it is straightforward that, if C is $(\geq r)$-satisfiable, then it is positively satisfiable. So, we need to prove that if concept C is positively satisfiable, then it is $(\geq r)$-satisfiable. Suppose that C is positively satisfiable, there exists a $[0,1]_*$-interpretation (with $* \in \{\Pi, G\}$) $\mathcal{I}$ and $a \in \Delta^{\mathcal{I}}$ and $s \in (0,1)$ such that $C^{\mathcal{I}}(a) = s$. Without loss of generality, we can suppose that $s < r$. It is well known (see for instance [SCE$^+$06]) that we can obtain a $[0,1]_*$-interpretation $\mathcal{I}'$ such that $C^{\mathcal{I}'}(a) = r$.

2. As in the Łukasiewicz case, concept C is positively satisfiable if and only if concept C is not 1-subsumed by $\perp$. □

Notice that, in language $[0,1]_{\text{Ł}}$-$\mathfrak{I}\mathcal{FL}_0$, a concept C can be positively satisfiable, but not $(\geq r)$-satisfiable, for some $r \in [0,1]$. As an example, consider concept $A \sqcap \neg A$. This is indeed positively satisfiable under $[0,1]_{\text{Ł}}$, but not, say, (≥ 0.7)-satisfiable.

Proposition 2.5.5. *Let $\mathbf{T} \in \{L_n, G_n\}$, then the following statements hold for language $\mathfrak{I}\mathcal{FL}_0\mathcal{C}^T$:*

1. *Concept r_m-subsumption, can be polynomially reduced to concept $(\geq 1 - r_{m-1})$-satisfiability, when $r_m > 0$.*

2. *For every $r \in T$, concept $(\geq r)$-satisfiability can be polynomially reduced to concept 1-satisfiability.*

3. *Concept positive satisfiability can be polynomially reduced to concept 1-satisfiability.*

Proof. 1. Concept C is r_m-subsumed by concept D if and only if concept $\sim (C \sqsupset D)$ is not $(\geq 1 - r_{m-1})$-satisfiable.

2. Concept C is $(\geq r)$-satisfiable if and only if concept $\overline{r} \sqsupset C$ is 1-satisfiable.

3. Concept C is positively satisfiable if and only if concept $\overline{r_1} \sqsupset C$ is 1-satisfiable, where r_1 is the lower truth value strictly greater than 0.

□

2.6 Relation to first order predicate logic

In [Bor96], Borgida provides a translation of DL concepts into first order classical logic. The relationship between FDL and first order fuzzy logic has been firstly described in [TM98]. A more systematic investigation on this subject has been undertaken in [GCAE10] and [CGCE10], where it is investigated the idea, presented in [Háj05] of a Fuzzy Description Logic tightly related to Mathematical Fuzzy Logic. In [TM98], [GCAE10] and [CGCE10] FDL is indeed presented as a fragment of MFL. Here we will present a quite different way to obtain the same translation and prove that it preserves the meaning of the expressions involved by defining their respective semantics from each other, according to the schema in Figure 2.3.

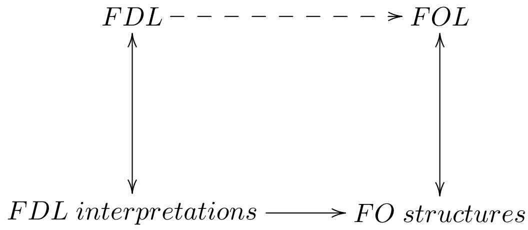

Figure 2.3: Relations to FOL

2.6.1 Concepts

Given a description signature $\mathcal{D} = \langle N_I, N_C, N_R \rangle$, we define the first order signature $\mathbf{s}_{\mathcal{D}} = N_I \cup N_C \cup N_R$, where

- N_I is the set of constant symbols,
- $N_C \cup N_R$ is the set of unary and binary predicates.

Let $\mathbf{l}$ be the propositional language of an extension $\mathcal{L}$ of MTL logic and Var a countable set of individual variables. Then, for every concept name $A \in N_C$, every role name $R \in N_R$ and every $x, y \in Var$, we can define the translations

$$\tau^x : N_C \longrightarrow Fm_{\mathbf{l}\forall,\mathbf{s}_{\mathcal{D}}}$$

and

$$\tau^{x,y} : N_R \longrightarrow Fm_{\mathbf{l}\forall,\mathbf{s}_{\mathcal{D}}}$$

of concept and role names, respectively, into the set of atomic first order formulas of the logic $\mathcal{L}\forall$, in the following way:

$$\begin{array}{rcl} \tau^x(A) & := & A(x) \\ \tau^{x,y}(R) & = & R(x,y). \end{array}$$

This translation can be inductively extended over the set of complex concept in the following way:

$$\begin{array}{rcl}
\tau^x(\bot) & := & \bot \\
\tau^x(\top) & := & \top \\
\tau^x(\overline{r}) & := & \overline{r} \\
\tau^x(\neg C) & := & \neg\tau^x(C) \\
\tau^x(\sim C) & := & \sim \tau^x(C) \\
\tau^x(\triangle C) & := & \triangle\tau^x(C) \\
\tau^x(C \boxtimes D) & := & \tau^x(C) \otimes \tau^x(D) \\
\tau^x(C \sqcap D) & := & \tau^x(C) \wedge \tau^x(D) \\
\tau^x(C \boxplus D) & := & \tau^x(C) \oplus \tau^x(D) \\
\tau^x(C \sqcup D) & := & \tau^x(C) \vee \tau^x(D) \\
\tau^x(C \sqsupset D) & := & \tau^x(C) \to \tau^x(D) \\
\tau^x(\forall R.C) & := & (\forall y)(\tau^{x,y}(R) \to \tau^y(C)), \text{ with } y \neq x \\
\tau^x(\exists R.C) & := & (\exists y)(\tau^{x,y}(R) \otimes \tau^y(C)), \text{ with } y \neq x
\end{array}$$

Notice that as a result of such translation, we obtain first order formulas $\tau^x(C)$ with only the free variable x.

Next we show, in Lemmas 2.6.1 and 2.6.3, that the translation preserves the same meaning of the original expression. We show this property through a definition of a first order structure and of an FDL interpretation from each other.

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an FDL interpretation, then we can define the first order structure $\mathbf{M}_{\mathcal{I}} = (M_{\mathcal{I}}, \{P_{\mathbf{M}_{\mathcal{I}}} : P \in N_C \cup N_R\}, \{c_{\mathbf{M}_{\mathcal{I}}} : c \in N_I\})$, where:

- $M_{\mathcal{I}} := \Delta^{\mathcal{I}}$,
- for each concept name $A \in N_C$, $A_{\mathbf{M}_{\mathcal{I}}}$ is the unary function $A_{\mathbf{M}_{\mathcal{I}}} : M_{\mathcal{I}} \longrightarrow T$, such that, for every $a \in M_{\mathcal{I}}$, it holds that $A_{\mathbf{M}_{\mathcal{I}}}(a) = A^{\mathcal{I}}(a)$,
- for each role name $R \in N_R$, $R_{\mathbf{M}_{\mathcal{I}}}$ is the binary function $R_{\mathbf{M}_{\mathcal{I}}} : M_{\mathcal{I}} \times M_{\mathcal{I}} \longrightarrow T$, such that, for every $a, b \in M_{\mathcal{I}}$, it holds that $R_{\mathbf{M}_{\mathcal{I}}}(a, b) = R^{\mathcal{I}}(a, b)$,
- for each individual $a \in N_I$, $a_{\mathbf{M}_{\mathcal{I}}}$ is an element of $M_{\mathcal{I}}$, such that $a_{\mathbf{M}_{\mathcal{I}}} = a^{\mathcal{I}}$.

Lemma 2.6.1. *Let C be a $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCED}^S$ concept. Then $\|\tau^x(C)\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}_{v([a/x])} = C^{\mathcal{I}}(a)$ for every object $a \in \Delta^{\mathcal{I}}$.*

Proof. The proof is by induction on the structure of complex concepts.

- For concept names and constant concepts it is straightforward by definition.
- Suppose that the statement holds for concepts C and D. Then

$$\begin{array}{rl}
& \|\tau^x(C \boxtimes D)\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}_{v([a/x])} = \\
= & \|\tau^x(C) \otimes \tau^x(D)\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}_{v([a/x])} = \\
= & \|\tau^x(C)\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}_{v([a/x])} * \|\tau^x(D)\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}_{v([a/x])} = \\
= & C^{\mathcal{I}}(a) * D^{\mathcal{I}}(a) = \\
= & (C \boxtimes D)^{\mathcal{I}}(a).
\end{array}$$

In the same way the statement can be proved also for constructors $\sqcap, \boxplus, \sqcup, \sqsupset, \sim, \triangle$ and $\neg$.

- Suppose that the statement holds for the role name R and for concept C. Then

$$\begin{aligned} & \|\tau^x(\forall R.C)\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})}_{v([a/x])} = \\ = \; & \|(\forall y)(\tau^{x,y}(R) \to \tau^y(C))\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})}_{v([a/x])} = \\ = \; & \inf_{y \in M_\mathcal{I}} \{\|\tau^{x,y}(R)\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})}_{v([a/x])} \Rightarrow \|\tau^y(C)\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})}_{v([a/x])}\} = \\ = \; & \inf_{y \in \Delta^\mathcal{I}} \{R^\mathcal{I}(a,y) \Rightarrow C^\mathcal{I}(y)\} = \\ = \; & (\forall R.C)^\mathcal{I}(a). \end{aligned}$$

In the same way the statement can be proved also for concept $\exists R.C$.

So, for every $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCED}^S$ concept C we have that $\|\tau^x(C)\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})}_{v([a/x])} = C^\mathcal{I}(a)$. □

On the other hand, let $\mathbf{M}$ be a first order structure such that $\mathbf{s}_\mathcal{D} = \mathbf{s}_\mathbf{M}$, then we can define the interpretation $\mathcal{I}_\mathbf{M} = (\Delta^{\mathcal{I}_\mathbf{M}}, \cdot^{\mathcal{I}_\mathbf{M}})$, where:

- $\Delta^{\mathcal{I}_\mathbf{M}} = M$,
- for each concept name $A \in N_C$, $A^{\mathcal{I}_\mathbf{M}}$ is the unary function $A^{\mathcal{I}_\mathbf{M}} \colon \Delta^{\mathcal{I}_\mathbf{M}} \longrightarrow T$, such that, for every $a \in \Delta^{\mathcal{I}_\mathbf{M}}$, it holds that $A^{\mathcal{I}_\mathbf{M}}(a) = A_\mathbf{M}(a)$,
- for each role name $R \in N_R$, $R^{\mathcal{I}_\mathbf{M}}$ is the binary function $A^{\mathcal{I}_\mathbf{M}} \colon \Delta^{\mathcal{I}_\mathbf{M}} \times \Delta^{\mathcal{I}_\mathbf{M}} \longrightarrow T$, such that, for every $a, b \in \Delta^{\mathcal{I}_\mathbf{M}}$, it holds that $R^{\mathcal{I}_\mathbf{M}}(a,b) = R_\mathbf{M}(a,b)$,
- for each individual $a \in N_I$, $a^{\mathcal{I}_\mathbf{M}}$ is an element of $\Delta^{\mathcal{I}_\mathbf{M}}$, such that $a^{\mathcal{I}_\mathbf{M}} = a_\mathbf{M}$.

As a straightforward consequence of the definitions of $\mathbf{M}_\mathcal{I}$ and $\mathcal{I}_\mathbf{M}$, we have the following lemma.

Lemma 2.6.2. *For every* $\mathbf{T}$*-interpretation* $\mathcal{I}$ *and every first order structure* $(\mathbf{M}, \mathbf{T})$ *it holds that*

- $\mathcal{I} = \mathcal{I}_{\mathbf{M}_\mathcal{I}}$,
- $\mathbf{M} = \mathbf{M}_{\mathcal{I}_\mathbf{M}}$.

From Lemma 2.6.2 and Lemma 2.6.1 we can prove a further consequence.

Lemma 2.6.3. *Let* C *be an* $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCED}^S$ *concept. Then we have that* $C^{\mathcal{I}_\mathbf{M}}(a) = \|\tau^x(C)\|^{(\mathbf{M},\mathbf{T})}_{v([a/x])}$ *for every object* $a \in M$.

Proof. From Lemma 2.6.1 we have that

$$C^{\mathcal{I}_{\mathbf{M}}}(a) = \|\tau^x(C)\|^{(\mathbf{M}_{\mathcal{I}_{\mathbf{M}}},\mathbf{T})}_{v([a/x])}.$$

From Lemma 2.6.2 we have that

$$\|\tau^x(C)\|^{(\mathbf{M}_{\mathcal{I}_{\mathbf{M}}},\mathbf{T})}_{v([a/x])} = \|\tau^x(C)\|^{(\mathbf{M},\mathbf{T})}_{v([a/x])}.$$

So,

$$C^{\mathcal{I}_{\mathbf{M}}}(a) = \|\tau^x(C)\|^{(\mathbf{M},\mathbf{T})}_{v([a/x])}.$$

□

Remark 2.6.4. The first order language considered could be built by means of a set Var of just two variables. The limitation to just two variables is enough in order to define the translation only for the kind of first order formulas that correspond to $\mathfrak{I}\mathcal{ALCED}^S$ concepts. In fact, in case of nested quantifier, like in the concept:

$$\forall R.\exists R.\forall R.A$$

we have that the translation is

$$\tau^x(\forall R.\exists R.\forall R.A) = (\forall y)(R(x,y) \to (\exists x)(R(y,x) \otimes (\forall y)(R(x,y) \to A(y))))$$

whose meaning, with respect to a structure $(\mathbf{M},\mathbf{T})$ is

$$\inf_{y\in M}\{R^{\mathbf{M}}(x,y) \Rightarrow \sup_{x\in M}\{R^{\mathbf{M}}(y,x) * \inf_{y\in M}\{R^{\mathbf{M}}(x,y) \Rightarrow A^{\mathbf{M}}(y)\}\}\}$$

and, since the inner variable "y" is closed, when a value for the outer function "inf" has to be calculated, this variable falls outside its scope.

Moreover, in case of conjugated quantified concepts, like

$$(\forall R.A) \boxtimes (\exists R.B)$$

we have that the translation is

$$\tau^x((\forall R.A) \boxtimes (\exists R.A)) = (\forall y)(R(x,y) \to A(y)) \otimes (\exists y)(R(x,y) \otimes B(y))$$

whose meaning, with respect to a structure $(\mathbf{M},\mathbf{T})$ is

$$\inf_{y\in M}\{R^{\mathbf{M}}(x,y) \Rightarrow A^{\mathbf{M}}(y)\} * \sup_{y\in M}\{R^{\mathbf{M}}(x,y) * B^{\mathbf{M}}(y)\}\}\}$$

where each appearance of variable "y" is closed inside the scope of a different quantifier and, for this reason, it does not fall inside the scope of the other quantifier. □

2.6.2 Fuzzy axioms

First of all, we utilize the translation $\tau^x(\cdot)$, introduced in Section 2.6.1 in order to obtain a corresponding translation τ from fuzzy axioms to first order formulas. Notice that, since the formulas obtained as a translation of axioms are closed, the super-index $\cdot^x$ does not make sense in this case and we will omit it.

Let x be an individual variable, then, for the fuzzy inclusion axioms, the translation is defined as follows:

$$\begin{aligned}
\tau(\langle C \sqsubseteq D \geq \overline{r}\rangle) &:= \overline{r} \to (\forall x)(\tau^x(C) \to \tau^x(D)),\\
\tau(\langle C \sqsubseteq D \leq \overline{r}\rangle) &:= (\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r},\\
\tau(\langle C \sqsubseteq D > \overline{r}\rangle) &:= \neg \triangle ((\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r}),\\
\tau(\langle C \sqsubseteq D < \overline{r}\rangle) &:= \neg \triangle (\overline{r} \to (\forall x)(\tau^x(C) \to \tau^x(D))),
\end{aligned}$$

For the fuzzy assertion axioms, the translation is defined as follows:

$$\begin{aligned}
\tau(\langle C(a) \geq \overline{r}\rangle) &:= \overline{r} \to \tau^x(C)[a/x],\\
\tau(\langle C(a) \leq \overline{r}\rangle) &:= \tau^x(C)[a/x] \to \overline{r},\\
\tau(\langle C(a) > \overline{r}\rangle) &:= \neg \triangle (\tau^x(C)[a/x] \to \overline{r}),\\
\tau(\langle C(a) < \overline{r}\rangle) &:= \neg \triangle (\overline{r} \to \tau^x(C)[a/x]),
\end{aligned}$$

For the fuzzy role assertion axioms, the translation is defined as follows:

$$\tau(\langle R(a,b) \geq \overline{r}\rangle) := \overline{r} \to \tau^{x,y}(R)[a/x, b/y]$$

As for concepts, here again it is possible to show that the translation preserves the meaning of the original expressions, through the same translation between first order structures and FDL interpretations given in Section 2.6.1.

Lemma 2.6.5. *Let $\langle \varphi \rhd r\rangle$ be a fuzzy axiom with $\rhd \in \{\geq, \leq, >, <\}$. Then a $\mathbf{T}$-interpretation $\mathcal{I}$ satisfies $\langle \varphi \rhd r\rangle$ if and only if $(\mathbf{M}_\mathcal{I}, \mathbf{T})$ 1-satisfies $\tau(\langle \varphi \rhd r\rangle)$.*

Proof. Let $\langle \varphi \rhd r\rangle$ be a fuzzy axiom and $\mathcal{I}$ a $\mathbf{T}$-interpretation, then

- If $\langle \varphi \rhd r\rangle = \langle C \sqsubseteq D \geq \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle C \sqsubseteq D \geq \overline{r}\rangle$ if and only if $\inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} \geq r$. By Lemma 2.6.1, we have that

$$\begin{aligned}
& r \leq \inf_{x\in\Delta^\mathcal{I}}\{C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\} =\\
= \quad & \inf_{x\in\Delta^\mathcal{I}}\{\|\tau^x(C)\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})} \Rightarrow \|\tau^x(D)\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})}\} =\\
= \quad & \inf_{x\in\Delta^\mathcal{I}}\{\|\tau^x(C) \to \tau^x(D)\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})}\} =\\
= \quad & \|(\forall x)(\tau^x(C) \to \tau^x(D))\|^{(\mathbf{M}_\mathcal{I},\mathbf{T})}.
\end{aligned}$$

So, since the residuated implication $\rightarrow$ defines an order, we have that structure $(\mathbf{M}_{\mathcal{I}}, \mathbf{T})$ satisfies $\overline{r} \rightarrow (\forall x)(\tau^x(C) \rightarrow \tau^x(D)) = \tau(\langle C \sqsubseteq D \geq \overline{r}\rangle)$. In the same way can be proved that the statement holds for axioms of type (2.2).

- If $\langle \varphi \rhd r\rangle = \langle C \sqsubseteq D > \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle C \sqsubseteq D > \overline{r}\rangle$ if and only if $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} > r$. Hence $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \nleq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C \sqsubseteq D \leq \overline{r}\rangle$. By the previous result we have that $(\mathbf{M}_{\mathcal{I}}, \mathbf{T})$ does not satisfy $\tau(\langle C \sqsubseteq D \leq \overline{r}\rangle) = (\forall x)(\tau^x(C) \rightarrow \tau^x(D)) \rightarrow \overline{r}$. Then $\|(\forall x)(\tau^x(C) \rightarrow \tau^x(D)) \rightarrow \overline{r}\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})} < 1$ and, therefore, $\|\neg \triangle (\forall x)(\tau^x(C) \rightarrow \tau^x(D)) \rightarrow \overline{r}\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})} = 1$. In the same way can be proved that the statement holds for axioms of type (2.4).

- If $\langle \varphi \rhd r\rangle = \langle C(a) \geq \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle C(a) \geq \overline{r}\rangle$ if and only if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq r$. By Lemma 2.6.1, we have that

$$\begin{aligned} & r \leq \\ \leq\ & C^{\mathcal{I}}(a^{\mathcal{I}}) = \\ =\ & \|\tau^x(C)\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}_{[a_{\mathbf{M}_{\mathcal{I}}}/x]} = \\ =\ & \|\tau^x(C)[a/x]\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}. \end{aligned}$$

So, since the residuated implication $\rightarrow$ defines an order, we have that structure $(\mathbf{M}_{\mathcal{I}}, \mathbf{T})$ satisfies $\overline{r} \rightarrow \tau^x(C)[a/x] = \tau(\langle C(a) \geq \overline{r}\rangle)$. In the same way can be proved that the statement holds for axioms of type (2.6).

- If $\langle \varphi \rhd r\rangle = \langle C(a) > \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle C(a) > \overline{r}\rangle$ if and only if $C^{\mathcal{I}}(a^{\mathcal{I}}) > r$. Hence $C^{\mathcal{I}}(a^{\mathcal{I}}) \nleq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C(a) \leq \overline{r}\rangle$. By the previous result we have that $(\mathbf{M}_{\mathcal{I}}, \mathbf{T})$ does not satisfy $\tau(\langle C(a) \leq \overline{r}\rangle) = \tau^x(C)[a/x] \rightarrow \overline{r}$. Then $\|\tau^x(C)[a/x] \rightarrow \overline{r}\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})} < 1$ and, therefore, $\|\neg \triangle \tau^x(C)[a/x] \rightarrow \overline{r}\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})} = 1$. In the same way can be proved that the statement holds for axioms of type (2.8).

- If $\langle \varphi \rhd r\rangle = \langle R(a, b) \geq \overline{r}\rangle$, then $\mathcal{I}$ satisfies $\langle R(a, b) \geq \overline{r}\rangle$ if and only if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq r$. By the definition of $(\mathbf{M}_{\mathcal{I}}, \mathbf{T})$, we have that

$$\begin{aligned} & r \leq \\ \leq\ & R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = \\ =\ & \|\tau^{x,y}(R)\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}_{[a_{\mathbf{M}_{\mathcal{I}}}/x, b_{\mathbf{M}_{\mathcal{I}}}/y]} = \\ =\ & \|\tau^{x,y}(R)[a/x, b/y]\|^{(\mathbf{M}_{\mathcal{I}},\mathbf{T})}. \end{aligned}$$

So, since the residuated implication $\rightarrow$ defines an order, we have that structure $(\mathbf{M}_{\mathcal{I}}, \mathbf{T})$ satisfies $\overline{r} \rightarrow \tau^{x,y}(R)[a/x, b/y] = \tau^{x,y}(\langle R(a, b) \geq \overline{r}\rangle)$.

So, for every fuzzy axiom $\langle\varphi \rhd r\rangle$ it holds that a **T**-interpretation $\mathcal{I}$ satisfies $\langle\varphi \rhd r\rangle$ if and only if $(\mathbf{M}_{\mathcal{I}}, \mathbf{T})$ satisfies $\tau(\langle\varphi \rhd r\rangle)$. □

Remark 2.6.6. In FDLs where the residuated negation is Gödel negation, as well as in FDLs based on finite-valued Łukasiewicz Logic there is no need of operator $\triangle$ in order to translate strict axioms, since this operator is definable within the language either as (2.22) in the former case and as:

$$\triangle x := x^{n-1}$$

in the latter.

Let φ be an axiom and $r \in T$. If the residuated negation is Gödel, then we have that

$$\begin{aligned}
\tau(\langle C \sqsubseteq D > \overline{r}\rangle) &:= \neg\neg \sim ((\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r}),\\
\tau(\langle C \sqsubseteq D < \overline{r}\rangle) &:= \neg\neg \sim (\overline{r} \to (\forall x)(\tau^x(C) \to \tau^x(D))),\\
\tau(\langle C(a) > \overline{r}\rangle) &:= \neg\neg \sim (\tau^x(C)[a/x] \to \overline{r}),\\
\tau(\langle C(a) < \overline{r}\rangle) &:= \neg\neg \sim (\overline{r} \to \tau^x(C)[a/x]),
\end{aligned}$$

If n is the cardinality of T, we have that

$$\begin{aligned}
\tau(\langle C \sqsubseteq D > \overline{r}\rangle) &:= \neg(((\forall x)(\tau^x(C) \to \tau^x(D)) \to \overline{r})^{n-1}),\\
\tau(\langle C \sqsubseteq D < \overline{r}\rangle) &:= \neg((\overline{r} \to (\forall x)(\tau^x(C) \to \tau^x(D)))^{n-1}),\\
\tau(\langle C(a) > \overline{r}\rangle) &:= \neg((\tau^x(C)[a/x] \to \overline{r})^{n-1}),\\
\tau(\langle C(a) < \overline{r}\rangle) &:= \neg((\overline{r} \to \tau^x(C)[a/x])^{n-1}),
\end{aligned}$$

2.6.3 Reasoning tasks

Now we can utilize the translation $\tau^x(\cdot)$, introduced in Section 2.6.1 and extended to fuzzy axioms in the previous subsection in order to obtain a corresponding translation of the reasoning tasks. In what follows, let C, D be two $\mathfrak{I}\mathcal{ALCE}$-concepts and $r, s \in T$.

- For concept r-satisfiability, we can consider the following three problems of first order logic:
  - C is $(\geq r)$-satisfiable if and only if the formula $\overline{r} \to \tau^x(C)$ is 1-satisfiable;
  - C is r-satisfiable if and only if formula $\tau^x(C)$ is r-satisfiable if and only if the formula $\overline{r} \leftrightarrow \tau^x(C)$ is 1-satisfiable;
  - C is positively satisfiable if and only if the formula $\neg\tau^x(C)$ is not a theorem.
- D r-subsumes C if and only if the formula $\overline{r} \to \tau^x(C \sqsupset D)$ is valid.

- A knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is consistent if and only if the closed formula $\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T} \cup \mathcal{A}} \tau(\langle \varphi \rhd r \rangle)$ is 1-satisfiable.

- C is r-satisfiable with respect to knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ if and only if the formula $\big(\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T} \cup \mathcal{A}} \tau(\langle \varphi \rhd r \rangle)\big) \wedge (\overline{r} \to \tau^x(C))$ is 1-satisfiable.

- Knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ entails the fuzzy axiom $\langle \varphi \rhd r \rangle$ if and only if the closed formula $\tau(\langle \varphi \rhd r \rangle)$ is a logical consequence of the set of closed formulas $\{\tau(\langle \psi \rhd s \rangle) \colon \langle \psi \rhd s \rangle \in \mathcal{T} \cup \mathcal{A}\}$.

- The best satisfiability degree of a concept C with respect to a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, translated to first order logic, is the problem of determining which is the higher value r with respect to which formula $\big(\bigwedge_{\langle \varphi \rhd s \rangle \in \mathcal{T} \cup \mathcal{A}} \tau(\langle \varphi \rhd r \rangle)\big) \wedge (\overline{r} \to \tau^x(C))$ is 1-satisfiable. With respect to the usual problems in first order logic, this problem can be considered as a family of problems, rather than as a single one, that is, one satisfiability problem for each $r \in T$.

- The best entailment degree of an axiom φ with respect to a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, translated to first order logic, is the problem of determining which is the higher value r with respect to which $\tau(\langle \varphi \rhd r \rangle)$ is a logical consequence of the set of formulas $\{\tau(\langle \psi \rhd s \rangle) \colon \langle \psi \rhd s \rangle \in \mathcal{T} \cup \mathcal{A}\}$. Again, this problem can be considered as one logical consequence problem for each $r \in T$.

2.7 Relation to multi-modal logic

In [Sch91] it is provided a translation of DL concepts into classical propositional multi-modal logic. The relationship between FDL and fuzzy multi-modal logic has been described in [CEG12] for the case of finite-valued Łukasiewicz Logic. In that paper the relationship between both formalisms is obtained through their respective relations with first order predicate logic. Here we present a alternative version of the result in [CEG12] by means of a more direct translation.

2.7.1 Concepts

Given a description signature $\mathcal{D} = \langle N_I, N_C, N_R \rangle$, we define the multi-modal language $\mathbf{l}_{\Box_\mathcal{D}} := \mathbf{l} \cup \{\Box_R, : R \in N_R\} \cup \{\Diamond_R, : R \in N_R\}$ over the set $At_\mathcal{D} = \{p_A \colon A \in N_C\}$ of propositional variables where

- $\mathbf{l}$ is the set of propositional connectives of any extension $\mathcal{L}$ of MTL logic,

- $\{\Box_R, : R \in N_R\} \cup \{\Diamond_R, : R \in N_R\}$ is a set of unary modal operators.

For every concept name $A \in N_C$ we can define the translation $\tau : N_C \longrightarrow At_\mathcal{D}$ from the set of concept names into the set of propositional variables of the language $\mathbf{l}_{\Box_\mathcal{D}}$, in the following way:

$$\tau(A) \quad := \quad p_A$$

This translation can be inductively extended over the set of complex concepts in the following way:

$$\begin{array}{rcl}
\tau(\bot) & := & \bot, \\
\tau(\top) & := & \top, \\
\tau(\overline{r}) & := & \overline{r}, \\
\tau(\neg C) & := & \neg\tau(C), \\
\tau(\sim C) & := & \sim \tau(C), \\
\tau(\triangle C) & := & \triangle\tau(C), \\
\tau(C \boxtimes D) & := & \tau(C) \otimes \tau(D) \\
\tau(C \sqcap D) & := & \tau(C) \wedge \tau(D) \\
\tau(C \boxplus D) & := & \tau(C) \oplus \tau(D) \\
\tau(C \sqcup D) & := & \tau(C) \vee \tau(D) \\
\tau(C \sqsupset D) & := & \tau(C) \rightarrow \tau(D) \\
\tau(\forall R.C) & := & \Box_R\tau(C) \\
\tau(\exists R.C) & := & \Diamond_R\tau(C)
\end{array}$$

Next we show that the translation preserves the meaning of the original expression through a definition of a Kripke model from an FDL interpretation. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an FDL interpretation, then we can define the **T**-valued Kripke model $\mathfrak{M}_{\mathcal{I}} = \langle W_{\mathcal{I}}, \{R_{\mathfrak{M}_{\mathcal{I}}} \colon R \in N_R\}, V_{\mathcal{I}}\rangle$, where:

- $W_{\mathcal{I}} = \Delta^{\mathcal{I}}$,
- for each role name $R \in N_R$, $R_{\mathfrak{M}_{\mathcal{I}}}$ is a **T**-valued accessibility relation on $W_{\mathcal{I}}$, i.e. a binary function $R_{\mathfrak{M}_{\mathcal{I}}} \colon W_{\mathcal{I}} \times W_{\mathcal{I}} \longrightarrow T$, such that, for every $a, b \in W_{\mathcal{I}}$, it holds that $R_{\mathfrak{M}_{\mathcal{I}}}(a,b) = R^{\mathcal{I}}(a,b)$,
- for each element $a \in W_{\mathcal{I}}$ and for every propositional variable $p_A \in At_{\mathcal{D}}$, it holds that $V_{\mathcal{I}}(p_A, a) = A^{\mathcal{I}}(a)$.

Lemma 2.7.1. *Let C be an $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCED}^S$ concept. Then, for every $x \in \Delta^{\mathcal{I}}$, it holds that $V_{\mathcal{I}}(\tau(C), a) = C^{\mathcal{I}}(a)$, for every object $a \in \Delta^{\mathcal{I}}$.*

Proof. The proof is by induction on the structure of complex concepts.

- For concept names and constant concepts it is straightforward by definition.
- Suppose that the statement holds for concepts C and D. Then

$$\begin{array}{rl}
& V_{\mathcal{I}}(\tau(C \boxtimes D), a) = \\
= & V_{\mathcal{I}}(\tau(C) \otimes \tau(D), a) = \\
= & V_{\mathcal{I}}(\tau(C), a) * V_{\mathcal{I}}(\tau(D), a) = \\
= & C^{\mathcal{I}}(a) * D^{\mathcal{I}}(a) = \\
= & (C \boxtimes D)^{\mathcal{I}}(a).
\end{array}$$

In the same way the statement can be proved also for constructors $\sqcap, \boxplus, \sqcup, \sqsupset, \sim, \triangle$ and $\neg$.

- Suppose that the statement holds for concept C. Then

$$
\begin{aligned}
& V_{\mathcal{I}}(\tau(\forall R.C), a) = \\
= \quad & V_{\mathcal{I}}(\Box_R \tau(C), a) = \\
= \quad & \inf_{y \in W_{\mathcal{I}}} \{R_{\mathfrak{M}_{\mathcal{I}}}(a, y) \Rightarrow V_{\mathcal{I}}(\tau(C), y)\} = \\
= \quad & \inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a, y) \Rightarrow C^{\mathcal{I}}(y)\} = \\
= \quad & (\forall R.C)^{\mathcal{I}}(a).
\end{aligned}
$$

In the same way the statement can be proved also for concept $\exists R.C$.

So, for every $\mathbf{T}$-$\mathfrak{I}\mathcal{ALCED}^S$ concept C and every $a \in \Delta^{\mathcal{I}}$ it holds that $C^{\mathcal{I}}(a) = V_{\mathcal{I}}(\tau(C), a)$. $\Box$

In the case of multi-modal logic it is also possible to provide a translation from multi-modal formulas into description concepts. Given a multi-modal language $\mathbf{l}_{\Box} = \mathbf{l} \cup \{\Box_R, : R \in N_R\} \cup \{\Diamond_R, : R \in N_R\}$, with I countable and a set of propositional variables $At = \{p_1, p_2, \ldots\}$, we define the description signature $\mathcal{D}_{\mathbf{l}_{\Box}} = \langle N_I^{\mathbf{l}_{\Box}}, N_C^{\mathbf{l}_{\Box}}, N_R^{\mathbf{l}_{\Box}} \rangle$, where

- $N_I^{\mathbf{l}_{\Box}} := \emptyset$,
- $N_C^{\mathbf{l}_{\Box}} := \{A_p : p \in At\}$,
- $N_R^{\mathbf{l}_{\Box}} := \{R_i : \Box_i \in \mathbf{l}_{\Box}\}$.

For every propositional variable $p \in At$ we can define the translation $\rho : At \longrightarrow N_C^{\mathbf{l}_{\Box}}$ from the set of propositional variable into the set of concept names of the signature $\mathcal{D}_{\mathbf{l}_{\Box}}$, in the following way:

$$\rho(p) \quad := \quad A_p$$

This translation can be inductively extended over the set of complex concepts in the following way:

$$
\begin{array}{rcl}
\rho(\bot) & := & \bot \\
\rho(\top) & := & \top \\
\rho(\overline{r}) & := & \overline{r} \\
\rho(\neg\varphi) & := & \neg\rho(\varphi) \\
\rho(\sim\varphi) & := & \sim\rho(\varphi) \\
\rho(\triangle\varphi) & := & \triangle\rho(\varphi) \\
\rho(\varphi\otimes\psi) & := & \rho(\varphi)\boxtimes\rho(\psi) \\
\rho(\varphi\wedge\psi) & := & \rho(\varphi)\sqcap\rho(\psi) \\
\rho(\varphi\oplus\psi) & := & \rho(\varphi)\boxplus\rho(\psi) \\
\rho(\varphi\vee\psi) & := & \rho(\varphi)\sqcup\rho(\psi) \\
\rho(\varphi\rightarrow\psi) & := & \rho(\varphi)\sqsupset\rho(\psi) \\
\rho(\Box_i\varphi) & := & \forall R_i.\rho(\varphi) \\
\rho(\Diamond_i\varphi) & := & \exists R_i.\rho(\varphi).
\end{array}
$$

As a straightforward consequence of the definitions of τ and ρ, we have the following lemma.

Lemma 2.7.2. *For every* $\mathbf{T}$-$\mathcal{ALCED}^S$ *concept* C *and every multi-modal formula* φ *it holds that:*

- $\rho(\tau(C)) = C$,
- $\tau(\rho(\varphi)) = \varphi$.

Again, it is possible to show that the translation preserves the meaning of the original expression through a definition of an FDL interpretation from a Kripke model. Let $\mathfrak{M} = \langle W, \{R_1, \ldots, R_n\}, V\rangle$ be a $\mathbf{T}$-valued Kripke model, then we can define the interpretation $\mathcal{I}_{\mathfrak{M}} = (\Delta^{\mathcal{I}_{\mathfrak{M}}}, \cdot^{\mathcal{I}_{\mathfrak{M}}})$, where:

- $\Delta^{\mathcal{I}_{\mathfrak{M}}} := W$,
- for each concept name $A_p \in N_C^{\mathbf{l}_\Box}$, $A_p^{\mathcal{I}_{\mathfrak{M}}}$ is the unary function $A_p^{\mathcal{I}_{\mathfrak{M}}} : \Delta^{\mathcal{I}_{\mathfrak{M}}} \longrightarrow T$, such that, for every $a \in \Delta^{\mathcal{I}_{\mathfrak{M}}}$, it holds that
$$A_p^{\mathcal{I}_{\mathfrak{M}}}(a) = V(p, a),$$
- for each role name $R_i \in N_R^{\mathbf{l}_\Box}$, $R_i^{\mathcal{I}_{\mathfrak{M}}}$ is the binary function $R_i^{\mathcal{I}_{\mathfrak{M}}} : \Delta^{\mathcal{I}_{\mathfrak{M}}} \times \Delta^{\mathcal{I}_{\mathfrak{M}}} \longrightarrow T$, such that, for every $a, b \in \Delta^{\mathcal{I}_{\mathfrak{M}}}$, it holds that
$$R_i^{\mathcal{I}_{\mathfrak{M}}}(a, b) = R_i(a, b).$$

As a straightforward consequence of the definitions of $\mathfrak{M}_{\mathcal{I}}$ and $\mathcal{I}_{\mathfrak{M}}$, we have the following lemma.

Lemma 2.7.3. *For every* $\mathbf{T}$*-interpretation* $\mathcal{I}$ *and every* $\mathbf{T}$*-valued Kripke model* $\mathfrak{M}$ *it holds that:*

- $\mathcal{I} = \mathcal{I}_{\mathfrak{M}_{\mathcal{I}}}$,

- $\mathfrak{M} = \mathfrak{M}_{\mathcal{I}_{\mathfrak{M}}}$.

From Lemma 2.7.2, Lemma 2.7.3 and Lemma 2.7.1 we can prove a further consequence.

Lemma 2.7.4. *Let φ be a multi-modal formula. Then, for every $w \in W$ it holds that $(\rho(\varphi))^{\mathcal{I}_{\mathfrak{M}}}(w) = V(\varphi, w)$.*

Proof. From Lemma 2.7.1 we have that

$$(\rho(\varphi))^{\mathcal{I}_{\mathfrak{M}}}(w) = V_{\mathcal{I}_{\mathfrak{M}}}(\tau(\rho(\varphi)), w).$$

From Lemma 2.7.2 we have that

$$V_{\mathcal{I}_{\mathfrak{M}}}(\tau(\rho(\varphi)), w) = V_{\mathcal{I}_{\mathfrak{M}}}(\varphi, w).$$

From Lemma 2.7.3 we have that

$$V_{\mathcal{I}_{\mathfrak{M}}}(\varphi, w) = V(\varphi, w).$$

So, $(\rho(\varphi))^{\mathcal{I}_{\mathfrak{M}}}(w) = V(\varphi, w)$. □

2.7.2 Fuzzy axioms

First of all, we utilize the translation $\tau(\cdot)$, introduced in Section 2.7 in order to obtain a corresponding translation of the fuzzy inclusion axioms proposed in Section 2.1.2. In this case, however, we need to use a multi-modal language that contains the universal modality $\Box_U$ (see page 15), as well as Delta operator $\triangle$ and truth constants.

$$\begin{aligned}
\tau(\langle C \sqsubseteq D \geq \overline{r}\rangle) &:= \Box_U(\overline{r} \to (\tau(C) \to \tau(D))) \\
\tau(\langle C \sqsubseteq D \leq \overline{r}\rangle) &:= \Box_U((\tau(C) \to \tau(D)) \to \overline{r}) \\
\tau(\langle C \sqsubseteq D > \overline{r}\rangle) &:= \Box_U \neg \triangle ((\tau(C) \to \tau(D)) \to \overline{r}) \\
\tau(\langle C \sqsubseteq D < \overline{r}\rangle) &:= \Box_U \neg \triangle (\overline{r} \to (\tau(C) \to \tau(D)))
\end{aligned}$$

Here again it is possible to show that the translation preserves the meaning of the original expressions. Note that, for formulas starting with the universal modality $\Box_U$ fuzzy axiom satisfiability coincide both with local and global satisfiability of its multi-modal translation.

Lemma 2.7.5. *Let $\langle \varphi \rhd r\rangle$ be a fuzzy inclusion axiom, with $\rhd \in \{\geq, \leq, >, <\}$. Then a **T**-interpretation $\mathcal{I}$ satisfies $\langle \varphi \rhd r\rangle$ if and only if $\mathfrak{M}_{\mathcal{I}}$ globally satisfies $\tau(\langle \varphi \rhd r\rangle)$ if and only if $\mathfrak{M}_{\mathcal{I}}$ locally satisfies $\tau(\langle \varphi \rhd r\rangle)$.*

Proof. Let $\langle \varphi \rhd r\rangle$ be a fuzzy axiom and $\mathcal{I}$ a **T**-interpretation.

- If $\langle \varphi \rhd r\rangle = \langle C \sqsubseteq D \geq \overline{r}\rangle$, then $\mathcal{I}$ satisfies it if and only if $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \geq r$. By Lemma 2.6.1, we have that

$$\begin{aligned} & r \leq \\ \leq\ & \inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} = \\ =\ & \inf_{x\in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C),x) \Rightarrow V_{\mathcal{I}}(\tau(D),x)\} = \\ =\ & \inf_{x\in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C) \rightarrow \tau(D),x)\}. \end{aligned}$$

Hence, for every $x \in W_{\mathcal{I}}$, it holds that $r \leq V_{\mathcal{I}}(\tau(C) \rightarrow \tau(D),x)$ and, therefore,

$$\begin{aligned} & 1 = \\ =\ & \inf_{x\in W_{\mathcal{I}}}\{r \Rightarrow V_{\mathcal{I}}(\tau(C) \rightarrow \tau(D),x)\} = \\ =\ & \inf_{w\in W_{\mathcal{I}}}\{R_U(w,x) \Rightarrow V_{\mathcal{I}}(r \rightarrow (\tau(C) \rightarrow \tau(D),x))\} = \\ =\ & V_{\mathcal{I}}(\Box_U(\overline{r} \rightarrow (\tau(C) \rightarrow \tau(D))),w) = \\ =\ & V_{\mathcal{I}}(\tau(\langle C \sqsubseteq D \geq \overline{r}\rangle),w), \end{aligned}$$

for every $w \in W_{\mathcal{I}}$. So, $\mathfrak{M}_{\mathcal{I}}$ both globally and locally 1-satisfies $\tau(\langle \varphi \rhd r\rangle)$. In the same way it can be proved that the statement holds for axioms of type (2.2).

- If $\langle \varphi \rhd r\rangle = \langle C \sqsubseteq D > \overline{r}\rangle$, then $\mathcal{I}$ satisfies it if and only if $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} > r$. Hence $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \not\leq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C \sqsubseteq D \leq \overline{r}\rangle$. By the previous result we have that $\mathfrak{M}_{\mathcal{I}}$ satisfies $\tau(\langle C \sqsubseteq D \leq \overline{r}\rangle) = \Box_U((\tau(C) \rightarrow \tau(D)) \rightarrow \overline{r})$, neither globally, nor locally. Then $V_{\mathcal{I}}(\Box_U((\tau(C) \rightarrow \tau(D)) \rightarrow \overline{r}),w) < 1$, for every $w \in W_{\mathcal{I}}$ and, therefore, $V_{\mathcal{I}}(\Box_U\neg \triangle ((\tau(C) \rightarrow \tau(D)) \rightarrow \overline{r}),w) = 1$. In the same way can be proved that the statement holds for axioms of type (2.4).

So, for every fuzzy inclusion axiom $\langle \varphi \rhd r\rangle$ it holds that a **T**-interpretation $\mathcal{I}$ satisfies $\langle \varphi \rhd r\rangle$ if and only if $\mathfrak{M}_{\mathcal{I}}$ satisfies $\tau(\langle \varphi \rhd r\rangle)$. □

In a language without a universal modality $\Box_U$ (but with Delta operator $\triangle$ and truth constants) we can not obtain multi-modal formulas as a translation of fuzzy axioms. Nevertheless, their satisfiability with respect to an interpretation $\mathcal{I}$ can be translated to either global or local satisfiability of certain multi-modal formulas with respect to model $\mathfrak{M}_{\mathcal{I}}$, depending on what kind of axiom has to be translated. So, in such a language, a translation of the fuzzy inclusion axioms can be obtained as follows:

Lemma 2.7.6. *For every* $\mathbf{T}$*-interpretation* $\mathcal{I}$ *the following equivalences hold:*

1. $\mathcal{I} \models \langle C \sqsubseteq D \geq \overline{r} \rangle \iff \mathfrak{M}_{\mathcal{I}} \models^{r}_{g} \tau(C) \to \tau(D)$
$\iff \mathfrak{M}_{\mathcal{I}} \models^{1}_{g} \overline{r} \to (\tau(C) \to \tau(D)),$

2. $\mathcal{I} \models \langle C \sqsubseteq D \leq \overline{r} \rangle \iff \mathfrak{M}_{\mathcal{I}} \models^{1}_{l} (\tau(C) \to \tau(D)) \to \overline{r},$

3. $\mathcal{I} \models \langle C \sqsubseteq D > \overline{r} \rangle \iff \mathfrak{M}_{\mathcal{I}} \models^{1}_{g} \neg \triangle ((\tau(C) \to \tau(D)) \to \overline{r}),$
$\iff \mathfrak{M}_{\mathcal{I}} \not\models^{1}_{l} (\tau(C) \to \tau(D)) \to \overline{r}$

4. $\mathcal{I} \models \langle C \sqsubseteq D < \overline{r} \rangle \iff \mathfrak{M}_{\mathcal{I}} \models^{1}_{l} \neg \triangle (\overline{r} \to (\tau(C) \to \tau(D))),$
$\iff \mathfrak{M}_{\mathcal{I}} \not\models^{r}_{g} \tau(C) \to \tau(D)$

Proof. Let $\mathcal{I}$ be a $\mathbf{T}$ interpretation, then:

1. We have that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D \geq \overline{r} \rangle$ if and only if $\inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \geq r$. By Lemma 2.7.1, we have that

$$\begin{aligned} & r \leq \\ \leq\ & \inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} = \\ =\ & \inf_{x \in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C), x) \Rightarrow V_{\mathcal{I}}(\tau(D), x)\} = \\ =\ & \inf_{x \in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C) \to \tau(D), x)\}. \end{aligned}$$

Hence, on the one hand, for every $x \in W_{\mathcal{I}}$, it holds that $r \leq V_{\mathcal{I}}(\tau(C) \to \tau(D), x)$, that is, $\mathfrak{M}_{\mathcal{I}} \models^{r}_{g} \tau(C) \to \tau(D)$. On the other hand, for every $x \in W_{\mathcal{I}}$, it holds that $V_{\mathcal{I}}(\overline{r} \to (\tau(C) \to \tau(D)), x) = 1$, that is, $\mathfrak{M}_{\mathcal{I}} \models^{1}_{g} \overline{r} \to (\tau(C) \to \tau(D))$.

2. We have that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D \leq \overline{r} \rangle$ if and only if $\inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \leq r$. By Lemma 2.7.1, we have that

$$\begin{aligned} & r \geq \\ \geq\ & \inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} = \\ =\ & \inf_{x \in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C), x) \Rightarrow V_{\mathcal{I}}(\tau(D), x)\} = \\ =\ & \inf_{x \in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C) \to \tau(D), x)\}. \end{aligned}$$

Hence, there exists $x \in W_{\mathcal{I}}$ such that $V_{\mathcal{I}}(\overline{r} \to (\tau(C) \to \tau(D)), x) = 1$, that is, $\mathfrak{M}_{\mathcal{I}} \models^{1}_{l} \overline{r} \to (\tau(C) \to \tau(D))$.

3. We have that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D > \overline{r}\rangle$ if and only if $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} > r$. Hence $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \nleq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C \sqsubseteq D \leq \overline{r}\rangle$. So, by item 2, we have that $\mathfrak{M}_{\mathcal{I}} \not\models^1_l (\tau(C) \to \tau(D)) \to \overline{r}$. Moreover, since $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \nleq r$, then, by Lemma 2.7.1, we have that

$$\begin{aligned} & r \ngeq \\ \ngeq\ & \inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} = \\ =\ & \inf_{x\in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C), x) \Rightarrow V_{\mathcal{I}}(\tau(D), x)\} = \\ =\ & \inf_{x\in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C) \to \tau(D), x)\}. \end{aligned}$$

Hence, for every $x \in W_{\mathcal{I}}$, it holds that $V_{\mathcal{I}}((\tau(C) \to \tau(D)) \to \overline{r}, x) < 1$, that is, $V_{\mathcal{I}}(\neg \triangle (\tau(C) \to \tau(D)) \to \overline{r}, x) = 1$. So, $\mathfrak{M}_{\mathcal{I}} \models^1_g \neg \triangle ((\tau(C) \to \tau(D)) \to \overline{r})$.

4. We have that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D < \overline{r}\rangle$ if and only if $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} < r$. Hence $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \ngeq r$ i.e. $\mathcal{I}$ does not satisfy axiom $\langle C \sqsubseteq D \geq \overline{r}\rangle$. So, by item 1, we have that $\mathfrak{M}_{\mathcal{I}} \not\models^r_g \tau(C) \to \tau(D)$. Moreover, since $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} < r$, then, by Lemma 2.7.1, we have that

$$\begin{aligned} & r > \\ >\ & \inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} = \\ =\ & \inf_{x\in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C), x) \Rightarrow V_{\mathcal{I}}(\tau(D), x)\} = \\ =\ & \inf_{x\in W_{\mathcal{I}}}\{V_{\mathcal{I}}(\tau(C) \to \tau(D), x)\}. \end{aligned}$$

Hence there exists $x \in W_{\mathcal{I}}$ such that $V_{\mathcal{I}}(\overline{r} \to (\tau(C) \to \tau(D)), x) < 1$, that is, $V_{\mathcal{I}}(\neg \triangle (\overline{r} \to (\tau(C) \to \tau(D)), x)) = 1$. So, $\mathfrak{M}_{\mathcal{I}} \models^1_l \neg \triangle (\overline{r} \to (\tau(C) \to \tau(D)))$. □

Differently from the case of first order logic and despite the fact that FDL interpretations can be a semantics for fuzzy assertions, within the multi-modal language it is not possible to translate fuzzy assertions like $\langle C(a) \geq r\rangle$. This is due to the fact that in multi-modal languages there is not a syntactic entity that can work as a translation for FDL individuals.

2.7.3 Reasoning tasks

Now we can utilize the translation $\tau(\cdot)$, introduced in Section 2.7 and extended to fuzzy axioms in the previous subsection in order to obtain a corresponding translation of the reasoning tasks. Nevertheless, analogously to the fact that we can not obtain a corresponding translation of fuzzy assertion axioms we can not obtain a translation to multi-modal logic of the problems related to knowledge bases where the ABox is non-empty. For this reason we will not consider in this section the knowledge base consistency problem when the ABox is not empty and will only consider problems related to knowledge bases $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{A} = \emptyset$. In what follows, let C, D be two concepts and $r, s \in T$.

- For concept r-satisfiability, we can consider the following three problems of multi-modal logic:
  - C is $(\geq r)$-satisfiable if and only if formula $\tau(C)$ is locally s-satisfiable for some $s \geq r$, if and only if formula $\overline{r} \to \tau(C)$ is locally 1-satisfiable;
  - C is r-satisfiable if and only if formula $\tau(C)$ is locally r-satisfiable if and only if formula $\overline{r} \leftrightarrow \tau(C)$ is locally 1-satisfiable.
  - C is positively satisfiable if and only if formula $\tau(\neg C)$ is not a theorem.
- D r-subsumes C if and only if formula $\overline{r} \to \tau(C \sqsupset D)$ is valid.
- A knowledge base $\mathcal{K} = \langle \mathcal{T} \rangle$ is consistent if and only if formula $\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T}} \tau(\langle \varphi \rhd r \rangle)$ is globally 1-satisfiable.
- C is r-satisfiable w.r.t. $\mathcal{K} = \langle \mathcal{T} \rangle$ if and only if there exists a $\mathbf{T}$-valued Kripke model $\mathfrak{M}$ such that both $\mathfrak{M} \models_g^1 \left(\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T}} \tau(\langle \varphi \rhd r \rangle) \right)$ and $\mathfrak{M} \models_l^r \tau(C)$.
- Knowledge base $\mathcal{K} = \langle \mathcal{T} \rangle$ entails the fuzzy axiom $\langle \varphi \rhd r \rangle$ if and only if formula $\tau(\langle \varphi \rhd r \rangle)$ is a global consequence of the set of formulas $\{\tau(\langle \psi \rhd s \rangle) \colon \langle \psi \rhd s \rangle \in \mathcal{T}\}$.
- The best satisfiability degree of a concept C with respect to a KB $\mathcal{K} = \langle \mathcal{T} \rangle$, translated to multi-modal logic, is the problem of determining which is the higher value r with respect to which there exists a $\mathbf{T}$-valued Kripke model $\mathfrak{M}$ such that both $\mathfrak{M} \models_g^1 \left(\bigwedge_{\langle \varphi \rhd r \rangle \in \mathcal{T}} \tau(\langle \varphi \rhd s \rangle) \right)$ and $\mathfrak{M} \models_l^1 \overline{r} \to \tau(C)$. With respect to the usual problems in multi-modal logic, this problem can be considered as a family of problems, rather than as a single one, that is, one satisfiability problem for each $r \in T$.
- The best entailment degree of an axiom φ with respect to a KB $\mathcal{K} = \langle \mathcal{T} \rangle$, translated to multi-modal logic, is the problem of determining which is the higher value r with respect to which $\tau(\langle \varphi \rhd r \rangle)$ is a global consequence of the set of formulas $\{\tau(\langle \psi \rhd s \rangle) \colon \langle \psi \rhd s \rangle \in \mathcal{T}\}$. Again, this problem can be considered as one logical consequence problem for each $r \in T$.

2.8 Related work

Since the first articles on FDL, it was evident that generalizing the formalism of DL to the fuzzy framework consists in generalizing its semantics. A first step in this sense is that of generalizing the semantics of atomic concepts and roles from crisp sets and relations to fuzzy sets and fuzzy relations respectively and the semantics of subsumption to the inclusion between fuzzy sets. Nevertheless this does not mean there is a wide agreement on how to generalize the semantics of complex concepts and, since the beginning of the research on FDL, several solutions have been proposed.

The first attempt in this direction is the one of [Yen91]. At that time the notation reported in Section 1.2.2 had not been fully adopted in the DL community and [Yen91] has been thought as a generalization of [BL84] where the so-called *Term Subsumption Languages* (TSL) is developed. The language studied in [Yen91], denoted "$\mathcal{FTSL}^-$", takes, as concept constructors, conjunction (: **and** $C_1, \ldots, C_n$), value restriction (: **all** $R\,C$), restricted existential quantification (: **some** $R\,\top$), modifiers (**:NOT**, **:VERY**, **:SLIGHTLY**, etc.) and an ancestor of concrete domains. The semantics underlying this first proposal was called *test score semantics* (see [Zad82]). This name just means that *scores* (what we nowadays call "truth values") are assigned to concepts after performing tests to the system. However, what is interesting, under our point of view, in the semantics used in [Yen91], is the truth functions used to calculate the truth values of complex concepts, in particular.

- It is suggested to use the min function to compute the value of a conjunction (: **and** $C_1, \ldots, C_n$) of concepts. Besides this suggestion, the author not only recognizes that any other t-norm can be used as the semantics of conjunction, but also that both lower and upper bounds for conjunctions can be computed considering min and Łukasiewicz t-norms as upper and lower bounds respectively.

- The semantics of value restriction (: **all** $R\,C$) is defined in two alternative ways. Through a fuzzy implication operator, as is done nowadays, and through the notion of conditional necessity from possibility theory. The author, however, adopts the second option.

The semantics defined in [Yen91] was enough general to leave open the adoption of a truth function for conjunction. Nevertheless, [Yen91] was inspired by practical purposes and his goal was providing a more refined tool for knowledge representation.

Later on, [TM98] has a more theoretic fashion. The evolution of the notation towards a logical-like abstraction, that can be seen in the DL community, influenced [TM98], which utilizes the same modern notation reported in Section 1.2.2. The language studied in this work was called $\mathcal{ALC}_{F_M}$ (the subindex F_M stands for infinitely many truth values). It presents, as concept constructors, conjunction $\sqcap$, disjunction $\sqcup$, value restriction $\forall R.C$, existential quantifier $\exists R.C$

and *manipulators* (what we call *modifiers*) M_iC. The authors of [TM98] utilize a translation of the FDL language to fuzzy first order logic[2] and provide a semantics to fuzzy first order logic that, through the translation, turns out to be the semantics of the FDL language, in accordance with the schema in Figure 2.4.

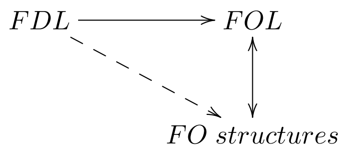

Figure 2.4: Relations to FOL in [TM98]

The choice of the truth functions for the logical connectives falls on min and max for conjunction and disjunction, respectively. The semantics for the existential quantifier is the one provided in Section 2.1.1 and it is the first place where it has been defined this way. This work is also the first in defining the semantics for value restriction $\forall R.C$ by means of the so-called *Kleene-Dienes implication*, defined on $[0, 1]$ as:

$$x \Rightarrow y := \max\{1 - x, y\}$$

which is a straightforward generalization of the classical one. In particular, if $\mathcal{I}$ is an FDL interpretation, the semantics of value restriction $\forall R.C$, based on Kleene-Dienes implication, is defined in the following way:

$$(\forall R.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(x, y), C^{\mathcal{I}}(y)\}\}.$$

Finally, for the semantics of manipulators M_iC, unary function on $[0, 1]$ were used, as in the framework of *fuzzy hedges* (see [CHN11] for details).

Until [TM98], the research on FDL has been quite limited, but in the same year Straccia published his first work on FDL, [Str98]. The language studied in this work is called (and, indeed, it is) $\mathcal{ALC}$ and the semantics adopted is the same as the one used in [TM98], plus a unary function that gives the semantics to concept complementation defined as:

$$\neg x := 1 - x.$$

The set of operations that includes $\min\{x, y\}, \max\{x, y\}, \max\{1 - x, y\}$ and $1 - x$ on the real unit interval is commonly denoted with the name of *Zadeh's semantics*. The strength of [Str98] and of its journal version [Str01], is that they set up a clear syntax and semantics, very close to the classical ones and relate each other without the intermediate step of first order logic, like in [TM98]. In

[2]The fuzzy first order logic considered in [TM98], however, was not the same calculus developed in the framework of Mathematical Fuzzy Logic.

this way Fuzzy Description Logic is set up as an autonomous discipline with a clearly defined syntax an semantics as the one provided in [Str01].

Moreover, [Str98] is the first place where fuzzy axioms have been defined this way, since in previous papers the notion of fuzzy axioms was not considered. In [Str98], just non-strict lower bound axioms are considered. In [Str01] have been introduced axioms stating a non-strict upper bound as well, but strict bound axioms are not considered. Since then, some works on FDL consider strict bound axioms, like [Str04a, SSP+05a, Str06, BS10b] and some others do not consider strict bound axioms, like [Str04b, Str05b, BS07, CGCE10, BBS11, BP11a, BP11f, CEG12].

Later, in [Str04b], the same author considers also the more general framework of semantics based on lattices that are supposed to be not necessarily chains. These works, indeed, opened the door to the possibility of expanding the language in order to cover the advances that had been done in the classical framework. A fuzzy semantics for concrete domains was introduced in [Str05c]. A semantics for unqualified number restriction, role hierarchies, inverse and transitive roles was introduced in [SSP+05a]. A semantics for nominals was introduced in [SSP+05b]. A semantics for qualified number restriction was introduced in [BDGR07].

However, due to the absence of a residuated implication, an FDL based on Zadeh's semantics is too weak and it can lead to counter-intuitive consequences. This fact has been pointed out in [Háj05], where the example of the assertion "all hotels near to the main square are expensive" is presented in order to highlight the consequences of using Kleene-Dienes implication in the semantics of value restriction. Such assertion can be formally expressed as

$$\forall \texttt{hasNear}.\texttt{Expensive}(\texttt{MainSquare}) \tag{2.23}$$

Here we will further develop Hájek's example. Consider the following fuzzy ABox $\mathcal{HOTELS}$:

- $\langle \texttt{hasNear(MainSquare,Hotel\_1)} = 0.9 \rangle$,
- $\langle \texttt{hasNear(MainSquare,Hotel\_2)} = 0.5 \rangle$,
- $\langle \texttt{hasNear(MainSquare,Hotel\_3)} = 0.1 \rangle$,
- $\langle \texttt{Expensive(Hotel\_1)} = 0.9 \rangle$,
- $\langle \texttt{Expensive(Hotel\_2)} = 0.5 \rangle$,
- $\langle \texttt{Expensive(Hotel\_3)} = 0.1 \rangle$,

The $\mathcal{HOTELS}$ ABox indeed depicts the ideal situation imagined by Hájek, where "for each hotel the degree of its being near to the main square equals the degree of its being expensive" and where "there is at least one hotel which is near to the main square in degree 0.5". In Figure 2.5 we report an interpretation that

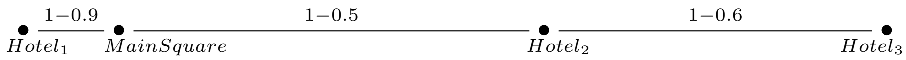

Figure 2.5: Interpretation satisfying $\mathcal{HOTELS}$

satisfies ABox $\mathcal{HOTELS}$, where the distance between `MainSquare` and `Hotel_x` is calculated as $1 -$ `hasNear(MainSquare,Hotel_x)`.

In this ideal situation the truth value of assertion (2.23) should be 1, because hotels are at least as expensive as they lie near the main square. Now, if the truth value of (2.23) is calculated using the truth function of any residuated implication, its value is indeed 1. In spite, using the truth function of Kleene-Dienes implication, the result is different. In fact, in every interpretation $\mathcal{I}$ that is a model of $\mathcal{HOTELS}$, we have that:

$$
\begin{aligned}
& (\forall \texttt{hasNear.Expensive(MainSquare)})^{\mathcal{I}} = \\
= \quad & \inf_{x \in \Delta^{\mathcal{I}}} \{\texttt{Near}^{\mathcal{I}}(\texttt{MainSquare}^{\mathcal{I}}, x) \Rightarrow \texttt{Expensive}^{\mathcal{I}}(x)\} \leq \\
\leq \quad & \inf\{\max\{1-0.9, 0.9\}, \max\{1-0.5, 0.5\}, \max\{1-0.1, 0.1\}\} = \\
= \quad & \inf\{0.9, 0.5, 0.9\} \\
= \quad & 0.5
\end{aligned}
$$

So, the truth value of assertion (2.23), using Kleene-Dienes implication, is at most 0.5 in a model of $\mathcal{HOTELS}$, against the intuition, reflected in $\mathcal{HOTELS}$, that its truth value should be 1.

But the example can go beyond this situation. Consider, in fact, the ABox $\mathcal{HOTELS}'$ obtained by adding to $\mathcal{HOTELS}$ the following set of assertions:

- $\langle$`hasNear(SideSquare,Hotel_1)` $= 0.1\rangle$,
- $\langle$`hasNear(SideSquare,Hotel_2)` $= 0.4\rangle$,
- $\langle$`hasNear(SideSquare,Hotel_3)` $= 0.4\rangle$,

In this new situation the truth value of assertion

$$\forall \texttt{hasNear.Expensive(SideSquare)} \tag{2.24}$$

should be no higher than the value of assertion (2.23), because `SideSquare` lies very far away from the most expensive hotel (`Hotel_1`) and there is one hotel whose degree of being near is higher than its degree of being expensive (`Hotel_3`). Indeed, in the situation depicted by ABox $\mathcal{HOTELS}'$, the main square is the square that has near the more expensive hotels and the side square is the one that lies nearer to the cheaper hotels. In Figure 2.5 we report an interpretation that satisfies ABox $\mathcal{HOTELS}'$.

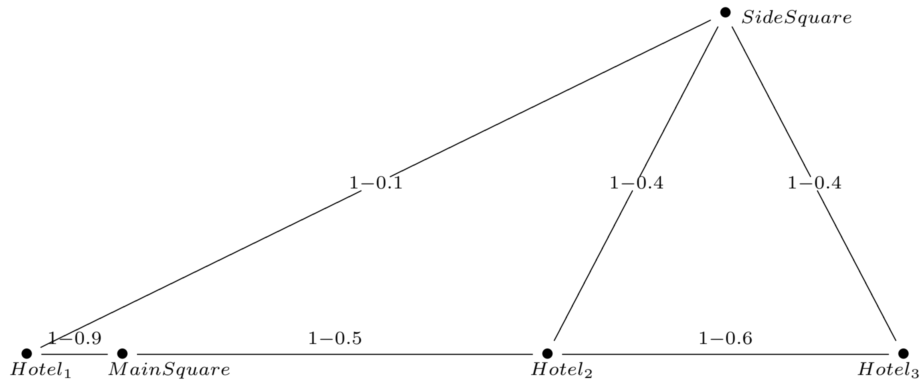

Figure 2.6: Interpretation satisfying $\mathcal{HOTELS}'$

For this reason, it appears counter intuitive the possibility that `SideSquare` can be an instance of concept $\forall$`hasNear.Expensive` in a degree higher than `SideSquare`. Again, if its truth value is calculated with the use of the truth function of any residuated implication, its value is indeed strictly less than 1. In spite, using the truth function of Kleene-Dienes implication, the result can be higher than the truth value of $\forall$`hasNear.Expensive(MainSquare)`. In fact, in every interpretation $\mathcal{I}$ that is a model of $\mathcal{HOTELS}'$, and where we have that:

$$\begin{aligned}
& (\forall \texttt{hasNear.Expensive(SideSquare)})^{\mathcal{I}} \\
= \ & \inf_{x \in \Delta^{\mathcal{I}}} \{\texttt{Near}^{\mathcal{I}}(\texttt{SideSquare}^{\mathcal{I}}, x) \Rightarrow \texttt{Expensive}^{\mathcal{I}}(x)\} \\
= \ & \inf\{\max\{1-0.1, 0.9\}, \max\{1-0.4, 0.5\}, \max\{1-0.4, 0.1\}\} \\
= \ & \inf\{0.9, 0.6, 0.6\} \\
= \ & 0.6 \\
> \ & 0.5 \\
= \ & (\forall \texttt{hasNear.Expensive(MainSquare)})^{\mathcal{I}}
\end{aligned}$$

So, the truth value of assertion (2.24) is greater than that of (2.23) in every model of $\mathcal{HOTELS}'$, against the intuition, reflected in $\mathcal{HOTELS}'$, that hotels should be more expensive around the main square.

For this reason, Hájek proposes, in [Háj05], a more general framework based on Mathematical Fuzzy Logic. As we have seen, with the only exception of [Yen91], the operation min is the only function adopted as a semantics for the conjunction concept constructor until [Háj05]. In this new framework, not only the semantics of conjunction is a t-norm, but it is also recovered the idea, firstly proposed in [TM98], of a tight relation between FDL and first order fuzzy logic that, in the meanwhile, has been defined in the general framework of Mathematical Fuzzy Logic (MFL) developing the basic ideas of [Háj98c]. The new framework proposed in [Háj05] inspired several successive works on FDL. Among the ones that consider a t-norm-based semantics we can find [Str05c] and, more recently, [CEB10] and [BP11b]. Among the ones that deepen the relationships between FDL and MFL we can find [GCAE10], [CGCE10] and [CEG12].

The new framework proposed in [Háj05] supposed also a re-thinking about the notation used in FDL. Indeed, the use of the same notation of DL for FDL has been based on the fact that, in order to generalize DLs to the multi-valued framework, it seemed enough to generalize the semantics of concepts and roles to fuzzy sets and fuzzy relations. With this idea it is obvious that the same concept constructors (and, with them, the same formal languages) could be maintained in a multi-valued framework. This formalization worked indeed well when the semantics adopted as underlying truth value algebra was the Zadeh's semantics. But, since [Háj05], some researchers on FDLs began to consider the use of residuated implications as formalized in the framework of MFL. However, adopting a framework based on MFL and maintaining the same notation as in the classical case, could produce a slight confusion. This is due to several reasons related to differences between the classical and the many-valued framework. Commonly, with some exceptions, such differences include the following items:

1. two kinds of conjunctions can be considered in the many-valued framework, with different mathematical properties, and the same holds for disjunction,

2. implication is, in general, not definable from other connectives,

3. the quantifiers are not definable from each other by means of the equivalence $\exists R.C \equiv \neg\forall R.\neg C$,

4. the disjunction is not definable from the residuated negation $\neg C := C \to \bot$ and the conjunction $\sqcap$.

All these items must be taken into account both when choosing the symbols denoting the constructors of our description languages and when building the hierarchy of fuzzy description languages, as we have already seen in Section 2.3. As an example recall that, in classical DLs, $\mathcal{ALE}$ is strictly contained in $\mathcal{ALC}$, while within many fuzzy DLs, by item 3 above, this is not the case.

In particular, we find worth discussing the case of implication. In classical DL, the implication is not usually a primitive concept constructor, even though implication is often implicitly used. This is due to the fact that the implication is definable from conjunction and negation. Nevertheless, in the logic MTL and many of its extensions, implication is in general not definable from other connectives. The first time that the concept constructor $\rightarrow$ for the implication is included in the definition of the language as a primitive connective has been in [Háj05]. Here we prefer to use the symbol $\sqsupset$ introduced in [CGCE10].

The introduction of a new symbol for implication allows to utilize a concept constructor that is not otherwise definable, even if quite useful in order to define, in BL and its extensions, other concept constructors like those for weak conjunction (whose semantics and symbol are the minimum and $\sqcap$ respectively), weak disjunction (whose semantics and symbol are the maximum and $\sqcup$ respectively) and residuated negation (whose definition is $C \sqsupset \bot$).

Another issue that could take great advantage from the use of a residuated implication is the semantics of concept subsumption. Since the first works on FDL, in fact, the semantics for subsumption between concepts is defined by means of the inclusion between fuzzy sets, that is, concept C is subsumed by concept D if and only if, for every interpretation $\mathcal{I}$ and every $x \in \Delta^{\mathcal{I}}$, it holds that $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$. If the truth function of the implication $\sqsupset$ is the residuum of the truth function of the strong conjunction $\boxtimes$, this is equivalent to say that concept $C \sqsupset D$ is valid or, equivalently, that concept $\neg(C \sqsupset D)$ is not positively satisfiable. If, otherwise, the truth function of the implication is $\max\{1-C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\}$ the above relation between fuzzy set inclusion and implication does not hold anymore. As an example, consider two concepts A and B. As a matter of fact, their conjunction $A \sqcap B$ is always subsumed by both concepts, that is, $A \sqcap B \sqsubseteq A$. In Zadeh's semantics, in fact, for every interpretation $\mathcal{I}$ and every $x \in \Delta^{\mathcal{I}}$, it holds that $(A \sqcap B)^{\mathcal{I}}(x) = \min\{A^{\mathcal{I}}(x), B^{\mathcal{I}}(x)\} \leq A^{\mathcal{I}}(x)$. Nevertheless, when the truth function of the implication is $\max\{1 - A^{\mathcal{I}}(x), B^{\mathcal{I}}(x)\}$ it is not true that concept $(A \sqcap B) \sqsupset A$ is valid. As a counter-example to the relationship between the notion of fuzzy set inclusion based on the order $\leq$ and Kleene-Dienes implication, consider interpretation $\mathcal{I}$ where:

- $\Delta^{\mathcal{I}} = \{a\}$,
- $A^{\mathcal{I}}(a) = B^{\mathcal{I}}(a) = 0.5$,

then,

$$
\begin{aligned}
& ((A \sqcap B) \sqsupset A)^{\mathcal{I}}(a) = \\
= \; & \max\{1 - \min\{A^{\mathcal{I}}(a), B^{\mathcal{I}}(a)\}, A^{\mathcal{I}}\} = \\
= \; & \max\{1 - \min\{0.5, 0.5\}, 0.5\} = \\
= \; & 0.5
\end{aligned}
$$

As we can see from the example of interpretation $\mathcal{I}$, the value of concept $A \sqcap B$ is less or equal than the value of concept A, but concept $(A \sqcap B) \sqsupset A$ is not a valid concept.

The advantage of considering a residuated implication, however not only consists in the fact that it behaves well with the inclusion between fuzzy concepts, but, above all, that by means of a residuated implication it is possible to define a graded notion of subsumption. By defining the semantics of subsumption as:

$$(C \sqsubseteq D)^{\mathcal{I}} := \inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\}$$

we are not just able to say whether concept C is totally subsumed in concept D, but, when it is not the case, we can also give a truth value to this subsumption. This is indeed a great increase in expressivity.

Chapter 3

Decidability

Decidability is a fundamental topic in classical DL. In FDL it is a very important topic as well. The study of decidability in FDL has brought to the generalization of classical algorithms, as well as to the design of new ones. Even though, however, decidability results for reasoning tasks without knowledge bases coincide with those of the classical framework, the same does not hold in presence of ontologies, when the set of truth values considered is infinite. For $\mathfrak{I}\mathcal{ALCE}$ language over most infinite t-norms, in fact, different undecidability results have been recently proved, that outline a significative difference with the classical case. In this chapter we deal with $\mathfrak{I}\mathcal{ALCE}$ language over product and Łukasiewicz infinite t-norms. The same problems for $\mathfrak{I}\mathcal{ALCE}$ language over finite t-norms will be addressed in the next chapter where we deal with the computational complexity of reasoning tasks without knowledge bases.

The content of the present chapter is the following. In Section 3.1 we report a reduction from $\mathfrak{I}\mathcal{ALCE}$ concepts to sets of propositional formulas provided in [Háj05]. Through such a reduction the decidability of the witnessed satisfiability and subsumption problems without knowledge bases over infinite-valued Łukasiewicz semantics has been proved in [Háj05]. We report it because we analyze it under several points of view throughout the present chapter and the next one. In Section 3.2, a quasi-witnessed extension of the reduction reported in Section 3.1 that deals with infinite interpretations of $\mathfrak{I}\mathcal{ALE}$ language over the standard product chain without ontologies is provided. Such extension allows to prove decidability of the quasi-witnessed positive satisfiability problem for the $\mathfrak{I}\mathcal{ALE}$ language over product t-norm. In Section 3.3 we give a brief account of how, by menas of the reductions given in the previous two sections, the decidability of the witnessed and quasi-witnessed subsumption problems without knowledge bases is proved for the same languages of Section 3.1 and Section 3.2. In Section 3.4, we provide an undecidability result for the knowledge base consistency problem of $\mathcal{ALC}$ language over the standard Łukasiewicz chain. Further results existing in the literature will be reported in Section 3.5.

3.1 Witnessed satisfiability and Łukasiewicz logic

Concept satisfiability is one of the simplest reasoning tasks in FDL and the one that is more studied in the logical counterpart. Since [Háj05] the attention of most researchers focussed on the satisfiability with respect to witnessed interpretations, that we will call, from now on, *witnessed satisfiability.*

In [Háj05] it is proved that concept witnessed r-satisfiability is a decidable problem for a Fuzzy Description Logic restricted to language $\mathfrak{I}\mathcal{ALCE}$, based on any t-norm. In order to achieve such result, in [Háj05] is defined an algorithm that, given a concept C_0, obtains a propositional theory P_{C_0}. We report the algorithm from [Háj05] in Definition 3.1.2. Before introducing the algorithm, we need some previous definitions from [Háj05].

Definition 3.1.1. 1. *Nesting degree of quantifiers* in C (or $C(a)$) is defined inductively:

- $nest(C) = 0$, if C is an concept name or a constant concept;
- if C is a concept, then $nest(\sim C) = nest(C)$;
- if C and D are concepts, then $nest(C \boxtimes D) = nest(C \sqsupset D) = \max(nest(C), nest(D))$;
- if C is a concept and R a role name, then $nest(\forall R.C) = nest(\exists R.C) = nest(C) + 1$.

2. *Generalized atoms* are quantified concepts, i.e. concepts of the form $\forall R.C$ or $\exists R.C$, where C is a propositional combination of concepts and generalized atoms; the latter will be called *generalized atoms* of C. We will also use the term generalized atom for instances of quantified concepts, the context will clarify the precise meaning.

Next we report Hájek's reduction from [Háj05].

Definition 3.1.2 ([Háj05, Definition 3])**.** Given $C_0(a_0)$ with nesting degree n, step 0 just transfers it to further processing in step 1; the constant a_0 has level 0. For $i > 0$ step i processes generalized atoms of formulas transferred from step $i-1$; they have the form $(QR.C)(b)$, where Q is $\forall$ or $\exists$, R is a role, C a concept with nesting degree $\leq n-i$ and b is a constant of level i. For each generalized atom α in question, do the following:

If α is $(\forall R.C)(b)$ then produce a new constant d_α and the axiom

$$(\forall R.C)(b) \equiv (R(b, d_\alpha) \sqsupset C(d_\alpha))$$

If α is $(\exists R.C)(b)$ then produce d_α and the axiom

$$(\exists R.C)(b) \equiv (R(b, d_\alpha) \boxtimes C(d_\alpha))$$

In both cases call the generated axioms *witnessing axioms* for α and d_α a constant *belonging to* R, b.

After this is done for all α in question (in the present step) consider each α once more and do the following:

If α is $(\forall R.C)(b)$ and d_β is any constant belonging to R, b and different from d_α, produce the axiom

$$(\forall R.C)(b) \sqsupset (R(b, d_\beta) \sqsupset C(d_\beta))$$

Similarly for α being $(\exists R.C)(b)$, produce

$$(R(b, d_\beta) \boxtimes C(d_\beta)) \sqsupset (\exists R.C)(b)$$

The collection of formulas produced in some step will be denoted by $P_{C_0}(a_0)$ or, simply P_{C_0}.

After proving that the given algorithm is correct and complete with respect to the problems considered, Hájek proves that, for language $\mathfrak{I}\mathcal{ALCE}$ over any t-norm, satisfiability, validity and subsumption with respect to witnessed interpretations coincide with the same problems with respect to finite interpretations. Hence, when these problems are restricted to witnessed interpretations, the FDLs considered enjoy the finite model property and are decidable.

Theorem 3.1.3 ([Háj05, Corollary 1])**.** *For every continuous t-norm $*$, witnessed r-satisfiability is a decidable problem in language $\mathfrak{I}\mathcal{ALCE}$.*

Depending on the t-norm $*$ considered, witnessed satisfiability, validity and subsumption do not need to coincide with the same problem with respect to unrestricted interpretations, but both notions indeed coincide in some cases, as we have seen in Section 1.1.3. In particular, under $[0,1]_{\text{Ł}}$, we have that unrestricted r-satisfiability is a decidable problem.

Theorem 3.1.4 ([Háj05, Theorem 1])**.** *r-satisfiability is a decidable problem in $[0,1]_L$-$\mathcal{ALC}$.*

If $\mathbf{T}$ is a finite BL-chain, then, trivially, witnessed satisfiability coincides with unrestricted satisfiability, for each notion of satisfiability considered. The same holds for validity and subsumption.

As remarked in [Háj05] and [HC06] by means of counter-examples, in the case of infinite-valued Gödel and product t-norms, there exist formulas that are satisfiable, but not in a witnessed interpretation.

3.2 Quasi-witnessed satisfiability and Product Logic

In [CEB10] it is proved that concept quasi-witnessed positive satisfiability is a decidable problem for a Fuzzy Description Logic restricted to language $\mathfrak{I}\mathcal{ALE}$, based on product t-norm.

We will use the notations Sat_{pos} and Sat to denote the sets of positively satisfiable and 1-satisfiable concepts, respectively; and we will write QSat_{pos} and QSat to denote the same sets restricted to quasi-witnessed interpretations. In particular, in [CEB10] has been proven the following theorem.

Theorem 3.2.1. *The sets* QSat_{pos} *and* QSat *are decidable.*

In Appendix A it is proved that quasi-witnessed positive satisfiability and unrestricted positive satisfiability indeed coincide under the standard product semantics. Hence unrestricted positive satisfiability is a decidable problem in language $[0,1]_{\Pi}\mathfrak{I}\mathcal{ALE}$. For the 1-satisfiability problem under standard product semantics, completeness with respect to quasi-witnessed models and, thus, decidability of language $\mathfrak{I}\mathcal{ALE}$ based on product t-norm are still open problems.

The proof of Theorem 3.2.1 follows the one provided in [Háj05] for the case of witnessed interpretations. We report here the whole proof.

The Reduction to the Propositional Case

In order to prove that positive satisfiability in a quasi-witnessed interpretation is decidable we are going to codify quasi-witnessed interpretations by means of finite number of formulas in the propositional product logic.

First of all, let us a fix an infinite set $\mathrm{Ind} = \{a_i : i \in \mathbb{N}\}$, whose elements will be called *individuals* or *constants*. With a little language abuse, throughout Section 3.2, an *assertion* will denote any propositional combination of expressions of the forms $C(a)$ and $R(a, b)$, where C is a concept, R is a role name and a, b are individuals. The definitions of the notions of nesting degree and generalized atoms can be found in Definition 3.1.1. In the next definition we provide a first version of a labeling function that we need in order to prove the result. An enhanced version of this function will be given in Definition 4.3.5, but for the purpose of the present section, the one in Definition 3.2.2 is enough.

Definition 3.2.2 (Labeling (first version))**.** Let C_0 be a concept. The *labeling function* is the function which associates to every occurrence D of a subconcept in C_0 an element of $\mathbb{N}^{\leq k}$ (where $k = nest(C_0)$) defined by the conditions:

1. $l(C_0)$ is the empty sequence ε,
2. if D is a propositional combination of concepts $D_1, \ldots, D_n$, then $l(D_i) := l(D)$ for every $i \leq n$.
3. if D is $\forall R.D'$ or $\exists R.D'$, then $l(D')$ is the concatenated sequence $l(D), i$, where i is the minimum number m such that the sequence $l(D), m$ has not been used to label any occurrence in C_0.

In order to illustrate the notions provided in Definition 3.1.1 and Definition 3.2.2, as well as further definitions, we propose an example that will be used throughout Chapter 4.

Example 3.2.3. *Consider the concept*

$$\texttt{Example} := \forall R.\exists R.A \sqcap \neg\forall R.(\exists R.A \boxtimes \exists R.A)$$

where A is an atomic concept. Then,

1. *concept* `Example` *has nesting degree 2.*
2. *the generalized atoms in* `Example` *are: $\forall R.\exists R.A$, $\forall R.(\exists R.A \boxtimes \exists R.A)$ and $\exists R.A$.*
3. *the labelling function associated with occurrences in* `Example` *is given by the genealogical tree*

$$\dfrac{\dfrac{\dfrac{A : 1,1}{\exists R.A : 1}}{\forall R.\exists R.A : \emptyset} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A : 2,1}{\exists R.A : 2} \quad \dfrac{A : 2,2}{\exists R.A : 2}}{\exists R.A \boxtimes \exists R.A : 2}}{\forall R.(\exists R.A \boxtimes \exists R.A) : \emptyset}}{\neg\forall R.(\exists R.A \boxtimes \exists R.A) : \emptyset}}{Example : \emptyset}$$

   Here we have used the notation $D : \sigma$ to indicate that the labelling of occurrence D is the sequence σ.

Next, for every concept C_0 we are going to recursively associate two finite sets P_{C_0} and Y_{C_0} of assertions.

Definition 3.2.4 (Algorithm)**.** Given a concept C_0, we construct finite sets P_{C_0} and Y_{C_0} of assertions. The construction takes steps $0, \ldots, n$ where n is the nesting degree of the concept C_0. At each step some generalized atoms are processed; and at each step we add some new constants from Ind and some new formulas to P_{C_0} and Y_{C_0} and we transfer some assertions of concepts for processing in the next step. The assertions produced in step i will have nesting degree $\leq n - i$; after step n is completed the algorithm stops.

At step 0, we simply transfer the assertion $C_0(d)$ to be further processed in step 1; and we say that constant d has level 0. For $i > 0$, step i selects the generalized atoms in formulas transferred from step $i-1$ and processes them. We know that the generalized atoms just selected have the form $QR.C(d_\sigma)$, where $Q \in \{\forall, \exists\}$, R is a role, C a concept with nesting degree $\leq n-i$, d_σ is a constant produced in the previous step and σ is the label of the generalized atom we are considering. For each generalized atom α, at step i we firstly do the following:

- If α is $\forall R.C(d_\sigma)$, then produce a new constant $d_{\sigma,m}$ and add to P_{C_0} the assertion
$$(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) \sqcup \neg\forall R.C(d_\sigma).$$
- If α is $\exists R.C(d_\sigma)$, then produce a new constant $d_{\sigma,m}$ and add to P_{C_0} the assertion:
$$(R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m})) \equiv \exists R.C(d_\sigma).$$

We will say that $d_{\sigma,m}$ is a *constant associated to* R, d_σ. Now, we consider each α of the present step and do the following:

- If α is $(\forall R.C)(d_\sigma)$ and $d_{\sigma,m}$ is any constant associated to R, d_σ, then add to P_{C_0} the assertion

$$\forall R.C(d_\sigma) \sqsupset (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})).$$

- If α is $(\exists R.C)(d_\sigma)$ and $d_{\sigma,m}$ is any constant associated to R, d_σ, then add to P_{C_0} the assertion

$$(R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m})) \sqsupset \exists R.C(d_\sigma).$$

- If α is $(\forall R.C)(d_\sigma)$, then add to Y_{C_0} the assertion

$$\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})).$$

Example 3.2.5. *Following Definition 3.2.4, the assertions belonging to* $P_{\texttt{Example}}$ *are:*

- $(\forall R.\exists R.A(d) \equiv (R(d, d_1) \sqsupset \exists R.A(d_1))) \sqcup \neg\forall R.\exists R.A(d)$,
- $(\forall R.(\exists R.A \boxtimes \exists R.A)(d) \equiv (R(d, d_2) \sqsupset (\exists R.A \boxtimes \exists R.A)(d_2))) \sqcup \neg\forall R.(\exists R.A \boxtimes \exists R.A)(d)$,
- $\forall R.\exists R.A(d) \sqsupset (R(d, d_2) \sqsupset A(d_2))$,
- $\forall R.(\exists R.A \boxtimes \exists R.A)(d) \sqsupset (R(d, d_1) \sqsupset (\exists R.A \boxtimes \exists R.A)(d_1))$,
- $\exists R.A(d_1) \equiv (R(d_1, d_{1,1}) \boxtimes A(d_{1,1}))$,
- $\exists R.A(d_2) \equiv (R(d_2, d_{2,1}) \boxtimes A(d_{2,1}))$,
- $\exists R.A(d_2) \equiv (R(d_2, d_{2,2}) \boxtimes A(d_{2,2}))$,
- $(R(d_2, d_{2,2}) \boxtimes A(d_{2,2})) \sqsupset \exists R.A(d_2)$,
- $(R(d_2, d_{2,1}) \boxtimes A(d_{2,1})) \sqsupset \exists R.A(d_2)$.

While the assertions belonging to $Y_{\texttt{Example}}$ *are:*

- $\neg\forall R.\exists R.A(d) \boxtimes (R(d, d_1) \sqsupset \exists R.A(d_1))$,
- $\neg\forall R.(\exists R.A \boxtimes \exists R.A)(d) \boxtimes (R(d, d_2) \sqsupset (\exists R.A \boxtimes \exists R.A)(d_2))$.

As it is said above our aim is to reduce our problem to one in the corresponding propositional calculus. Here we will consider this propositional logic using as variables the set

$$At := \{p_{R(a,b)} : R \text{ is a role name and } a, b \in \text{Ind}\} \;\cup$$

$$\{p_{C(a)} : C \text{ atomic or quantified concept and } a \in \text{Ind}\}.$$

We stress that we are taking a concrete fix set as variables. Nevertheless, for a particular concept C_0 it is clear that a finite subset At_{C_0} of At would be enough. Using that all assertions are indeed propositional combinations of expressions of the form $C(a)$ and $R(a,b)$, the following definition is meaningful. This definition tells us that we can look at assertions as propositional formulas with variables in At.

Definition 3.2.6. The map pr associates to every assertion a formula in the propositional logic (with the variables given above) according to the following clauses:

1. $pr(A(a)) = p_{A(a)}$ if A is an atomic concept or generalized atom,
2. $pr(R(a,b)) = p_{R(a,b)}$ if R is a role name and $a, b \in \text{Ind}$,
3. $pr(\bot(a)) = \bot$,
4. $pr(\top(a)) = \top$
5. $pr((C \boxtimes D)(a)) = pr(C(a)) \otimes pr(D(a))$,
6. $pr((C \sqsupset D)(a)) = pr(C(a)) \rightarrow pr(D(a))$.

If P is a set of assertions, then $pr(P)$ is $\{pr(\alpha) \colon \alpha \in P\}$.

Example 3.2.7. *If $P_{\texttt{Example}}$ is the set defined in the Example 3.2.5, then, following Definition 3.2.6, propositional formulas belonging to $pr(P_{\texttt{Example}})$ are:*

- $(p_{\forall R.\exists R.A(d)} \equiv (p_{R(d,d_1)} \rightarrow p_{\exists R.A(d_1)})) \vee \neg p_{\forall R.\exists R.A(d)}$,
- $(p_{\forall R.(\exists R.A \boxtimes \exists R.A)(d)} \equiv (p_{R(d,d_2)} \rightarrow (p_{\exists R.A \boxtimes \exists R.A)(d_2)})) \vee \neg p_{\forall R.(\exists R.A \boxtimes \exists R.A)(d)}$,
- $p_{\forall R.\exists R.A(d)} \rightarrow (p_{R(d,d_2)} \rightarrow p_{A(d_2)})$,
- $p_{\forall R.(\exists R.A \boxtimes \exists R.A)(d)} \rightarrow (p_{R(d,d_1)} \rightarrow p_{(\exists R.A \boxtimes \exists R.A)(d_1)})$,
- $p_{\exists R.A(d_1)} \equiv (p_{R(d_1,d_{1,1})} \otimes p_{A(d_{1,1})})$,
- $p_{\exists R.A(d_2)} \equiv (p_{R(d_2,d_{2,1})} \otimes p_{A(d_{2,1})})$,
- $p_{\exists R.A(d_2)} \equiv (p_{R(d_2,d_{2,2})} \otimes p_{A(d_{2,2})})$,
- $(p_{R(d_2,d_{2,2})} \otimes p_{A(d_{2,2})}) \rightarrow p_{\exists R.A(d_2)}$,
- $(p_{R(d_2,d_{2,1})} \otimes p_{A(d_{2,1})}) \rightarrow p_{\exists R.A(d_2)}$.

On the other hand, propositional formulas belonging to $pr(Y_{\texttt{Example}})$ are:

- $\neg p_{\forall R.\exists R.A(d)} \otimes (p_{R(d,d_1)} \rightarrow p_{\exists R.A(d_1)})$,

- $\neg p_{\forall R.(\exists R.A \boxtimes \exists R.A)(d)} \otimes (p_{R(d,d_2)} \to p_{(\exists R.A \boxtimes \exists R.A)(d_2)})$.

The next and crucial step in the proof is the following result. We leave the proofs of each one of the directions for Proposition 3.2.12 and Proposition 3.2.18.

Proposition 3.2.8. *Let C_0 be a concept, and let P_{C_0} and Y_{C_0} be the two finite sets associated by Definition 3.2.4. For every $r \in [0,1]$, the following statements are equivalent:*

1. *C_0 is satisfiable with truth value r in a quasi-witnessed interpretation,*

2. *there is some propositional evaluation e over the set At such that $e(pr(C(d_0))) = r$, $e(pr(P_{C_0})) = 1$, and $e(\psi) \neq 1$ for every $\psi \in pr(Y_{C_0})$.*

From now on we will say that a propositional evaluation e is *quasi-witnessing relatively to* C_0 (*quasi-witnessing*, for short) when it satisfies that $e(pr(P_{C_0})) = 1$, and $e(\psi) \neq 1$ for every $\psi \in pr(Y_{C_0})$.

As a consequence of this last proposition we are now able to prove Theorem 3.2.1. This is so because by Proposition 3.2.8 we know that $C \in$ QSat if and only if $\bigvee pr(Y_{C_0})$ is not logical consequence, in the corresponding propositional calculus, of the set $\{pr(C(d_0))\} \cup pr(P_{C_0})$.

Hence, we have a reduction of this problem to the semantic consequence problem, with a finite number of hypothesis, in the corresponding propositional calculus. This problem can be formalized as the problem of deciding, given two propositional formulas φ and ψ, whether ψ is a semantic consequence of φ, i.e., whether each propositional evaluation which gives value 1 to φ, also gives value 1 to ψ (see Section 1.1.1). In [Háj06, Theorem 3] it is proved that such problem is decidable for the expansion of product logic with truth constants, but, since a formula without truth constants can be considered as a formula of the expanded language in which do not appear truth constants, this result also holds for the product logic without truth constants. Thus, the proof of Proposition 3.2.8 is the only missing step in order to prove Theorem 3.2.1.

From FDL interpretations to propositional evaluations

The purpose of this subsection is to show the downwards implication of Proposition 3.2.8. Let us assume that for a given concept C_0, there is a quasi-witnessed interpretation $\mathcal{I}$ and an object a such that $C_0^{\mathcal{I}}(a) = r$ for some $r \in [0,1]$. The following definition tells us a way to obtain a propositional evaluation satisfying the requirements in Proposition 3.2.8.

Definition 3.2.9. Let $\mathcal{I}$ be a quasi-witnessed interpretation, a an object of the domain and C_0 a concept. Let us consider P_{C_0}, Y_{C_0} as the sets of assertions obtained from the concept C_0 by applying Definition 3.2.4. We assume that the individual a_0 has been interpreted in $\mathcal{I}$ as the object a; and for each step, we assume that constants in previous steps have been interpreted in $\mathcal{I}$. For each generalized atom α processed in that step, do the following:

(∀1) If $\alpha = \forall R.C(d_\sigma)$ and there exists $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then interpret the constant $d_{\sigma,m}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

(∀2) If $\alpha = \forall R.C(d_\sigma)$ and there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\ R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then choose an element $u \in \Delta^{\mathcal{I}}$ such that $0 < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) < 1$. Such an element exists since, being $\mathcal{I}$ a quasi-witnessed interpretation, we have, on the one hand, that, for each $u \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) > 0$ and, on the other hand, if there was no element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) < 1$, then $\inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\} = 1 = R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u)$ against the supposition. Once chosen the element u, interpret the constant $d_{\sigma,m}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

(∃) If $\alpha = \exists R.C(d_\sigma)$, then choose an element $u \in \Delta^{\mathcal{I}}$ witnessing α and interpret the constant $d_{\sigma,m}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

Finally, for every generalized atom and every atomic formula α, occurring in $P_{C_0} \cup Y_{C_0}$, define $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$.

Using and modifying an example reported in [BS09], we provide the following instance of the above definition.

Example 3.2.10. *Consider the interpretation $\mathcal{I}$ such that:*

1. $\Delta^{\mathcal{I}} = \{a, b, c, g, f\} \cup \{g_i \mid i \in \mathbb{N}\backslash\{0\}\}$,
2. *there is a binary relation r such that*
   - $r(b, c) = r(g, f) = 1$,
   - $r(a, b) = r(a, g) = 0.5$,
   - $r(a, g_i) = 0.5^i$,
   - $r(x, y) = 0$, *when x, y is any other pair of elements of the domain.*
3. *there is a unary predicate s such that*
   - $s(c) = s(f) = 0.5$
   - $s(x) = 0$, *for any other element x of the domain.*

So, if we take

- $R^{\mathcal{I}} = r$,
- $A^{\mathcal{I}} = s$,
- $d^{\mathcal{I}} = a$,
- $d_1^{\mathcal{I}} = b$,

- $d^{\mathcal{I}}_{1,1} = c$,

- $d^{\mathcal{I}}_{2} = g$

- $d^{\mathcal{I}}_{2,1} = d^{\mathcal{I}}_{2,2} = f$,

then it is easy to check that:

1. $\mathcal{I}$ *is a quasi-witnessed model of concept* `Example`,

2. $e_{\mathcal{I}}(pr(\varphi)) = 1$, *for each* $\varphi \in P_{\texttt{Example}}$,

3. $e_{\mathcal{I}}(pr(\psi)) < 1$, *for each* $\psi \in Y_{\texttt{Example}}$.

In Lemma 3.2.11 and Proposition 3.2.12, we are going to prove that all propositional evaluations obtained in this way are quasi-witnessing.

Lemma 3.2.11. *Let* $\mathcal{I}$ *be a quasi-witnessed interpretation,* C_0 *a concept, and let us consider* P_{C_0}, Y_{C_0} *as the sets of assertions obtained from the concept* C_0 *by applying Definition 3.2.4. Then, the propositional evaluation* $e_{\mathcal{I}}$ *is quasi-witnessing relatively to* C_0.

Proof. We will show the result considering, case by case, the five kinds of propositions we can find in $pr(P_{C_0})$ and $pr(Y_{C_0})$.

1. Consider the assertion $(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) \sqcup \neg\forall R.C(d_\sigma)$, then:

   ($\forall$1) If, following Definition 3.2.9, we have interpreted the constant $d_{\sigma,m}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then we have that

$$
\begin{aligned}
& e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) \sqcup \neg\forall R.C(d_\sigma))) = \\
= \ & (e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) \equiv \\
& (e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m}))) \Rightarrow e_{\mathcal{I}}(pr(C(d_{\sigma,m}))))) \vee \neg e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = \\
= \ & ((\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}) \equiv (R^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}, d^{\mathcal{I}}_{\sigma,m}) \Rightarrow C^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma,m}))) \vee \neg(\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}) = \\
= \ & (\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}) \equiv (R^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}, d^{\mathcal{I}}_{\sigma,m}) \Rightarrow C^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma,m})) = \\
= \ & 1
\end{aligned}
$$

   ($\forall$2) If there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, since $\mathcal{I}$ is a quasi-witnessed interpretation,

$$
\begin{aligned}
& e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) \sqcup \neg\forall R.C(d_\sigma))) = \\
= \;& (e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) \equiv \\
& e_{\mathcal{I}}((pr(R(d_\sigma, d_{\sigma,m}))) \Rightarrow e_{\mathcal{I}}(pr(C(d_{\sigma,m}))))) \vee \neg e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = \\
= \;& ((\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \equiv (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,m}^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(d_{\sigma,m}^{\mathcal{I}}))) \vee \neg(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = \\
= \;& \neg(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = \\
= \;& 1
\end{aligned}
$$

In both cases we have that

$$
e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) \sqcup \neg\forall R.C(d_\sigma))) = 1.
$$

2. Consider the assertion $\exists R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m}))$. Then, by Definition 3.2.9, we have that

$$
\begin{aligned}
& e_{\mathcal{I}}(pr(\exists R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m})) = \\
= \;& e_{\mathcal{I}}(pr(\exists R.C(d_\sigma))) \equiv e_{\mathcal{I}}((pr(R(d_\sigma, d_{\sigma,m}))) * e_{\mathcal{I}}(pr(C(d_{\sigma,m})))) = \\
= \;& \exists R.C^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \equiv (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,m}^{\mathcal{I}}) * C^{\mathcal{I}}(d_{\sigma,m}^{\mathcal{I}})) = \\
= \;& 1.
\end{aligned}
$$

3. Consider the assertion $\forall R.C(d_\sigma) \sqsupset (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))$. Since $(\forall R.C(d_\sigma))^{\mathcal{I}} = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, by Definition 3.2.9 we have that

$$
\begin{aligned}
& e_{\mathcal{I}}(pr(\forall R.C(d_\sigma) \sqsupset (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})))) = \\
= \;& e_{\mathcal{I}}(pr(\forall R.C(d_\sigma)) \Rightarrow (e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m})) \Rightarrow e_{\mathcal{I}}(pr(C(d_{\sigma,m}))) = \\
= \;& (\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \Rightarrow (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,m}^{\mathcal{I}}) \Rightarrow C(d_{\sigma,m}^{\mathcal{I}})) = \\
= \;& 1.
\end{aligned}
$$

4. Consider the assertion $(R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m})) \sqsupset \exists R.C(d_\sigma)$. Since $(\exists R.C(d_\sigma))^{\mathcal{I}} = \sup_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) * C^{\mathcal{I}}(d)\}$, then, by Definition 3.2.9 we have that

$$
\begin{aligned}
& e_{\mathcal{I}}(pr((R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) \sqsupset \exists R.C(d_\sigma)) = \\
= \;& e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m}))) * e_{\mathcal{I}}(pr(C(d_{\sigma,m}))) \Rightarrow e_{\mathcal{I}}(pr(\exists R.C(d_\sigma))) = \\
= \;& (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,m}^{\mathcal{I}}) * C(d_{\sigma,m}^{\mathcal{I}})) \Rightarrow \exists R.C^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = \\
= \;& 1.
\end{aligned}
$$

5. Consider the assertion $\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))$, then:

   (∀1) if, following Definition 3.2.9, we have interpreted the constant $d_{\sigma,m}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then we have that if, on the one hand, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma))) = 1$, then $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = 0$ and, therefore, by Definition 3.2.9, $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) = 0$. Hence $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})))) = 0 < 1$. If, on the other hand, $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) = 1$, then, by assumption, $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = 1$ and, therefore, again, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma)\boxtimes (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})))) = 0 < 1$.

   (∀2) if there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, since $\mathcal{I}$ is a quasi-witnessed interpretation, we have that, necessarily, $\forall R^{\mathcal{I}}.C^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = 0$ and, hence, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma))) = 1$. However, since, by Definition 3.2.9, we have interpreted the constant $d_{\sigma,m}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $0 < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) < 1$ and, therefore, we have that $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m}))) < 1$. So, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \boxtimes (R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})))) < 1$.

Hence, for every proposition $pr(\varphi) \in pr(P_{C_0})$, it holds that $e_{\mathcal{I}}(pr(\varphi)) = 1$ and for every proposition $pr(\psi) \in pr(Y_{C_0})$, it holds that $e_{\mathcal{I}}(pr(\psi)) < 1$ and, therefore, $e_{\mathcal{I}}$ is a quasi-witnessing propositional evaluation. □

Proposition 3.2.12. *Let $\mathcal{I}$ be a quasi-witnessed interpretation, $C_0(a_0)$ a $\mathfrak{I}\mathcal{ALE}$-assertion and P_{C_0}, Y_{C_0} the sets of assertions produced from $C_0(a_0)$ applying Definition 3.2.4, then, for every $\alpha \in P_{C_0} \cup Y_{C_0}$, it holds that $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$.*

Proof. We will prove the Lemma by induction on the structure of α.

1. If α is an atomic formula, it is straightforward from Definition 3.2.9.

2. If α is a generalized atom, it is straightforward from Lemma 3.2.11.

3. If α is of the form $\delta \star \gamma$ where δ, γ are either atomic formulas or generalized atoms, $\star$ is a concept constructor and $\hat{\star}$ is the respective algebraic operation, suppose, by inductive hypothesis, that $e_{\mathcal{I}}(pr(\delta)) = \delta^{\mathcal{I}}$ and $e_{\mathcal{I}}(pr(\gamma)) = \gamma^{\mathcal{I}}$. Hence,

$$\begin{aligned} & e_{\mathcal{I}}(pr(\alpha)) = \\ = \;& e_{\mathcal{I}}(pr(\delta \star \gamma)) = \\ = \;& e_{\mathcal{I}}(pr(\delta))\hat{\star}e_{\mathcal{I}}(pr(\gamma)) = \\ = \;& \delta^{\mathcal{I}}\hat{\star}\gamma^{\mathcal{I}} = \\ = \;& (\delta \star \gamma)^{\mathcal{I}} = \alpha^{\mathcal{I}}. \end{aligned}$$

Hence, for every proposition $pr(\alpha)$ in $pr(P_{C_0} \cup Y_{C_0})$, it holds that

$$e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}.$$

In particular,

$$e_{\mathcal{I}}(pr(C_0(a_0))) = C_0^{\mathcal{I}}(a_0^{\mathcal{I}})$$

as we wanted to prove. □

This finishes the proof of the downwards implication of Proposition 3.2.8.

From propositional evaluations to DL interpretations

The aim of this subsection is to prove the upwards implication of Proposition 3.2.8. Let us assume that there is a propositional evaluation quasi-witnessing relatively to C_0 such that $e(pr(C_0(d))) = r$ for some $r \in [0, 1]$. First of all, we provide a way to obtain a quasi-witnessed interpretation from a quasi-witnessing propositional evaluation with the above features.

Definition 3.2.13. Let $C_0(a_0)$ be an assertion, P_{C_0} and Y_{C_0} be the sets of concepts and axioms produced from $C_0(a_0)$ applying Definition 3.2.4, let $pr(P_{C_0})$, $pr(Y_{C_0})$ be the sets of propositions obtained by applying Definition 3.2.6 and let e be a quasi-witnessing propositional evaluation. Then we define a witnessed interpretation $\mathcal{I}_e^w$ as follows:

1. $\Delta^{\mathcal{I}_e^w}$ is the set of all constants d_σ occurring in formulas of $P_{C_0} \cup Y_{C_0}$.
2. For each atomic concept A, let:
   (a) $A^{\mathcal{I}_e^w}(d_\sigma) = e(pr(A(d_\sigma)))$, where $\sigma = l(A)$, if $pr(A(d_\sigma))$ occurs in $pr(P_{C_0})$,
   (b) $A^{\mathcal{I}_e^w}(d_\sigma) = 0$, otherwise.
3. For each role R let:
   (a) $R^{\mathcal{I}_e^w}(d_\sigma, d_{\sigma,m}) = e(pr(R(d_\sigma, d_{\sigma,m})))$, if $pr(R(d_\sigma, d_{\sigma,m}))$ occurs in $pr(P_{C_0})$,
   (b) $R^{\mathcal{I}_e^w}(d_\sigma, d_{\sigma,m}) = 0$, otherwise.

In order to illustrate Definition 3.2.13, we provide an example of the witnessed interpretation arising from $pr(P_{\texttt{Example}})$ and $pr(Y_{\texttt{Example}})$.

Example 3.2.14. *Let e be a propositional evaluation such that*

- $p_{R(d,d_1)} = p_{R(d,d_2)} = 0.5,$
- $p_{R(d_1,d_{1,1})} = p_{R(d_2,d_{2,1})} = p_{R(d_2,d_{2,2})} = 1,$
- $p_{A(d_{1,1})} = p_{A(d_{2,1})} = p_{A(d_{2,2})} = 0.5.$

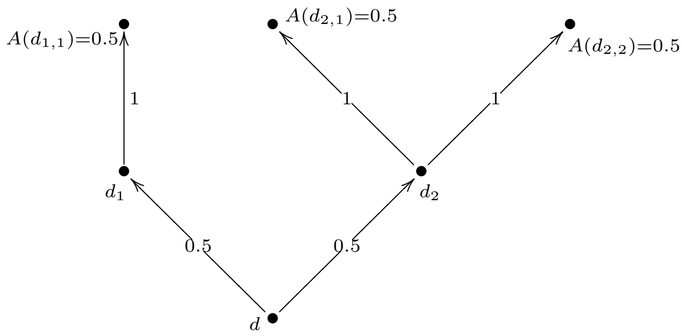

Figure 3.1: A witnessed interpretation for concept *Example*

As we have seen in the previous section, this is indeed a quasi-witnessing propositional evaluation. Moreover, following Definition 3.2.13, we obtain the interpretation depicted in Figure 3.1

We point out that this interpretation, however, is not a model of the concept `Example`*.*

The structure that can be built using the guidelines in Definition 3.2.13 is a witnessed interpretation which would be enough in case we were only interested on witnessed interpretations. But in order to encompass all quasi-witnessed interpretations we need the following extension of the above interpretation.

Definition 3.2.15. Let $C_0(a_0)$ be an assertion, P_{C_0} and Y_{C_0} be the sets of first order formulas produced from $C_0(a_0)$ applying Definition 3.2.4. and $d := a_0$. Let $pr(P_{C_0})$, $pr(Y_{C_0})$ be the sets of propositions obtained by applying Definition 3.2.6 and let e be a quasi-witnessing propositional evaluation; finally let $\mathcal{I}_e^w$ be the interpretation defined in Definition 3.2.13. Then we define the first order interpretation $\mathcal{I}_e$ as the following expansion of $\mathcal{I}_e^w$:

1. The domain $\Delta^{\mathcal{I}_e}$ is obtained by adding to $\Delta^{\mathcal{I}_e^w}$ an infinite set of new individuals $\{d_\sigma^i | i \in \mathbb{N}\backslash\{0\}\}$, for each $d_\sigma \in \Delta^{\mathcal{I}_e^w}$, but not for d.

2. If A is an atomic concept, and $pr(A(d_\sigma^i))$ occurs in $pr(P_{C_0})$, then
$$A^{\mathcal{I}_e}(d_\sigma^i) = (A^{\mathcal{I}_e}(d_\sigma))^i.$$

3. For each role R:
   (a) if R appears in an universally quantified formula, then:
      i. if $e(pr(\forall R.C(d_\sigma))) \neq e(pr(R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_\alpha)))$, then:
         A. $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}^i) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}))^i$, for every $i \in \mathbb{N}\backslash\{0\}$,
         B. $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,m}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}))^j$, for every $i, j \in \mathbb{N}\backslash\{0\}$,

ii. if $e(pr(\forall R.C(d_\sigma))) = e(pr(R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})))$, then

$$R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,m}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}))^j,$$

for every $i, j \in \mathbb{N}\backslash\{0\}$, if $i = j$ and

$$R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,m}^j) = 0,$$

otherwise,

(b) if R appears in an existentially quantified formula, then

$$R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,m}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}))^j,$$

for every $i, j \in \mathbb{N}\backslash\{0\}$, if $i = j$ and

$$R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,m}^j) = 0,$$

otherwise.

In order to illustrate Definition 3.2.15, we provide an example of the quasi-witnessed interpretation arising from $pr(P_{\texttt{Example}})$ and $pr(Y_{\texttt{Example}})$.

Example 3.2.16. *Let e be the same propositional evaluation as in the previous example, then, following Definition 3.2.13, we obtain the interpretation depicted in Figure 3.2.*

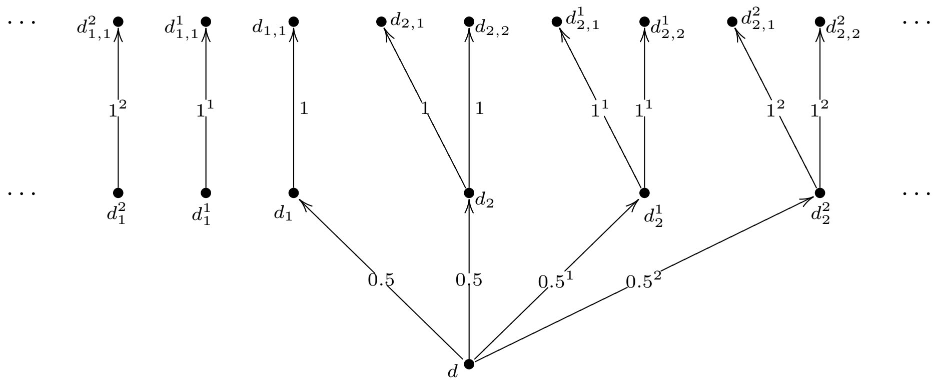

Figure 3.2: A quasi-witnessed interpretation for concept *Example*

In this case it is worth pointing out that this interpretation is indeed a model of `Example`.

Lemma 3.2.17. *Let $D(d_\sigma) \in Sub(C_0)$ and e a quasi-witnessing propositional evaluation, then, for each $i \in \mathbb{N}\backslash\{0\}$, it holds that*

$$D^{\mathcal{I}_e}(d_\sigma^i) = (D^{\mathcal{I}_e}(d_\sigma))^i.$$

Proof. The proof is by induction on the nesting degree of C_0.

(0) An assertion with nesting degree equal to 0 is either an atomic concept or a propositional combination of atomic concepts:

1. If C_0 is an atomic concept, then it is straightforward from Definition 3.2.15.
2. Let $C_0 = E\hat{\star}F$, where E, F are atomic concepts and $\hat{\star} \in \{\Rightarrow, *\}$. Suppose, by inductive hypothesis, that the claim holds for two concepts E, F, then:

$$\begin{aligned} & (E^{\mathcal{I}_e}\hat{\star}F^{\mathcal{I}_e})(d^i_\sigma) = \\ = \;& E^{\mathcal{I}_e}(d^i_\sigma)\hat{\star}F^{\mathcal{I}_e}(d^i_\sigma) = \\ = \;& (E^{\mathcal{I}_e}(d_\sigma))^i\hat{\star}(F^{\mathcal{I}_e}(d_\sigma))^i = \\ = \;& (E^{\mathcal{I}_e}(d_\sigma)\hat{\star}F^{\mathcal{I}_e}(d_\sigma))^i = \\ = \;& (E^{\mathcal{I}_e}\hat{\star}F^{\mathcal{I}_e}(d_\sigma))^i \end{aligned}$$

(k+1) Let $D(d_\sigma)$ be a generalized atom with nesting degree equal to $k+1$ and suppose, by inductive hypothesis, that, for each generalized atom $E(d_{\sigma,m})$ with nesting degree equal to k, it holds that $E^{\mathcal{I}_e}(d^i_{\sigma,m}) = (E^{\mathcal{I}_e}(d_{\sigma,m}))^i$, then:

1. If $D(d_\sigma) = \exists R.E(d_\sigma)$, then, by Definition 3.2.15,

$$\begin{aligned} & D^{\mathcal{I}_e}(d^i_\sigma) = \\ = \;& \sup_{d\in\Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d^i_\sigma, d) * E^{\mathcal{I}_e}(d) = \\ = \;& R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,m}) * E^{\mathcal{I}_e}(d^i_{\sigma,m}) \end{aligned}$$

   and, by inductive hypothesis, Definition 3.2.4 and Definition 3.2.15,

$$\begin{aligned} & R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,m}) * E^{\mathcal{I}_e}(d^i_{\sigma,m}) = \\ = \;& (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}))^i * (E^{\mathcal{I}_e}(d_{\sigma,m}))^i = \\ = \;& (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) * E^{\mathcal{I}_e}(d_{\sigma,m}))^i = \\ = \;& (D^{\mathcal{I}_e}(d_\sigma))^i. \end{aligned}$$

2. If $D(d_\sigma) = \forall R.E(d_\sigma)$, and $e(pr(\forall R.E(d_\sigma))) = (R(d_\sigma, d_{\sigma,m}) \sqsupset E(d_{\sigma,m}))$, then, by Definition 3.2.15,

$$\begin{aligned} & E^{\mathcal{I}_e}(d^i_\sigma) = \\ = \;& \inf_{d\in\Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d^i_\sigma, d) \Rightarrow E^{\mathcal{I}_e}(d) = \\ = \;& R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,m}) \Rightarrow E^{\mathcal{I}_e}(d^i_{\sigma,m}) \end{aligned}$$

and, by inductive hypothesis, Definition 3.2.4 and Definition 3.2.15,

$$\begin{aligned} & R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,m}) \Rightarrow E^{\mathcal{I}_e}(d^i_{\sigma,m}) = \\ = \quad & (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}))^i \Rightarrow (E^{\mathcal{I}_e}(d_{\sigma,m}))^i = \\ = \quad & (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) \Rightarrow E^{\mathcal{I}_e}(d_{\sigma,m}))^i = \\ = \quad & (D^{\mathcal{I}_e}(d_\sigma))^i. \end{aligned}$$

3. If $D(d_\sigma) = \forall R.E(d_\sigma)$, and $e(pr(\forall R.E(d_\sigma))) \neq (R(d_\sigma, d_{\sigma,m}) \sqsupset E(d_{\sigma,m}))$, then, by Definition 3.2.15,

$$D^{\mathcal{I}_e}(d_\sigma) = 0$$

and, therefore, by Definition 3.2.15,

$$\begin{aligned} & D^{\mathcal{I}_e}(d^i_\sigma) = \\ = \quad & \inf_{d \in \Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d^i_\sigma, d) \Rightarrow E^{\mathcal{I}_e}(d)\} = \\ = \quad & \inf_{j \in \mathbb{N} \setminus \{0\}} \{R^{\mathcal{I}_e}(d^i_\sigma, d^j_{\sigma,m}) \Rightarrow E^{\mathcal{I}_e}(d^j_{\sigma,m})\} = \\ = \quad & 0 = \\ = \quad & (D^{\mathcal{I}_e}(d_\sigma))^i. \end{aligned}$$

So, in every case we have that $D^{\mathcal{I}_e}(d^i_\sigma) = (D^{\mathcal{I}_e}(d_\sigma))^i$. □

Proposition 3.2.18. *Let e be a quasi-witnessing propositional evaluation, then, for every assertion α,*

$$e(pr(\alpha)) = \alpha^{\mathcal{I}_e}.$$

Proof. The proof is by induction on the nesting degree of α.

(0) An assertion with nesting degree equal to 0 is either an atomic concept or a propositional combination of atomic concepts:

1. If α is an atomic concept, then it is straightforward from Definition 3.2.13.
2. Let $\alpha = C \Box D$, where C, D are concepts, $\Box \in \{\sqsupset, \boxtimes\}$ and let $\star\{\rightarrow, \otimes\}$ and $\hat{\star} \in \{\Rightarrow, *\}$. Suppose that the inductive hypothesis holds for two concepts C, D, then, by Definition 3.2.6 we have that, for each concept constructor $\star$:

$$\begin{aligned} & (C \Box D)^{\mathcal{I}_e} = \\ = \quad & C^{\mathcal{I}_e} \hat{\star} D^{\mathcal{I}_e} = \\ = \quad & e(pr(C)) \hat{\star} e(pr(D)) = \\ = \quad & e(pr(C) \star pr(D)) = \\ = \quad & e(pr(C \Box D)) \end{aligned}$$

(k+1) Let α be a generalized atom with nesting degree equal to $k+1$ and suppose, by inductive hypothesis, that, for each generalized atom β with nesting degree $\leq n$, occurring within the scope of the quantifier of α, it holds that $e(pr(\beta)) = \beta^{\mathcal{I}_e}$.

1. If $\alpha = \exists R.C(d_\sigma)$, then, since e is quasi-witnessing we have that

$$e(pr(\alpha)) = e(pr(R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m})))$$

and, by Definition 3.2.13 and inductive hypothesis, we have that

$$e(pr(R(d_\sigma, d_{\sigma,m}) \boxtimes C(d_{\sigma,m}))) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) * C^{\mathcal{I}_e}(d_{\sigma,m}).$$

Let $d \in \Delta^{\mathcal{I}_e}$ be any constant different from $d_{\sigma,m}$, then either d is associated to to R, d_σ or not.

In the first case, since e is quasi-witnessing and $e(pr(R(d_\sigma, d) \boxtimes C(d))) \Rightarrow e(pr(\alpha)) = 1$, then

$$R^{\mathcal{I}_e}(d_\sigma, d) * C^{\mathcal{I}_e}(d) \leq e(pr(\alpha)).$$

In the second case, by Definition 3.2.13,

$$\begin{aligned} & R^{\mathcal{I}_e}(d_\sigma, d) * C^{\mathcal{I}_e}(d) = \\ = \; & 0 * C^{\mathcal{I}_e}(d) = \\ = \; & 0 \leq \\ \leq \; & e(pr(\alpha)). \end{aligned}$$

So, in each case,

$$\begin{aligned} & e(pr(\alpha)) = \\ = \; & R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) * C^{\mathcal{I}_e}(d_{\sigma,m}) = \\ = \; & \sup_{d \in \Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d_\sigma, d) * C^{\mathcal{I}_e}(d)\} = \\ = \; & \alpha^{\mathcal{I}_e}. \end{aligned}$$

2. If $\alpha = \forall R.C(a)$ and $e(pr(\alpha)) = e(pr(R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})))$, then, by Definition 3.2.13 and inductive hypothesis, we have that

$$e(pr(\alpha)) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,m}).$$

Let $d \in \Delta^{\mathcal{I}_e}$ be any constant different from $d_{\sigma,m}$, then either d is associated to R, d_σ or not.

In the first case, since e is quasi-witnessing and $e(pr(\alpha)) \Rightarrow e(pr(R(d_\sigma, d) \sqsupset C(d))) = 1$, then

$$e(pr(\alpha)) \leq R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d).$$

In the second case, by Definition 3.2.15,

$$\begin{aligned} & R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d) = \\ = \;& 0 \Rightarrow C^{\mathcal{I}_e}(d) = \\ = \;& 1 \geq \\ \geq \;& e(pr(\alpha)). \end{aligned}$$

So, in each case,

$$\begin{aligned} & e(pr(\alpha)) = \\ = \;& R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,m}) = \\ = \;& \inf_{d \in \Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d)\} = \\ = \;& \alpha^{\mathcal{I}_e}. \end{aligned}$$

3. If $\alpha = \forall R.C(a)$ and $e(pr(\alpha)) \neq e(pr(R(d_\sigma, d_{\sigma,m}) \sqsupset C(d_{\sigma,m})))$, then, since e is quasi-witnessing we have that $0 = e(pr(\alpha))$ and, by Definition 3.2.13 and inductive hypothesis, we have that $e(pr(\alpha)) \neq R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,m})$. Again since e is quasi-witnessing (look at the set Y) we have that

$$R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,m}) < 1$$

and, by the above assumptions, we have that

$$R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,m}) > 0.$$

Since, by Lemma 3.2.17, we have that, for each $i \in \mathbb{N}\backslash\{0\}$,

$$R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d^i_{\sigma,m}) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,m}))^i,$$

then

$$\begin{aligned} & e(pr(\alpha)) = \\ = \;& 0 = \\ = \;& \inf_{i \in \mathbb{N}\backslash\{0\}} \{(R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,m}))^i\} = \\ = \;& \inf_{i \in \mathbb{N}\backslash\{0\}} \{R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,m}) \Rightarrow C^{\mathcal{I}_e}(d^i_{\sigma,m})\} = \\ = \;& \inf_{d \in \Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d)\} = \\ = \;& \alpha^{\mathcal{I}_e}. \end{aligned}$$

The result is straightforward for propositional combinations of atomic concepts and generalized atoms with nesting degree equal to $k + 1$.

In particular, $e(pr(C_0(a_0))) = C_0^{\mathcal{I}_e}(a_0)$. $\square$

This finishes the last step in the proof of Proposition 3.2.8, and so the last step in the proof of Theorem 3.2.1.

3.3 Concept subsumption and other problems

In this section we report further results that can be obtained exploiting the methods proposed in Section 3.1 and Section 3.2. Moreover, exploiting the reductions provided in Section 2.5.1 we obtain decidability of other notions of satisfiability.

3.3.1 The case of $[0,1]_{\text{Ł}}$-$\mathcal{ALC}$

The same procedures proposed in [Háj05], for the case of witnessed interpretations and Łukasiewicz semantics were used by Hájek to prove decidability of concept 1-subsumption for language $[0,1]_{\text{Ł}}$-$\mathcal{ALC}$. [Háj05] indeed deals with *validity of formulas*, but recall that a concept C is r-subsumed by a concept D if and only if concept $\top$ is r-subsumed by concept $C \sqsupset D$. So, it is straightforward that, concept $C \sqsupset D$ is valid, if and only if $\top$ is 1-subsumed by $C \sqsupset D$, if and only if C is 1-subsumed by D. The 1-subsumption problem is reduced to the propositional entailment problem. In [Háj05, Theorem 1] it is indeed stated that a concept C_0 is valid iff $pr(P_{C_0})$ entails $pr(C_0)$. This, clearly means that

$$\text{concept } C \text{ is 1-subsumed by concept } D \iff pr(T_{C \sqsupset D}) \text{ entails } pr(C \sqsupset D)$$

In [Háj05] is indeed proved the following result.

Theorem 3.3.1 ([Háj05],Theorem 1)**.** *Concept 1-subsumption is a decidable problem in* $[0,1]_{Ł}$-$\mathcal{ALC}$.

Moreover, exploiting Proposition 2.5.3, we can obtain the following results from Theorem 3.1.4 and Theorem 3.3.1.

Corollary 3.3.2. *Concept* $(\geq r)$*-satisfiability and positive satisfiability are decidable problems in* $[0,1]_{Ł}$-$\mathcal{ALC}$.

3.3.2 The case of $[0,1]_{\Pi}$-$\mathfrak{I}\mathcal{ALE}$

For the case of quasi-witnessed interpretations and product t-norm the procedure proposed in Section 3.2 can be used as well. The reason is, again, that the logical consequence problem is decidable for Product Logic and, as for positive satisfiability, we can reduce the problem of deciding whether a concept C is subsumed by concept D in degree 1, to the logical consequence problem in propositional Product Logic. In fact, we have that

$$C \sqsubseteq D \iff pr(P_{C \sqsupset D}) \text{ entails } pr((C \sqsupset D)(d_0)) \vee \bigvee pr(Y_{C \sqsupset D})$$

In Appendix A it is proven that quasi-witnessed validity and unrestricted validity indeed coincide under the standard product semantics. Hence unrestricted validity is a decidable problem in $[0,1]_\Pi$. In this way, in [CEB10] has been proven the following theorem.

Theorem 3.3.3. *Concept 1-subsumption is a decidable problem in* $[0,1]_\Pi$-$\mathfrak{I}\mathcal{ALE}$.

Moreover, exploiting Proposition 2.5.4, we can obtain the following results from Theorem 3.2.1 and Theorem 3.3.3.

Corollary 3.3.4. *Concept* $(\geq r)$*-satisfiability positive satisfiability and positive satisfiability are decidable problems in* $[0,1]_\Pi$-$\mathfrak{I}\mathcal{ALE}$.

3.4 Knowledge base consistency in Łukasiewicz logic

The suspicion that general KB consistency was not a decidable problem with an infinite (non-idempotent) set of truth values, began when in [BBS11] was proved the failure of the finite model property for language $\mathfrak{I}\mathcal{ALCE}$ based on Łukasiewicz and product t-norms. Its decidability kept being suspicious with [BP11a]. Nevertheless, until then, there was no direct evidence that those problems were undecidable.

The first result on undecidability can be found in [BP11b] for language $\mathfrak{I}\mathcal{ALCE}$ based on product t-norm with respect to witnessed interpretations. The proof in [BP11b] consists in a reduction of the *Post Correspondence Problem* (PCP) to the general KB consistency for language $\mathfrak{I}\mathcal{ALCE}$ based on product t-norm. Subsequently, using the same methods as [BP11b] and [BP11f], this author together with U. Straccia achieved an undecidability proof for language $\mathfrak{I}\mathcal{ALCE}$ based on Łukasiewicz t-norm with respect to witnessed interpretations.

Our proof consists of a reduction of the *reverse* of the PCP and follows conceptually the one in [BP11a, BP11b, BP11f]. PCP is well known to be undecidable [Pos46], so is the reverse PCP, as shown next. Let $s \in \mathbb{N} \setminus \{0,1\}$ be fixed in the rest of Section 3.4.

Definition 3.4.1 (PCP)**.** An instance φ of the *Post Correspondence Problem* (PCP) is defined in the following way: let $v_1,\ldots,v_p$ and $w_1,\ldots,w_p$ be two finite lists of words over an alphabet $\Sigma = \{1,\ldots,s\}$. A *solution* to φ is a non-empty sequence $i_1, i_2, \ldots, i_k$, with $1 \leq i_j \leq p$ such that $v_{i_1}v_{i_2}\ldots v_{i_k} = w_{i_1}w_{i_2}\ldots w_{i_k}$. The decision problem then is to decide, given φ, whether a solution to φ exists or not.

For the sake of our purpose, we will rely on a variant of the PCP, which we call *Reverse* PCP (RPCP). Essentially, words are concatenated from right to left rather than from left to right.

Definition 3.4.2 (RPCP)**.** An instance φ of the *Reverse Post Correspondence Problem* (RPCP) is defined in the following way: let $v_1, \dots, v_p$ and $w_1, \dots, w_p$ be two finite lists of words over an alphabet $\Sigma = \{1, \dots, s\}$. A *solution* to φ is a non-empty sequence $i_1, i_2, \dots, i_k$, with $1 \leq i_j \leq p$ such that $v_{i_k} v_{i_{k-1}} \dots v_{i_1} = w_{i_k} w_{i_{k-1}} \dots w_{i_1}$. The decision problem then is to decide, given φ, whether a solution to φ exists or not.

For a word $\mu = i_1 i_2 \dots i_k \in \{1, \dots, p\}^*$ we will use v_μ, w_μ to denote the words $v_{i_k} v_{i_{k-1}} \dots v_{i_1}$ and $w_{i_k} w_{i_{k-1}} \dots w_{i_1}$. We denote the empty string as ε and define v_ε as ε. The alphabet Σ consists of the first s positive integers. We can thus view every word in Σ^* as a natural number represented in base $s + 1$ in which 0 never occurs. Using this intuition, we will use the number 0 to encode the empty word.

Now we show that the reduction from PCP to RPCP is a very simple matter and it can be done through the transformation of the instance lists to the lists of their palindromes defined as follows: let $\Sigma = \{1, \dots, s\}$ be an alphabet and $v = t_1 t_2 \dots t_{|v|}$ a word over Σ, with $t_i \in \Sigma$, for $1 \leq i \leq |v|$, then the function

$$pal \colon \Sigma^* \to \Sigma^*$$

is defined by

$$pal(v) = t_{|v|} t_{|v|-1} \dots t_1.$$

We will say that $pal(v)$ is the *palindrome* of v.

Lemma 3.4.3. *Let $v_1, \dots, v_p$ and $w_1, \dots, w_p$ be two finite lists of words over an alphabet $\Sigma = \{1, \dots, s\}$. For every non-empty sequence $i_1, i_2, \dots, i_k$, with $1 \leq i_j \leq p$ it holds that*

$$\begin{array}{rcl} v_{i_1} v_{i_2} \dots v_{i_k} & = & w_{i_1} w_{i_2} \dots w_{i_k} \\ & \textit{iff} & \\ pal(v_{i_k}) pal(v_{i_{k-1}}) \dots pal(v_{i_1}) & = & pal(w_{i_k}) pal(w_{i_{k-1}}) \dots pal(w_{i_1}) \ . \end{array}$$

Proof. First we prove by induction on k, that, for every sequence $v = v_{i_1} v_{i_2} \dots v_{i_k}$ of words over Σ, it holds that $pal(v) = pal(v_{i_k}) pal(v_{i_{k-1}}) \dots pal(v_{i_1})$.

- The case $k = 1$ is straightforward.
- Let $v = v_{i_1} v_{i_2} \dots v_{i_k}$ and suppose, by inductive hypothesis, that $pal(v_{i_1} v_{i_2} \dots v_{i_{k-1}}) = pal(v_{i_{k-1}}) pal(v_{i_{k-2}}) \dots pal(v_{i_1})$. It follows that $pal(v) = pal(v_{i_1} v_{i_2} \dots v_{i_{k-1}}, v_{i_k}) = pal(v_{i_k}) pal(v_{i_{k-1}}) \dots pal(v_{i_1})$.

Since the palindrome of a word is unique, we have that, if $v_{i_1} v_{i_2} \dots v_{i_k} = w_{i_1} w_{i_2} \dots w_{i_k}$, then $pal(v_{i_1} v_{i_2} \dots v_{i_k}) = pal(w_{i_1} w_{i_2} \dots w_{i_k})$ and, thus, $pal(v_{i_k}) pal(v_{i_{k-1}}) \dots pal(v_{i_1}) = pal(w_{i_k})\ pal(w_{i_{k-1}}) \dots\ pal(w_{i_1})$. □

Corollary 3.4.4. *RPCP is undecidable.*

Proof. The proof is based on the reduction of PCP to RPCP. For every instance $\varphi = (v_1, w_1), \ldots, (v_p, w_p)$ of PCP, let f be the function

$$f(\varphi) = (pal(v_1), pal(w_1)), \ldots, (pal(v_p), pal(w_p)) \ .$$

Clearly f is a computable function. Moreover, φ has a solution if and only if there exists a non-empty sequence $i_1, i_2, \ldots, i_k$, with $1 \leq i_j \leq p$ such that $v_{i_1} v_{i_2} \ldots v_{i_k} = w_{i_1} w_{i_2} \ldots w_{i_k}$, that is, by Lemma 3.4.3,

$$pal(v_{i_k}) pal(v_{i_{k-1}}) \ldots pal(v_{i_1}) = pal(w_{i_k}) pal(w_{i_{k-1}}) \ldots pal(w_{i_1})$$

i.e., $f(\varphi)$ has a solution. Therefore, $\varphi \in PCP$ has a solution if and only if $f(\varphi) \in RPCP$ has a solution. □

3.4.1 Undecidability of general KB satisfiability

We show the undecidability of the KB satisfiability problem for language $\mathcal{ALC}$ over $[0,1]_{\text{Ł}}$, by a reduction of RPCP to the cited problem.

Given an instance φ of RPCP, we will construct a Knowledge Base $\mathcal{O}_\varphi$ that is satisfiable iff φ has no solution.

In order to do this we will encode words v over the alphabet Σ as rational numbers $0.v$ in $[0,1]$ in base $s+1$; the empty word will be encoded by the number 0.

So, let us define the following TBoxes:

$$\mathcal{T} \quad := \{ \quad \langle V \equiv V_1 \boxplus V_2 \geq 1 \rangle, \langle W \equiv W_1 \boxplus W_2 \geq 1 \rangle \quad \}$$

and for $1 \leq i \leq p$

$$\begin{aligned}
\mathcal{T}_\varphi^i \quad := \{ \quad & \langle \top \sqsubseteq \exists R_i.\top \geq 1 \rangle, \\
& \\
& \langle V \sqsubseteq (s+1)^{|v_i|} \cdot \forall R_i.V_1 \geq 1 \rangle, \\
& \langle (s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq V \geq 1 \rangle, \\
& \langle W \sqsubseteq (s+1)^{|w_i|} \cdot \forall R_i.W_1 \geq 1 \rangle, \\
& \langle (s+1)^{|w_i|} \cdot \exists R_i.W_1 \sqsubseteq W \geq 1 \rangle \\
& \\
& \langle \top \sqsubseteq \forall R_i.V_2 \geq 0.v_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.\neg V_2 \geq 1 - 0.v_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.W_2 \geq 0.w_i \rangle, \\
& \langle \top \sqsubseteq \forall R_i.\neg W_2 \geq 1 - 0.w_i \rangle, \\
& \\
& \langle A \sqsubseteq (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \forall R_i.A \geq 1 \rangle \\
& \langle (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \exists R_i.A \sqsubseteq A \geq 1 \rangle \quad \} \ .
\end{aligned}$$

Now, let

$$\mathcal{T}_\varphi = \mathcal{T} \cup \bigcup_{i=1}^{p} \mathcal{T}_\varphi^i \ .$$

Further we define the ABox $\mathcal{A}$ as follows:

$$\mathcal{A} \ := \ \{\langle \neg V(a) \geq 1\rangle, \langle \neg W(a) \geq 1\rangle, \langle A(a) \geq 0.01\rangle, \langle \neg A(a) \geq 0.99\rangle\} \ .$$

Finally, we define

$$\mathcal{O}_\varphi := (\mathcal{T}_\varphi, \mathcal{A}) \ .$$

We now define the interpretation

$$\mathcal{I}_\varphi := (\Delta^{\mathcal{I}_\varphi}, \cdot^{\mathcal{I}_\varphi})$$

as follows:

- $\Delta^{\mathcal{I}_\varphi} = \{1, \ldots, p\}^*$
- $a^{\mathcal{I}_\varphi} = \varepsilon$
- $V^{\mathcal{I}_\varphi}(\varepsilon) = W^{\mathcal{I}_\varphi}(\varepsilon) = 0$,
- $A^{\mathcal{I}_\varphi}(\varepsilon) = 0.01$,
- $V_i^{\mathcal{I}_\varphi}(\varepsilon) = W_i^{\mathcal{I}_\varphi}(\varepsilon) = 0$, for $1 \leq i \leq 2$,
- for all $\mu, \mu' \in \Delta^{\mathcal{I}_\varphi}$ and $1 \leq i \leq p$

$$R_i^{\mathcal{I}_\varphi}(\mu, \mu') = \begin{cases} 1, & \text{if } \mu' = \mu i \\ 0, & \text{otherwise} \end{cases}$$

- for every $\mu \in \Delta^{\mathcal{I}_\varphi}$, where $\mu = i_1 i_2 \ldots i_k \neq \varepsilon$
  - $V^{\mathcal{I}_\varphi}(\mu) = 0.v_\mu$,
  - $W^{\mathcal{I}_\varphi}(\mu) = 0.w_\mu$
  - $A^{\mathcal{I}_\varphi}(\mu) = 0.01 \cdot (s+1)^{-\sum_{j \in \{i_1, i_2, \ldots, i_k\}} \max\{|v_j|, |w_j|\}}$
  - $V_1^{\mathcal{I}_\varphi}(\mu) = 0.v_{\bar{\mu}} \cdot (s+1)^{-|v_{i_k}|}$, where $\bar{\mu} = i_1 i_2 \ldots i_{k-1}$ (last index i_k is dropped from μ, and we assume that $0.\varepsilon$ is 0),
  - $W_1^{\mathcal{I}_\varphi}(\mu) = 0.w_{\bar{\mu}} \cdot (s+1)^{-|w_{i_k}|}$, where $\bar{\mu} = i_1 i_2 \ldots i_{k-1}$ (last index i_k is dropped from μ, and we assume that $0.\varepsilon$ is 0),
  - $V_2^{\mathcal{I}_\varphi}(\mu) = 0.v_{i_k}$,
  - $W_2^{\mathcal{I}_\varphi}(\mu) = 0.w_{i_k}$.

It is easy to see that $\mathcal{I}_\varphi$ is a witnessed model of $\mathcal{O}_\varphi$ (note that *e.g.*, $(\forall R_i.V_1)^{\mathcal{I}_\varphi}(\mu) = V_1^{\mathcal{I}_\varphi}(\mu i)$). [1]

Moreover, as in [BP11a] it is possible to prove that, for every witnessed model $\mathcal{I}$ of $\mathcal{O}_\varphi$, there is a mapping g from $\mathcal{I}_\varphi$ to $\mathcal{I}$.

Lemma 3.4.5. *Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_\varphi$. Then there exists a function $g : \Delta^{\mathcal{I}_\varphi} \longrightarrow \Delta^{\mathcal{I}}$ such that, for every $\mu \in \Delta^{\mathcal{I}_\varphi}$, $C^{\mathcal{I}_\varphi}(\mu) = C^{\mathcal{I}}(g(\mu))$ holds for every concept name C and $R_i^{\mathcal{I}_\varphi}(\mu, \mu i) = R_i^{\mathcal{I}}(g(\mu), g(\mu i))$ holds for every i, with $1 \leq i \leq p$.*

Proof. Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_\varphi$. We will build the function g inductively on the length of μ.

(ε) Since $\mathcal{I}$ is a model of $\mathcal{O}_\varphi$, then there is an element $\delta \in \Delta^{\mathcal{I}}$ such that $a^{\mathcal{I}} = \delta$. Since $\mathcal{I}$ is a model of $\mathcal{A}_\varphi$, setting $g(\varepsilon) = \delta$, we have that

$$V^{\mathcal{I}_\varphi}(\varepsilon) = 0 = V^{\mathcal{I}}(g(\varepsilon))$$

and the same holds for concept W. Moreover, since $\mathcal{I}$ is a model of $\mathcal{T}_\varphi$, we have that $V^{\mathcal{I}}(\delta) = (V_1 \boxplus V_2)^{\mathcal{I}}(\delta)$ and, therefore

$$V_1^{\mathcal{I}_\varphi}(\varepsilon) = 0 = V_1^{\mathcal{I}}(g(\varepsilon))$$

and the same holds for V_2, W_1 and W_2. On the other hand, we have that

$$A^{\mathcal{I}_\varphi}(\varepsilon) = 0.01 = A^{\mathcal{I}}(g(\varepsilon)),$$

as well. So, $g(\varepsilon) = \delta$ satisfies the condition of the lemma.

(μi) Let now μ be such that $g(\mu)$ has already been defined. Now, since $\mathcal{I}$ is a witnessed model and satisfies axiom $\langle \top \sqsubseteq \exists R_i.\top \geq 1\rangle$, then for all i, with $1 \leq i \leq p$, there exists a $\gamma \in \Delta^{\mathcal{I}}$ such that $R_i^{\mathcal{I}}(g(\mu), \gamma) = 1$. So, setting $g(\mu i) = \gamma$ we get

$$1 = R_i^{\mathcal{I}_\varphi}(\mu, \mu i) = R_i^{\mathcal{I}}(g(\mu), g(\mu i)).$$

Furthermore, by induction hypothesis, we can assume that

$$V^{\mathcal{I}}(g(\mu)) = 0.v_\mu$$

and

$$W^{\mathcal{I}}(g(\mu)) = 0.w_\mu.$$

[1] However, $\mathcal{I}_\varphi$ is not a strongly witnessed model of $\mathcal{O}_\varphi$.

Since $\mathcal{I}$ satisfies axiom $\langle V \sqsubseteq (s+1)^{|v_i|} \cdot \forall R_i.V_1 \geq 1\rangle$, then

$$\begin{array}{rl} & 0.v_\mu = \\ = & V^{\mathcal{I}}(g(\mu)) \leq (s+1)^{|v_i|} \cdot (\forall R_i.V_1)^{\mathcal{I}}(g(\mu)) = \\ = & (s+1)^{|v_i|} \cdot \inf_{\gamma \in \Delta^{\mathcal{I}}} \{R_i^{\mathcal{I}}(g(\mu), \gamma) \Rightarrow V_1^{\mathcal{I}}(\gamma)\} \leq \\ \leq & (s+1)^{|v_i|} \cdot (R_i^{\mathcal{I}}(g(\mu), \mu i) \Rightarrow V_1^{\mathcal{I}}(\mu i)) = \\ = & (s+1)^{|v_i|} \cdot V_1^{\mathcal{I}}(g(\mu i)). \end{array}$$

Since $\mathcal{I}$ satisfies axiom $\langle (s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq V \geq 1\rangle$, then

$$\begin{array}{rl} & 0.v_\mu = \\ = & V^{\mathcal{I}}(g(\mu)) \geq (s+1)^{|v_i|} \cdot (\exists R_i.V_1)^{\mathcal{I}}(g(\mu)) = \\ = & (s+1)^{|v_i|} \cdot \sup_{\gamma \in \Delta^{\mathcal{I}}} \{R_i^{\mathcal{I}}(g(\mu), \gamma) * V_1^{\mathcal{I}}(\gamma)\} \geq \\ \geq & (s+1)^{|v_i|} \cdot (R_i^{\mathcal{I}}(g(\mu), \mu i) * V_1^{\mathcal{I}}(\mu i)) = \\ = & (s+1)^{|v_i|} \cdot V_1^{\mathcal{I}}(g(\mu i)). \end{array}$$

Therefore,

$$(s+1)^{|v_i|} \cdot V_1^{\mathcal{I}}(g(\mu i)) = 0.v_\mu$$

and

$$V_1^{\mathcal{I}}(g(\mu i)) = 0.v_\mu \cdot (s+1)^{-|v_i|} = V_1^{\mathcal{I}_\varphi}(\mu i).$$

Similarly, it can be shown that

$$W_1^{\mathcal{I}}(g(\mu i)) = 0.w_\mu \cdot (s+1)^{-|w_i|} = W_1^{\mathcal{I}_\varphi}(\mu i).$$

Since $\mathcal{I}$ satisfies axioms $\langle \top \sqsubseteq \forall R_i.V_2 \geq 0.v_i\rangle$ and $\langle \top \sqsubseteq \forall R_i.\neg V_2 \geq 1-0.v_i\rangle$, it follows that $(\forall R_i.V_2)^{\mathcal{I}}(g(\mu)) \geq 0.v_i$ and $(\forall R_i.\neg V_2)^{\mathcal{I}}(g(\mu)) \geq 1 - 0.v_i$. Therefore, for $R_i^{\mathcal{I}}(g(\mu), g(\mu i)) = 1$ we have

$$V_2^{\mathcal{I}}(g(\mu i)) = 0.v_i = V_2^{\mathcal{I}_\varphi}(\mu i).$$

Similarly, it can be shown that

$$W_2^{\mathcal{I}_\varphi}(\mu i) = 0.w_i = W_2^{\mathcal{I}}(g(\mu i)).$$

Now, since $\mathcal{I}$ satisfies axiom $\langle V \equiv V_1 \boxplus V_2 \geq 1\rangle$, then,

$$\begin{array}{rl} & V^{\mathcal{I}}(g(\mu i)) = \\ = & V_1^{\mathcal{I}}(g(\mu i)) + V_2^{\mathcal{I}}(g(\mu i)) = \\ = & 0.v_\mu \cdot (s+1)^{-|v_i|} + 0.v_i = \\ = & 0.v_i v_\mu = \\ = & V^{\mathcal{I}_\varphi}(\mu i). \end{array}$$

Finally, by inductive hypothesis, assume that

$$A^{\mathcal{I}}(g(\mu)) = A^{\mathcal{I}_\varphi}(\mu) = 0.01 \cdot (s+1)^{-\sum_{j\in\{i_1,i_2,\ldots,i_k\}} \max\{|v_j|,|w_j|\}} ,$$

where $\mu = i_1 i_2 \ldots i_k$.

Since $\mathcal{I}$ satisfies axioms $\langle A \sqsubseteq (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \forall R_i.A \geq 1\rangle$, we have that

$$\begin{array}{rl} & A^{\mathcal{I}}(g(\mu)) \leq \\ \leq & (s+1)^{\max\{|v_i|,|w_i|\}} \cdot (\forall R_i.A)^{\mathcal{I}}(g(\mu)) \leq \\ \leq & (s+1)^{\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}}(g(\mu i)). \end{array}$$

Likewise, since $\mathcal{I}$ satisfies axioms $\langle (s+1)^{\max\{|v_i|,|w_i|\}} \cdot \exists R_i.A \sqsubseteq A \geq 1\rangle$, we have that

$$\begin{array}{rl} & A^{\mathcal{I}}(g(\mu)) \geq \\ \geq & (s+1)^{\max\{|v_i|,|w_i|\}} \cdot (\exists R_i.A)^{\mathcal{I}}(g(\mu)) \geq \\ \geq & (s+1)^{\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}}(g(\mu i)) \end{array}$$

and, thus,

$$A^{\mathcal{I}}(g(\mu)) = (s+1)^{\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}}(g(\mu i)) .$$

Therefore,

$$\begin{array}{rl} & A^{\mathcal{I}}(g(\mu i)) = \\ = & (s+1)^{-\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}}(g(\mu)) = \\ = & (s+1)^{-\max\{|v_i|,|w_i|\}} \cdot A^{\mathcal{I}_\varphi}(\mu) = \\ = & (s+1)^{-\max\{|v_i|,|w_i|\}} \cdot 0.01 \cdot (s+1)^{-\sum_{j\in\{i_1,i_2,\ldots,i_k\}} \max\{|v_j|,|w_j|\}} = \\ = & 0.01 \cdot (s+1)^{-(\max\{|v_i|,|w_i|\}+\sum_{j\in\{i_1,i_2,\ldots,i_k\}} \max\{|v_j|,|w_j|\})} = \\ = & 0.01 \cdot (s+1)^{-\sum_{j\in\{i_1,i_2,\ldots,i_k,i\}} \max\{|v_j|,|w_j|\}} = \\ = & A^{\mathcal{I}_\varphi}(\mu i) , \end{array}$$

which completes the proof. □

From Lemma 3.4.5 it follows that if the RPCP instance φ has a solution μ, for some $\mu \in \{1, \ldots, p\}^+$, then $v_\mu = w_\mu$ and, thus, $0.v_\mu = 0.w_\mu$. Therefore, every witnessed model $\mathcal{I}$ of $\mathcal{O}_\varphi$ contains an element $\delta = g(\mu)$ such that

$$V^{\mathcal{I}}(\delta) = V^{\mathcal{I}_\varphi}(\mu) = 0.v_\mu = 0.w_\mu = W^{\mathcal{I}_\varphi}(\mu) = W^{\mathcal{I}}(\delta).$$

Conversely, from the definition of $\mathcal{I}_\varphi$, if φ has no solution, then there is no μ such that $0.v_\mu = 0.w_\mu$, *i.e.*, for every μ it holds that

$$V^{\mathcal{I}_\varphi}(\mu) \neq W^{\mathcal{I}_\varphi}(\mu).$$

However, as $\mathcal{O}_\varphi$ is always satisfiable, it does not yet help us to decide the RPCP. We next extend $\mathcal{O}_\varphi$ to $\mathcal{O}'_\varphi$ in such a way that an instance φ of the RPCP has a solution iff the ontology $\mathcal{O}'_\varphi$ is not witnessed satisfiable and, thus, establish that the KB satisfiability problem is undecidable. To this end, consider

$$\mathcal{O}'_\varphi := (\mathcal{T}'_\varphi, \mathcal{A}) \ ,$$

where

$$\mathcal{T}'_\varphi := \mathcal{T}_\varphi \cup \bigcup_{1 \leq i \leq p} \{\langle \top \sqsubseteq \forall R_i.(\neg(V \leftrightarrow W) \boxplus \neg A) \geq 1\rangle\} \ .$$

The intuition here is the following. If there is a solution for RPCP then, as a consequence of Lemma 3.4.5, there is a point δ in which the value of V and W coincide under $\mathcal{I}$. That is, the value of $\neg(V \leftrightarrow W)$ is 0 and, thus, the value of $\neg(V \leftrightarrow W) \boxplus \neg A$ is less than 1. So, $\mathcal{I}$ cannot satisfy the new GCI in $\mathcal{T}'_\varphi$ and, thus, $\mathcal{O}'_\varphi$ is not satisfiable. On the other hand, if there is no solution to the RPCP then in $\mathcal{I}_\varphi$ there is no point in which V and W coincide and, thus, $\neg(V \leftrightarrow W) > 0$. Moreover, we will show that the value of $\neg(V \leftrightarrow W)$ in all points is strictly greater than A and, as $A \boxplus \neg A$ is 1, so also $\neg(V \leftrightarrow W) \boxplus \neg A$ will be 1 in any point. Hence, $\mathcal{I}_\phi$ is a model of the additional axiom in $\mathcal{T}'_\varphi$, *i.e.*, $\mathcal{O}'_\varphi$ is satisfiable.

Proposition 3.4.6. *The instance φ of the RPCP has a solution iff the ontology $\mathcal{O}'_\varphi$ is not witnessed satisfiable.*

Proof. Assume first that φ has a solution $\mu = i_1 \ldots i_k$ and let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_\varphi$. Let $\bar{\mu} = i_1 i_2 \ldots i_{k-1}$ (last index i_k is dropped from μ). Then by Lemma 3.4.5 it follows that there are nodes $\delta, \delta' \in \Delta^{\mathcal{I}}$ such that $\delta = g(\mu)$, $\delta' = g(\bar{\mu})$, with $V^{\mathcal{I}}(\delta) = V^{\mathcal{I}_\varphi}(\mu) = W^{\mathcal{I}_\varphi}(\mu) = W^{\mathcal{I}}(\delta)$ and $R^{\mathcal{I}}_{i_k}(\delta', \delta) = 1$. Then $(V \leftrightarrow W)^{\mathcal{I}}(\delta) = 1$. Since $(\neg A)^{\mathcal{I}}(\delta) < 1$, then $(\neg(V \leftrightarrow W) \boxplus \neg A)^{\mathcal{I}}(\delta) < 1$. Hence there is i, with $1 \leq i \leq p$, such that

$$(\forall R_i.(\neg(V \leftrightarrow W) \boxplus \neg A))^{\mathcal{I}}(\delta') < 1.$$

So, axiom $\langle \top \sqsubseteq \forall R_i.(\neg(V \leftrightarrow W) \boxplus \neg A) \geq 1\rangle$ is not satisfied and, therefore, $\mathcal{O}'_\varphi$ is not satisfiable.

For the converse, assume that φ has no solution. On the one hand we know that $\mathcal{I}_\varphi$ is a model of $\mathcal{O}_\varphi$. On the other hand, since φ has no solution, then there is no $\mu = i_1 \ldots i_k$ such that $v_\mu = w_\mu$ (*i.e.*, $0.v_\mu = 0.w_\mu$) and, therefore, there is no $\mu \in \Delta^{\mathcal{I}_\varphi}$ such that $V^{\mathcal{I}_\varphi}(\mu) = W^{\mathcal{I}_\varphi}(\mu)$. Consider $\mu \in \Delta^{\mathcal{I}_\varphi}$ and i, with $1 \leq i \leq p$ and assume, without loss of generality, that $V^{\mathcal{I}_\varphi}(\mu i) < W^{\mathcal{I}_\varphi}(\mu i)$. Then

$$\begin{aligned}
& (V \leftrightarrow W)^{\mathcal{I}_\varphi}(\mu i) = \\
= \;& (V^{\mathcal{I}_\varphi}(\mu i) \Rightarrow W^{\mathcal{I}_\varphi}(\mu i)) * (W^{\mathcal{I}_\varphi}(\mu i) \Rightarrow V^{\mathcal{I}_\varphi}(\mu i)) = \\
= \;& 1 * (W^{\mathcal{I}_\varphi}(\mu i) \Rightarrow V^{\mathcal{I}_\varphi}(\mu i)) = \\
= \;& W^{\mathcal{I}_\varphi}(\mu i) \Rightarrow V^{\mathcal{I}_\varphi}(\mu i) = \\
= \;& 1 - W^{\mathcal{I}_\varphi}(\mu i) + V^{\mathcal{I}_\varphi}(\mu i) = \\
= \;& 1 - (W^{\mathcal{I}_\varphi}(\mu i) - V^{\mathcal{I}_\varphi}(\mu i)) = \\
= \;& 1 - (0.w_{\mu i} - 0.v_{\mu i}) \leq \\
\leq \;& 1 - 0.01 \cdot (s+1)^{-\max\{|v_{\mu i}|, |w_{\mu i}|\}} \leq \\
\leq \;& 1 - 0.01 \cdot (s+1)^{-\sum_{j \in \{i_1, i_2, \ldots, i_k, i\}} \max\{|v_j|, |w_j|\}} = \\
= \;& (\neg A)^{\mathcal{I}_\varphi}(\mu i) \ .
\end{aligned}$$

Therefore, $(\neg(V \leftrightarrow W))^{\mathcal{I}_\varphi}(\mu i) \geq A^{\mathcal{I}_\varphi}(\mu i)$. As $A^{\mathcal{I}_\varphi}(\mu i) \veebar (\neg A)^{\mathcal{I}_\varphi}(\mu i) = 1$, it follows that for every $\mu \in \Delta^{\mathcal{I}_\varphi}$ and i, with $1 \leq i \leq p$, it holds that $(\forall R_i.(\neg(V \leftrightarrow W) \boxplus \neg A))^{\mathcal{I}_\varphi}(\mu) = 1$ and, therefore, $\mathcal{I}_\varphi$ is a witnessed model of $\mathcal{O}'_\varphi$. □

By Proposition 3.4.6, we have a reduction of a RPCP to a KB satisfiability problem. Note that all roles are crisp.

Theorem 3.4.7. *The knowledge base satisfiability problem is undecidable for $[0,1]_L$-$\mathcal{ALC}$ with GCIs. The result also holds if crisp roles are assumed.*

3.4.2 Knowledge Base consistency w.r.t. finite interpretations

In this section we address a sub problem of the previous one. That is, deciding whether a KB has a finite interpretation.

In [BP11b] it is provided a proof of undecidability for language $[0,1]_\Pi$-$\mathfrak{I}\mathcal{ALCE}$ respect to strongly witnessed interpretations. Using the same methods as in [BP11b] and [BP11f], we will prove that KB satisfiability with respect to finite interpretations for language $[0,1]_{\text{Ł}}$-$\mathcal{ALC}$ is undecidable.

As in [BP11b], given an instance φ of RPCP, we provide an ontology $\tilde{\mathcal{O}}_\varphi$ and prove that it has a finite model iff φ has a solution. We now define a TBox $\tilde{\mathcal{T}}$ as follows:

$$\begin{aligned}
\tilde{\mathcal{T}} \;\; := \{ \;\; & \langle V \equiv V_1 \boxplus V_2 \geq 1 \rangle, \langle W \equiv W_1 \boxplus W_2 \geq 1 \rangle, \\
& \langle \neg(V \leftrightarrow W) \sqsubseteq C_1 \sqcup \ldots \sqcup C_p \geq 1 \rangle \quad \} \ ,
\end{aligned}$$

and TBoxes $\tilde{\mathcal{T}}^i_\varphi$ as follows:

$$
\begin{aligned}
\tilde{\mathcal{T}}^i_\varphi \quad := \{ \quad & \langle C_i \equiv \exists R_i.\top \geq 1\rangle, \\
& \langle \top \sqsubseteq C_i \sqcup \neg C_i \geq 1\rangle, \\
\\
& \langle (C_i \sqsupset V) \sqsubseteq (s+1)^{|v_i|} \cdot \forall R_i.V_1 \geq 1\rangle, \\
& \langle (s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq (C_i \sqsupset V) \geq 1\rangle, \\
& \langle (C_i \sqsupset W) \sqsubseteq (s+1)^{|w_i|} \cdot \forall R_i.W_1 \geq 1\rangle, \\
& \langle (s+1)^{|w_i|} \cdot \exists R_i.W_1 \sqsubseteq (C_i \sqsupset W) \geq 1\rangle, \\
\\
& \langle \top \sqsubseteq \forall R_i.V_2 \geq 0.v_i\rangle, \\
& \langle \top \sqsubseteq \forall R_i.\neg V_2 \geq 1 - 0.v_i\rangle, \\
& \langle \top \sqsubseteq \forall R_i.W_2 \geq 0.w_i\rangle, \\
& \langle \top \sqsubseteq \forall R_i.\neg W_2 \geq 1 - 0.w_i\rangle \quad \}
\end{aligned}
$$

Now, let

$$\tilde{\mathcal{T}}_\varphi = \tilde{\mathcal{T}} \cup \bigcup_{i=1}^{p} \tilde{\mathcal{T}}^i_\varphi \ .$$

Further we define the ABox $\tilde{\mathcal{A}}_\varphi$ as follows:

$$\tilde{\mathcal{A}}_\varphi \quad := \quad \{\langle \neg V(a) \geq 1\rangle, \langle \neg W(a) \geq 1\rangle, \langle (C_1 \sqcup \ldots \sqcup C_p)(a) \geq 1\rangle\} \ .$$

Finally,

$$\tilde{\mathcal{O}}_\varphi := (\tilde{\mathcal{T}}_\varphi, \tilde{\mathcal{A}}_\varphi) \ .$$

Proposition 3.4.8. *The instance φ of the RPCP has a solution iff the $[0,1]_L$-$\mathcal{ALC}$ ontology $\tilde{\mathcal{O}}_\varphi$ has a finite model.*

Proof. **($\Rightarrow$)** Let $\mu = i_1 \ldots i_k$ be a solution of φ and let $suf(\mu)$ be the set of all suffixes of μ [2]. We build the finite interpretation $\tilde{\mathcal{I}}_\varphi$ as follows:

- $\Delta^{\tilde{\mathcal{I}}_\varphi} := suf(\mu)$,
- $a^{\tilde{\mathcal{I}}_\varphi} = \varepsilon$,
- $V^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = W^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 0$,
- $V_i^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = W_i^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 0$, for $1 \leq i \leq 2$,
- for all $\nu \in \Delta^{\tilde{\mathcal{I}}_\varphi}$, $V^{\tilde{\mathcal{I}}_\varphi}(\nu) = 0.v_\nu$ and $W^{\tilde{\mathcal{I}}_\varphi}(\nu) = 0.w_\nu$

[2] A *suffix* of a string $t_1t_2 \ldots t_n$ is a string $t_{n-m+1} \ldots t_n$ $(0 \leq m \leq n)$, which is the empty string ε for $m = 0$.

- for all $\nu, \nu' \in \Delta^{\tilde{\mathcal{I}}_\varphi}$ and $1 \leq i \leq p$

$$R_i^{\tilde{\mathcal{I}}_\varphi}(\nu, \nu') = \begin{cases} 1, & \text{if } \nu' = i\nu \\ 0, & \text{otherwise} \end{cases}$$

- for all $\nu \in \Delta^{\tilde{\mathcal{I}}_\varphi}$ and $1 \leq i \leq p$,

$$C_i^{\tilde{\mathcal{I}}_\varphi}(\nu) = \begin{cases} 1, & \text{if } i\nu \in suf(\mu) \\ 0, & \text{otherwise} \end{cases}$$

- for all $\nu \in \Delta^{\tilde{\mathcal{I}}_\varphi}$ and $1 \leq i \leq p$ such that $i\nu \in suf(\mu)$
  - $V_1^{\tilde{\mathcal{I}}_\varphi}(i\nu) = 0.v_\nu \cdot (s+1)^{-|v_i|}$,
  - $W_1^{\tilde{\mathcal{I}}_\varphi}(i\nu) = 0.w_\nu \cdot (s+1)^{-|w_i|}$,
  - $V_2^{\tilde{\mathcal{I}}_\varphi}(i\nu) = 0.v_i$,
  - $W_2^{\tilde{\mathcal{I}}_\varphi}(i\nu) = 0.w_i$.

We show now that $\tilde{\mathcal{I}}_\varphi$ is a model $\tilde{\mathcal{O}}_\varphi$. Since $V^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 0.v_\varepsilon = 0$ and $W^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 0.w_\varepsilon = 0$, then the first two axioms in $\tilde{\mathcal{A}}_\varphi$ are satisfied. Since there is $1 \leq i \leq p$ such that $i\varepsilon = i \in suf(\mu)$, then

$$C_i^{\tilde{\mathcal{I}}_\varphi}(\varepsilon) = 1$$

and, therefore, the third axiom in $\tilde{\mathcal{A}}_\varphi$ is satisfied.

We now show that the axioms in $\tilde{\mathcal{T}}$ and each $\tilde{\mathcal{T}}_\varphi^i$, with $1 \leq i \leq p$ are satisfied for every $\nu \in suf(\mu)$. So, let $\nu \in suf(\mu) \setminus \{\mu\}$. Then there is $1 \leq i \leq p$ such that $i\nu \in suf(\mu)$ and, therefore, by the definition of $\tilde{\mathcal{I}}_\varphi$, $C_i^{\tilde{\mathcal{I}}_\varphi}(\nu) = 1$ and $R_i^{\tilde{\mathcal{I}}_\varphi}(\nu, i\nu) = 1$. Therefore,

$$(C_i \sqsupset V)^{\tilde{\mathcal{I}}_\varphi}(\nu) = V^{\tilde{\mathcal{I}}_\varphi}(\nu)$$

from which it follows that every axiom in $\tilde{\mathcal{T}}_\varphi^i$ is satisfied by $\tilde{\mathcal{I}}_\varphi$ (the proof is the same as for $\mathcal{I}_\varphi$ satisfying $\mathcal{T}_\varphi^i$). E.g., note that $V^{\tilde{\mathcal{I}}_\varphi}(\nu) = 0.v_\nu = (s+1)^{|v_i|} \cdot V_1^{\tilde{\mathcal{I}}_\varphi}(i\nu)$ and, thus, both $\langle (C_i \sqsupset V) \sqsubseteq (s+1)^{|v_i|} \cdot \forall R_i.V_1 \geq 1 \rangle$ and $\langle (s+1)^{|v_i|} \cdot \exists R_i.V_1 \sqsubseteq (C_i \sqsupset V) \geq 1 \rangle$ are satisfied.

Moreover, for every $j \neq i$ and $\nu' \in suf(\mu)$, it holds that $C_j^{\tilde{\mathcal{I}}_\varphi}(\nu) = 0$ and $R_j^{\tilde{\mathcal{I}}_\varphi}(\nu, \nu') = 0$ and, therefore every axiom in $\tilde{\mathcal{T}}_\varphi^j$ is satisfied as well (note that *e.g.*, $(\forall R_j.V_1)^{\tilde{\mathcal{I}}_\varphi}(\nu) = 1$). This last argument holds for μ as well.

Finally, consider $\tilde{\mathcal{T}}_\varphi$. It is easy to check that the first two axioms are satisfied in every $\nu \in suf(\mu)$. For the third axiom, if $\nu \in suf(\mu) \setminus \{\mu\}$, then there is $1 \leq i \leq p$ such that $C_i^{\tilde{\mathcal{I}}_\varphi}(\nu) = 1$ and, then, the axiom is trivially satisfied. Otherwise, if $\nu = \mu$, since μ is a solution for φ, then $(\neg(V \leftrightarrow W))^{\tilde{\mathcal{I}}_\varphi}(\mu) = 0$ and, then, the axiom is trivially satisfied as well.

(⇐) For the converse, suppose that φ has no solution and let $\mathcal{I}$ be a model of $\tilde{\mathcal{O}}_\varphi$. By absurd, let us assume that $\mathcal{I}$ is finite and, thus, *witnessed*.

Now, since $\mathcal{I}$ is a model of axioms $\langle \neg V(a) \geq 1 \rangle$ and $\langle \neg W(a) \geq 1 \rangle$, then there is a node $a^{\mathcal{I}} = \delta \in \Delta^{\mathcal{I}}$, such that $V^{\mathcal{I}}(\delta) = W^{\mathcal{I}}(\delta) = 0$.

Moreover, since $\mathcal{I}$ is a model of axioms $\langle V \equiv V_1 \boxplus V_2 \geq 1 \rangle$ and $W \equiv W_1 \boxplus W_2$, then

$$V_1^{\mathcal{I}}(\delta) = V_2^{\mathcal{I}}(\delta) = W_1^{\mathcal{I}}(\delta) = W_2^{\mathcal{I}}(\delta) = 0$$

as well.

Next, we prove by induction that for every $n \in \mathbb{N}$ there is an element $\delta_{i_n} \in \Delta^{\mathcal{I}}$ such that:

- $V^{\mathcal{I}}(\delta_{i_n}) = 0.v_{i_n} \ldots v_{i_1}$,
- $W^{\mathcal{I}}(\delta_{i_n}) = 0.w_{i_n} \ldots w_{i_1}$,

and $|\{\delta, \delta_{i_1}, \ldots, \delta_{i_n}\}| = n+1$ (all elements are distinct). As a consequence, $\Delta^{\mathcal{I}}$ cannot be finite, contrary to the assumption that $\mathcal{I}$ is finite.

(1) Since $\mathcal{I}$ is a witnessed model, it satisfies axiom $\langle (C_1 \sqcup \ldots \sqcup C_p)(a) \geq 1 \rangle$. So, there is i, such that $C_i^{\mathcal{I}}(\delta) = 1$. Let $i_1 = i$. Since $\mathcal{I}$ satisfies axiom $\langle C_{i_1} \equiv \exists R_{i_1}.\top \geq 1 \rangle$, then there is $\delta' \in \Delta^{\mathcal{I}}$ such that $R_{i_1}^{\mathcal{I}}(\delta, \delta') = 1$. Let $\delta_{i_1} = \delta'$. Since $\mathcal{I}$ satisfies axiom $\langle (s+1)^{|v_{i_1}|} \cdot \exists R_{i_1}.V_1 \sqsubseteq (C_{i_1} \sqsupset V) \geq 1 \rangle$, then

$$\begin{aligned}
& 0 = \\
= \; & (1 \Rightarrow 0) = \\
= \; & (C_{i_1}(\delta) \Rightarrow V)^{\mathcal{I}}(\delta) \geq \\
\geq \; & (s+1)^{|v_{i_1}|} \cdot \sup_{\delta' \in \Delta^{\mathcal{I}}} \{R_{i_1}^{\mathcal{I}}(\delta, \delta') * V_1^{\mathcal{I}}(\delta')\} \geq \\
\geq \; & R_{i_1}^{\mathcal{I}}(\delta, \delta_{i_1}) * V_1^{\mathcal{I}}(\delta_{i_1}) = \\
= \; & 1 * V_1^{\mathcal{I}}(\delta_{i_1}) = \\
= \; & V_1^{\mathcal{I}}(\delta_{i_1}).
\end{aligned}$$

Hence,

$$V_1^{\mathcal{I}}(\delta_{i_1}) = 0.$$

In the same way it can be proved that

$$W_1^{\mathcal{I}}(\delta_{i_1}) = 0.$$

Since $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq \forall R_{i_1}.V_2 \geq 0.v_{i_1} \rangle$, we have that

$$\begin{aligned} & 0.v_{i_1} \leq \\ \leq \;& (R_{i_1}^{\mathcal{I}}(\delta, \delta_{i_1}) \Rightarrow V_2^{\mathcal{I}}(\delta_{i_1})) = \\ = \;& (1 \Rightarrow V_2^{\mathcal{I}}(\delta_{i_1})) = \\ = \;& V_2^{\mathcal{I}}(\delta_{i_1}). \end{aligned}$$

Since $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq \forall R_{i_1}.\neg V_2 \geq 1 - 0.v_{i_1} \rangle$, it follows that

$$\begin{aligned} & 1 - 0.v_{i_1} \leq \\ \leq \;& (R_{i_1}^{\mathcal{I}}(\delta, \delta_{i_1}) \Rightarrow \neg V_2^{\mathcal{I}}(\delta_{i_1})) = \\ = \;& (1 \Rightarrow \neg V_2^{\mathcal{I}}(\delta_{i_1})) = \\ = \;& \neg V_2^{\mathcal{I}}(\delta_{i_1}) = \\ = \;& 1 - V_2^{\mathcal{I}}(\delta_{i_1}) \end{aligned}$$

and therefore,

$$V_2^{\mathcal{I}}(\delta_{i_1}) \leq 0.v_{i_1}.$$

So,

$$V_2^{\mathcal{I}}(\delta_{i_1}) = 0.v_{i_1}.$$

In the same way it can be proved that

$$W_2^{\mathcal{I}}(\delta_{i_1}) = 0.w_{i_1}.$$

Finally, since $\mathcal{I}$ satisfies axiom $\langle V \equiv V_1 \boxplus V_2 \geq 1 \rangle$, then

$$V^{\mathcal{I}}(\delta_{i_1}) = V_1^{\mathcal{I}}(\delta_{i_1}) \veebar V_2^{\mathcal{I}}(\delta_{i_1}) = 0 \veebar 0.v_{i_1} = 0.v_{i_1}.$$

In the same way it can be proved that

$$W^{\mathcal{I}}(\delta_{i_1}) = 0.w_{i_1}.$$

Moreover, since $V^{\mathcal{I}}(\delta) = 0 \neq 0.v_{i_1} = V^{\mathcal{I}}(\delta_{i_1})$, then $\delta \neq \delta_{i_1}$ and, thus,

$$|\{\delta, \delta_{i_1}\}| = 2,$$

which completes the case.

(n+1) Let $n > 1$ and suppose, by inductive hypothesis, that, for every $j \leq n$, the above conditions hold.

Since φ has no solution, then $v_{i_n} \dots v_{i_1} \neq w_{i_n} \dots w_{i_1}$ and, therefore, by inductive hypothesis, $V^{\mathcal{I}}(\delta_{i_n}) = 0.v_{i_n} \dots v_{i_1} \neq 0.w_{i_n} \dots w_{i_1} = W^{\mathcal{I}}(\delta_{i_n})$. Hence $(V \leftrightarrow W)^{\mathcal{I}}(\delta_{i_n}) < 1$ and, therefore, $\neg(V \leftrightarrow W)^{\mathcal{I}}(\delta_{i_n}) > 0$. So, since $\mathcal{I}$ satisfies axiom $\langle \neg(V \leftrightarrow W) \sqsubseteq C_1 \sqcup$

$\ldots \sqcup C_p \geq 1\rangle$, we have that $(C_1 \sqcup \ldots \sqcup C_p\})^{\mathcal{I}}(\delta_{i_n}) > 0$ and, thus, there is i such that $C_i^{\mathcal{I}}(\delta_{i_n}) > 0$. Therefore, as $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq C_i \sqcup \neg C_i \geq 1\rangle$, we have that $C_i^{\mathcal{I}}(\delta_{i_n}) = 1$.

Now, let $i_{n+1} = i$. Since $\mathcal{I}$ satisfies axiom $\langle C_{i_{n+1}} \equiv \exists R_{i_{n+1}}.\top \geq 1\rangle$, then there is $\delta' \in \Delta^{\mathcal{I}}$ such that $R_{i_{n+1}}^{\mathcal{I}}(\delta_{i_n}, \delta') = 1$. So, let $\delta_{i_{n+1}} = \delta'$. Since $\mathcal{I}$ satisfies axiom $\langle (C_{i_{n+1}} \sqsupset V) \sqsubseteq (s+1)^{|v_{i_{n+1}}|} \cdot \forall R_{i_{n+1}}.V_1 \geq 1\rangle$, then

$$\begin{aligned}
& 0.v_{i_n} \ldots v_{i_1} = \\
= \;& (1 \Rightarrow 0.v_{i_n} \ldots v_{i_1}) = \\
= \;& (C_{i_n} \Rightarrow V)^{\mathcal{I}}(\delta_{i_n}) \leq \\
\leq \;& (s+1)^{|v_{i_{n+1}}|} \cdot \inf_{\delta' \in \Delta^{\mathcal{I}}} \{R_{i_{n+1}}^{\mathcal{I}}(\delta_{i_n}, \delta') \Rightarrow V_1^{\mathcal{I}}(\delta')\} \leq \\
\leq \;& R_{i_{n+1}}^{\mathcal{I}}(\delta_{i_n}, \delta_{i_{n+1}}) \Rightarrow V_1^{\mathcal{I}}(\delta_{i_{n+1}}) = \\
= \;& V_1^{\mathcal{I}}(\delta_{i_{n+1}}).
\end{aligned}$$

On the other hand, since $\mathcal{I}$ satisfies axiom $\langle (s+1)^{|v_{i_{n+1}}|} \cdot \exists R_{i_{n+1}}.V_1 \sqsubseteq (C_{i_{n+1}} \sqsupset V) \geq 1\rangle$, then

$$\begin{aligned}
& 0.v_{i_n} \ldots v_{i_1} = \\
= \;& (1 \Rightarrow 0.v_{i_n} \ldots v_{i_1}) = \\
= \;& (C_{i_n} \Rightarrow V)^{\mathcal{I}}(\delta_{i_n}) \geq \\
\geq \;& (s+1)^{|v_{i_{n+1}}|} \cdot \sup_{\delta' \in \Delta^{\mathcal{I}}} \{R_{i_{n+1}}^{\mathcal{I}}(\delta_{i_n}, \delta') * V_1^{\mathcal{I}}(\delta')\} \geq \\
\geq \;& (s+1)^{|v_{i_{n+1}}|} \cdot (R_{i_{n+1}}^{\mathcal{I}}(\delta_{i_n}, \delta_{i_{n+1}}) * V_1^{\mathcal{I}}(\delta_{i_{n+1}})) = \\
= \;& (s+1)^{|v_{i_{n+1}}|} \cdot V_1^{\mathcal{I}}(\delta_{i_{n+1}}).
\end{aligned}$$

So, $0.v_{i_n} \ldots v_{i_1} = (s+1)^{|v_{i_{n+1}}|} \cdot V_1^{\mathcal{I}}(\delta_{i_{n+1}})$ and, thus,

$$V_1^{\mathcal{I}}(\delta_{i_{n+1}}) = (s+1)^{-|v_{i_{n+1}}|} \cdot 0.v_{i_n} \ldots v_{i_1}.$$

In the same way it can be proved that

$$W_1^{\mathcal{I}}(\delta_{i_{n+1}}) = 0.w_{i_n} \ldots w_{i_1} \cdot (s+1)^{-|w_{i_{n+1}}|}.$$

Since $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq \forall R_{i_{n+1}}.V_2 \geq 0.v_{i_{n+1}}\rangle$, we get

$$\begin{aligned}
& 0.v_{i_{n+1}} \leq \\
\leq \;& R_{i_{n+1}}^{\mathcal{I}}(\delta_{i_n}, \delta_{i_{n+1}}) \Rightarrow V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = \\
= \;& 1 \Rightarrow V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = \\
= \;& V_2^{\mathcal{I}}(\delta_{i_{n+1}}).
\end{aligned}$$

Similarly, since $\mathcal{I}$ satisfies axiom $\langle \top \sqsubseteq \forall R_{i_{n+1}}.\neg V_2 \geq 1 - 0.v_{i_{n+1}} \rangle$, we get

$$\begin{aligned} & 1 - 0.v_{i_{n+1}} \leq \\ \leq\ & R^{\mathcal{I}}_{i_{n+1}}(\delta_{i_n}, \delta_{i_{n+1}}) \Rightarrow \neg V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = \\ =\ & 1 \Rightarrow \neg V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = \\ =\ & \neg V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = \\ =\ & 1 - V_2^{\mathcal{I}}(\delta_{i_{n+1}}) \end{aligned}$$

and therefore, $V_2^{\mathcal{I}}(\delta_{i_{n+1}}) \leq 0.v_{i_{n+1}}$. So,

$$V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = 0.v_{i_{n+1}}.$$

In the same way it can be proved that

$$W_2^{\mathcal{I}}(\delta_{i_{n+1}}) = 0.w_{i_{n+1}}.$$

Finally, since $\mathcal{I}$ satisfies axiom $\langle V \equiv V_1 \boxplus V_2 \geq 1 \rangle$, then

$$\begin{aligned} & V^{\mathcal{I}}(\delta_{i_{n+1}}) = \\ =\ & V_1^{\mathcal{I}}(\delta_{i_{n+1}}) \veebar V_2^{\mathcal{I}}(\delta_{i_{n+1}}) = \\ =\ & ((s+1)^{-|v_{i_{n+1}}|} \cdot 0.v_{i_n} \dots v_{i_1}) \veebar 0.v_{i_{n+1}} = \\ =\ & 0.v_{i_{n+1}} \dots 0.v_{i_1}. \end{aligned}$$

In the same way it can be proved that

$$W^{\mathcal{I}}(\delta_{i_{n+1}}) = 0.w_{i_{n+1}} \dots 0.w_{i_1}.$$

Moreover, since, by inductive hypothesis, for every $j \leq n$,

$$V^{\mathcal{I}}(\delta_{i_j}) = 0.v_{i_j} \dots v_{i_1} \neq 0.v_{i_{n+1}} \dots v_{i_j} \dots v_{i_1} = V^{\mathcal{I}}(\delta_{i_{n+1}}),$$

then $\delta_{i_j} \neq \delta_{i_{n+1}}$.
Furthermore, as

$$V^{\mathcal{I}}(\delta) = 0 \neq V^{\mathcal{I}}(\delta_{i_{n+1}}),$$

then $\delta \neq \delta_{i_{n+1}}$ and, thus,

$$|\{\delta, \delta_{i_1}, \dots, \delta_{i_{n+1}}\}| = n + 2,$$

which completes the case.

So, $\tilde{\mathcal{O}}_\varphi$ has no finite model. □

By Proposition 3.4.8, we have a reduction of a RPCP to a finite satisfiability problem. Again, note that all roles are crisp. Therefore,

Theorem 3.4.9. *The knowledge base finite satisfiability problem is undecidable for $[0,1]_L$-$\mathcal{ALC}$ with GCIs. The result also holds if crisp roles are assumed.*

3.4.3 Further consequences

Exploiting the reduction provided in Proposition 2.5.2, from Theorem 3.4.7 and Theorem 3.4.9 it is easy to prove the following result.

Corollary 3.4.10. *In language $[0,1]_L$-$\mathcal{ALC}^T$ the following are undecidable problems:*

- *Concept r-satisfiability with respect to a KB.*
- *Concept $\geq r$-satisfiability with respect to a KB.*
- *Concept positive satisfiability with respect to a KB.*
- *Entailment of an axiom by a KB.*

3.5 Related work

Decidability has been a central matter since the beginning of the research in FDL. As in the classical case, the first problem to be dealt with has been concept subsumption. In order to deal with this problem a structural subsumption based procedure has been employed. These kinds of algorithm perform a comparison in the syntactic structure of two given concept description after having transformed them in a suitable normal form. The main examples of these researches are [Yen91] and [TM98]. In [Yen91] it is proved the decidability for the concept subsumption problem of a language denoted $\mathcal{FTSL}^-$, whose set of concept constructor and underlying truth value algebra have been discussed in Section 2.8. The main result in [TM98] is the decidability proof for the concept subsumption problem of a language denoted $\mathcal{ALC}_{F_M}$.

In [Str98] it is proved the decidability of the entailment problem with respect to empty TBoxes of language $\mathcal{ALC}$ under Zadeh's semantics. The procedure used is based on the recursive production of a set of constraints until either a clash is produced or the set of constraint is complete, in the sense that no further rule can be applied to the existing set of constraints.

In [Str04a] it is proved the decidability of KB consistency of language $\mathcal{ALC}$ with role hierarchies, over Zadeh's semantics. The procedure used in [Str04a] is a finite reduction of fuzzy $\mathcal{ALC}$ to classical $\mathcal{ALC}$.

In [Str05a] it is used a procedure based on a reduction to the Mixed Integer Linear Programming problem in order to prove decidability of language $\mathcal{ALC}$ extended with concrete domains and fuzzy modifiers under Zadeh's semantics.

Until [Háj05], the algebra of truth values considered is mainly the real unit interval [0,1] with operations max, min, $1 - x$ and Kleene-Dienes implication (what we call Zadeh's semantics). [Háj05] is the first work that considers a t-norm based semantics. The main result proved in [Háj05] is the decidability of the satisfiability and subsumption problems for language $\mathcal{ALC}$ over a standard algebra restricted to witnessed models. The result is achieved by means of a reduction to satisfiability and logical consequence of the corresponding propositional calculus. The details of this work are reported in Section 3.1.

In the last years there have been several works dealing with (un)decidability FDLs over a BL-chain determined by a continuous t-norm.

In [BS10a] it is proved the decidability of KB consistency of language $\mathcal{ALCE}$ with role hierarchies, over a finite t-norm. The result is achieved by means of a computable reduction to classical DL. By means of the same procedure, [BDGRS09] proves the decidability of KB witnessed consistency for a quite expressive FDL language over $[0,1]_G$. This is, currently, the only decidability result for a reasoning task involving general fuzzy concept inclusions for an FDL language over an infinite t-norm.

The paper [BBS11] proves that languages $[0,1]_Ł$-$\mathcal{ALC}$ and $[0,1]_\Pi$-$\Im\mathcal{ALCE}$ do not enjoy the finite model property when general TBoxes are considered. Despite this work does not prove any undecidability result, it casts doubts on the decidability of the previously considered algorithms.

The first actual undecidability result is given in [BP11a], where undecidability of KB consistency for language $[0,1]_\Pi$-$\Im\mathcal{ALCE}$ is proved. The method used in this work as well as in the subsequent works on undecidability is a reduction to PCP similar to the one explained in Section 3.4. However, since in this work, in order to prove undecidability it is needed the use of axioms of type 2.3, the authors think that the result is not enough to prove undecidability of the same problem without this kind of axioms.

Nevertheless the full undecidability of KB consistency for language $[0,1]_\Pi$-$\Im\mathcal{ALCE}$ was proved soon later. The result is indeed proved a few time after in [BP11b] and [BP11c]. Note that in these works the undecidability result is proved for the cited problem when restricted both to witnessed interpretations and to strongly witnessed interpretations.

In [BP11f] it is proved the undecidability of concept satisfiability with respect to a KB for language $\mathcal{ALC}$ over a De Morgan chain containing the standard Łukasiewicz chain as a subalgebra. In the same paper [BP11f] it is also proved that if the De Morgan lattice considered as underlying algebra of truth values is finite, the same problem is indeed decidable. The result is achieved through a recursive reduction to a decidable problem from automata theory.

It is worth briefly mentioning the following recent works. In [BDGRS12] a quite expressive FDL over a set of operations that join Gödel t-norm with Zadeh's operations is presented. In this paper the decidability of the KB consistency problem is proved through a recursive reduction to classical DL when the set of truth values is fixed in advance and finite. In [BP12c] a systematic study of undecidability in FDL is undertaken. Moreover sufficient conditions for proving undecidability in FDL are provided. In [BDP12] it is proved that KB consistency for an FDL with role constructors and individuals over any t-norm with Gödel negation is linearly reducible to classical DL. In [BP12a] it is proved undecidability of KB consistency with respect to unrestricted interpretations for a really basic FDL over Łukasiewicz t-norm. In [BP12b] different results are provided for quite expressive FDLs over complete residuated De Morgan lattices. Among them there are the undecidability result of the infinite-valued case and different decidability results for reasoning tasks over finite lattices. A really interesting feature of this paper is the use of tableaux algorithms for the decidability results.

Chapter 4

Computational complexity

Until now, the works that faced the of computational complexity of FDLs have been able only to deal with FDLs based on a finite chain $\mathbf{T}$ of truth values. This is due to the fact that the reasoning tasks for FDLs based on infinite algebras either have been proved to be undecidable problems, or the reductions considered to prove the decidability are not in polynomial time. In this chapter we deal with the problem of concept r-satisfiability with respect to empty knowledge bases for language $\mathfrak{I}\mathcal{ALCED}^T$ and prove that, when the algebra of truth values $\mathbf{T}$ considered is a finite MTL-chain, the PSPACE bound that characterize the same problem in classical $\mathcal{ALC}$ is preserved. Characterizing the exact complexity requires a lower bound and an upper bound argument. Throughout the chapter we provide full proofs for the upper bound arguments. The proofs of the lower bound argument are only sketched.

The content of the present chapter is the following. In Section 4.1 we show that the reductions reported in Chapter 3 in order to prove decidability of $[0,1]_{\text{Ł}}$-$\mathcal{ALC}$ and $[0,1]_{\Pi}$-$\mathfrak{I}\mathcal{ALE}$ are not polynomial. In Section 4.2 we prove that formula r-satisfiability is PSPACE-complete for the minimal Modal Logic over Ł_n-frames. The procedure used in this case is a generalization of an already known procedure used to prove PSPACE-completeness for the classical modal system $\mathbf{K}$ and is particularly suited to prove the result for the finite-valued Łukasiewicz case because relies on the properties of involutive negation. The proof of the lower bound argument is only sketched because it is the same proof as for the classical case. In Section 4.3 we generalize the previous result and prove that concept r-satisfiability with respect to empty knowledge bases is PSPACE-complete for language $\mathfrak{I}\mathcal{ALCED}^T$ over any finite MTL-chain $\mathbf{T}$. The procedure used in this case is novel and it is based on a PSPACE implementation of the reduction provided in Definition 3.1.2. The procedure provided in Section 4.3 is uniform in the sense that it does not depend on the algebra of truth values $\mathbf{T}$ considered, as long as it is a finite MTL-chain. We omit the argument for the lower bound argument because it is based on the same idea as the former. The result provided in Section 4.2 is clearly a consequence of the result given in Section 4.3. Nevertheless, I find that the former is worth to be reported because the proof is based on a different procedure than the latter and it shows a way to generalize procedures used in the framework of classical Modal Logic to the finite-valued Łukasiewicz case. It will be a matter of further investigation a confrontation of the execution speed of both procedures that belong to PSPACE. Despite the fact that concept satisfiability is a problem that has been proved to be decidable for languages $\mathcal{ALC}$ under infinite-valued Łukasiewicz semantics and $\mathfrak{I}\mathcal{ALE}$ under infinite-valued product semantics, its complexity in both cases is still an open problem. Finally, other results existing in the literature will be reported in Section 4.4.

4.1 Some remarks on Hájek's reduction

In [Háj05] it is proved the decidability of the concept satisfiability problem for the language $\mathfrak{I}\mathcal{ALCED}^T$ under infinite-valued Łukasiewicz semantics. The proof strategy, commented in Section 3.1, consists in reducing this problem, for a given concept C, to the satisfiability in the infinite-valued Łukasiewicz propositional logic of the set of formulas P_C already introduced in Definition 3.1.2. In this section we stress that this reduction is indeed non-polynomial. Proving this fact is a tedious task, but this is the actual explanation why the naive approach does not help in identifying the complexity class. In the following sections we will overcome this difficulty by using alternative algorithms to solve the problem.

The rest of the section is devoted to show that the cardinality of the set P_C grows at least factorially in the size of concept C. To this aim, we consider the concepts that are obtained by applying the translation $\rho(\cdot)$ given in Section 2.7.1 to the modal formulas $\varphi^{\mathcal{B}}(m)$ (one for each $m \in \mathbb{N}$) reported in [BdRV01, p. 384][1]. In other words, for every natural number m, the concept $\rho(\varphi^{\mathcal{B}}(m))$ is the one in the signature with atomic concepts $\{A_1, A_2, \ldots\} \cup \{B_1, B_2, \ldots\}$ and atomic role R obtained as the conjunction of the following concepts:

$$
\begin{array}{lllllll}
\text{(i)} & A_0 & & & & & \\
\text{(ii)} & \forall R^{(m)}.(A_i \sqsupset (\boxtimes_{i \neq j} \neg A_j)) & & (0 \leq i \leq m) & & & \\
\text{(iii)} & C_0 & \boxtimes \forall R.C_1 & \boxtimes \forall R^2.C_2 & \boxtimes \forall R^3.C_3 & \boxtimes \ldots \boxtimes \forall R^{m-1}.C_{m-1} \\
\text{(iv)} & & \forall R.D_1 & \boxtimes \forall R^2.D_1 & \boxtimes \forall R^3.D_1 & \boxtimes \ldots \boxtimes \forall R^{m-1}.D_1 \\
 & & & \boxtimes \forall R^2.D_2 & \boxtimes \forall R^3.D_2 & \boxtimes \ldots \boxtimes \forall R^{m-1}.D_2 \\
 & & & & \boxtimes \forall R^3.D_2 & \boxtimes \ldots \boxtimes \forall R^{m-1}.D_2 \\
 & & & & & \vdots \\
 & & & & & \boxtimes \forall R^{m-1}.D_{m-1}
\end{array}
$$

[1] In [BdRV01] these formulas are used to prove that classical Modal Logic lacks the polysize model property.

where

$$C_i \quad := \quad A_i \sqsupset (\exists R.(A_{i+1} \boxtimes B_{i+1}) \boxtimes \exists R.(A_{i+1} \boxtimes \neg B_{i+1}))$$

and

$$D_i \quad := \quad (B_i \sqsupset \forall R.B_i) \boxtimes (\neg B_i \sqsupset \neg\forall R.B_i)$$

Here we have used the convention that $\forall R^i.E$ is a shorthand for $\overbrace{\forall R.\ldots\forall R}^{i \text{ times}}.E$ and $\forall R^{(m)}.E$ is a shorthand for $E \boxtimes \forall R.E \boxtimes \forall R^2.E \boxtimes \ldots \boxtimes \forall R^m.E$.

Now, as pointed out in [BdRV01], the size of $\rho(\varphi^{\mathcal{B}}(m))$ is quadratic in m. If we apply the algorithm provided in [Háj05] or the one explained in Section 3.2, the cardinality of the set $P_{\rho(\varphi^{\mathcal{B}}(m))}$ of sentences that are produced by the algorithm at some step can be bounded by means of the following observations. Let us denote by $|i|$ the number of generalized atoms of $\rho(\varphi^{\mathcal{B}}(m))$ whose label has cardinality i.

1. In the first step we have that, for each generalized atom $E(d)$ (with $l(E) = \emptyset$) of $\rho(\varphi^{\mathcal{B}}(m))$ the algorithm deterministically produces a new constant d_a and a sentence which says that d_a is a witness for such $E(d)$. So, for each generalized atom, we have a new element in $P_{\rho(\varphi^{\mathcal{B}}(m))}$.

2. In the second step, for each new produced constant d_b which is not a witness for E, the algorithm deterministically produces a new sentence which says that d_b is not a witness for E. Since d_b turns out to be the witness of a generalized atom $E'(d)$ of $P_{\rho(\varphi^{\mathcal{B}}(m))}$ which share the same label with E, another sentence will be produced which says that d_a is not a witness for $E'(d)$. So, we have that $|0|^2$ new elements are in $P_{\rho(\varphi^{\mathcal{B}}(m))}$.

3. After steps 1 and 2 the algorithm has produced:

   (a) an amount of $|0|$ new constants $d_1, \ldots, d_{|0|}$,

   (b) an amount of $|1|$ new generalized atoms for each new constant d_a, with $1 \leq a \leq |0|$.

   Hence, since each set of generalized atoms identified by the same new constant is processed by the algorithm as in step 2, and this does not happen when two generalized atoms do not share the same new constant, an amount of $|0| \cdot |1|^2$ is added to $P_{\rho(\varphi^{\mathcal{B}}(m))}$.

4. Again, after step 3 the algorithm has produced:

   (a) an amount of $|0| \cdot |1|$ new constants $d_{1,1}, \ldots, d_{1,|1|}, d_{2,1}, \ldots, d_{2,|1|}, \ldots, d_{|0|,1}, \ldots, d_{|0|,|1|}$,

   (b) an amount of $|2|$ new generalized atoms for each new constant d_a, with $1 \leq a \leq |0| \cdot |1|$.

Hence, applying again the idea in item 3, we obtain that an amount of $|0| \cdot |1| \cdot |2|^2$ is added to $P_{\rho(\varphi^{\mathcal{B}}(m))}$,

5. Repeat the same idea until the algorithm processes the whole concept $\rho(\varphi^{\mathcal{B}}(m))$.

So, at the end of the process, when no more generalized atoms are produced to be further processed, the cardinal of the resulting propositional theory $P_{\rho(\varphi^{\mathcal{B}}(m))}$ can be described by function $f(m)\colon \mathbb{N} \longrightarrow \mathbb{N}$:

$$
\begin{aligned}
f(m) \quad := \quad & \Big(\sum_{i=1}^{m} i\Big)^2 + \\
& \Big(\sum_{i=1}^{m} i\Big) \cdot \Big(\Big(\sum_{i=1}^{m} i\Big) - 2\Big)^2 + \\
& \Big(\sum_{i=1}^{m} i\Big) \cdot \Big(\Big(\sum_{i=1}^{m} i\Big) - 2\Big) \cdot \Big(\Big(\sum_{i=1}^{m} i\Big) - 2 - 3\Big)^2 + \\
& \vdots \\
& \Big(\sum_{i=1}^{m} i\Big) \cdot \Big(\Big(\sum_{i=1}^{m} i\Big) - 2\Big) \cdot \ldots \cdot \Big(\ldots\Big(\sum_{i=1}^{m} i\Big) - 2 - 3 \ldots - m\Big)^2
\end{aligned}
$$

which can be shown to be strictly greater than function $m!$, for each m. We will prove this by induction on m:

1. Consider, as base cases, $m = 2$ and $m = 3$. In the first case we have that $m! = 2 \cdot 1 = 2$, while $f(m) = (1+2)^2 + (1+2) \cdot ((1+2) - 2)^2 = 12$. In the second case we have that $m! = 3 \cdot 2 \cdot 1 = 6$, while $f(m) = (1+2+3)^2 + (1+2+3) \cdot ((1+2+3) - 3)^2 + (1+2+3) \cdot ((1+2+3) - 3) \cdot ((1+2+3) - 3 - 2)^2 = 36 + 54 + 18 = 108$.

2. Suppose, by induction hypothesis, that $m > 3$ and $f(m) > m!$, we have to show that $f(m+1) > (m+1)!$. We know that each summand in $f(m)$ is a product of a finite number of factors. Let us denote by $F_{f(m)}$ the set of factors in the last addend in $f(m)$, then the function $g(j)\colon \{1, \ldots, m\} \longrightarrow F_{f(m)}$, defined by:

$$
g(j) = \begin{cases} \sum_{i=1}^{m} i, & \text{if } j = m, \\ \ldots \Big(\sum_{i=1}^{m} i\Big) - 2 \ldots - ((m+1) - j), & \text{if } j < m \end{cases}
$$

is a bijection between the factors in $m!$ and the factors in $F_{f(m)}$. Since for every $1 < j \leq m$ we have that $j < g(j)$ and, for $j = 1$, we have that $j = g(j)$, then $\prod F_{f(m)} = \prod_{j=1}^{m} g(j) > m!$.

Using that

$$\sum_{i=1}^{m} i = \frac{m\cdot(m+1)}{2}$$

and that, since $m > 3$,

$$\left(\sum_{i=1}^{m} i\right) - 2 > 2 \cdot (m-1),$$

then

$$\begin{aligned}
& g(m) \cdot g(m-1) = \\
= \;& \sum_{i=1}^{m} i \cdot \left(\left(\sum_{i=1}^{m} i\right) - 2\right) > \\
> \;& \frac{m \cdot (m+1)}{2} \cdot 2 \cdot (m-1) = \\
= \;& m \cdot (m+1) \cdot (m-1) > \\
> \;& m \cdot m \cdot (m-1)
\end{aligned}$$

Then,

$$\prod F_{f(m)} \quad = \quad \left(\prod_{j=1}^{m-2} g(j)\right) \cdot \left(\prod_{j=m-1}^{m} g(j)\right) \quad > \quad m! \cdot m$$

and, thus, by induction hypothesis,

$$\begin{aligned}
& f(m+1) > \\
> \;& f(m) + \prod F_{f(m)} > \\
> \;& m! + m! \cdot m = \\
= \;& m! \cdot (m+1) = \\
= \;& (m+1)!
\end{aligned}$$

This finishes the proof that the reduction considered grows faster than factorial function, in particular it is non-polynomial.

4.2 Modal Logic over $Ł_n$

A first step towards understanding the complexity of the concept r-satisfiability problem in finite-valued fuzzy description logics is the work presented in this section, (published in [BCE11]). Throughout this Section 4.2 we assume that the algebra $\mathbf{T}$ of truth values is a finite Łukasiewics chain and the language used is that of Modal Logic with only one modality, with the Delta operator $\triangle$ and a truth constant $\overline{r}$ for each $r \in \mathbf{T}$. Since, as proved in Section 2.7, there is a close connection between the expressive powers of the minimal n-valued Łukasiewicz Modal Logic and the one of the description logic Ł$_n$-$\mathcal{ALC}$ without knowledge base, the result here reported can be translated to our framework. The result we are going to prove in this section is the following.

Theorem 4.2.1. *For every $n \in \mathbb{N}$ and every $r \in L_n$,*

- *the set of r-satisfiable formulas over Kripke L_n-models is PSPACE-complete,*
- *the set of valid formulas over Kripke L_n-models is PSPACE-complete.*

The same complexity result is attained when we add the Delta operator and/or the canonical truth constants. And also we get the same complexity when we only deal with crisp Kripke models.

In the rest of this section we prove this last theorem. Since

- φ is modally valid iff $\varphi \vee \overline{s}$ is not modally s-satisfiable (where s is the penultimate element of Ł$_n$, i.e., $s = \frac{n-2}{n-1}$), and
- φ is modally r-satisfiable iff $\overline{r} \leftrightarrow \varphi$ is modally satisfiable,

it will be enough to prove PSPACE-completeness of the modal satisfiability problems. It may seem that this trick needs the use of canonical constants in the language, but by McNaughton theorem (see [CDM00, Corollary 3.2.8], we can also reduce r-satisfiability to satisfiability without the help of canonical constants; for example, we notice that

- φ is modally 0.75-satisfiable, iff
- $\varphi^2 \leftrightarrow \neg(\varphi^2)$ is modally satisfiable,

Thus, by the inclusion relationships among the sets of modally satisfiable formulas it will be enough to prove that

1. $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}^{L_n}_{n,\triangle})$ and $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}^{L_n}_{n,\triangle})$ are in PSPACE,
2. $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}_n)$ and $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}_n)$ are PSPACE-hard.

We remind that, in Section 1.1.2, we defined

- $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}_n)$ as the set of all 1-satisfiable formulas in the logic of all Kripke frames valued over the n-valued Łukasiewicz chain,
- $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}_n)$ as the set of all 1-satisfiable formulas in the logic of all crisp Kripke frames valued over the n-valued Łukasiewicz chain.

Following the same pattern, we have considered in the above statements.

- $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}^{L_n}_{n,\triangle})$ as the set of all 1-satisfiable formulas in the logic of all Kripke frames valued over the n-valued Łukasiewicz chain with delta operator and a truth constant $\overline{r}$ for each $r \in L_n$,

- $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}_{n,\triangle}^{L_n})$ as the set of all 1-satisfiable formulas in the logic of all crisp Kripke frames valued over the n-valued Łukasiewicz chain with delta operator and a truth constant $\overline{r}$ for each $r \in L_n$.

The next two subsections are devoted to prove each one of the above complexity statements.

4.2.1 PSPACE upper bound

We start giving a PSPACE algorithm for solving $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}_{n,\triangle}^{L_n})$, and later we will see that this algorithm can be slightly modified to compute $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}_{n,\triangle}^{L_n})$. Our algorithm follows a similar approach to the one given in [BdRV01, p. 383–388]. We remind the fact that all formulas considered in this section may contain the Delta operator and truth constants.

Definition 4.2.2. Let Γ be a set of modal formulas, and $Sub(\Gamma)$ be the set of its subformulas. We define the *closure* of Γ, in symbols $Cl(\Gamma)$, as the set

$$(Sub(\Gamma) \cup \{\Box\neg\sigma : \Diamond\sigma \in Sub(\Gamma)\} \cup \{\Diamond\neg\sigma : \Box\sigma \in Sub(\Gamma)\})^+,$$

where the superscript $^+$ refers to the process of deleting all occurrences of two consecutive negation symbols (i.e., $\neg\neg$). When $Cl(\Gamma) = \Gamma$ we will say that Γ is *closed*.

Note that if Γ is finite, then so is $Cl(\Gamma)$.

Definition 4.2.3. Let Γ be a closed set of modal formulas. We define the sequence $(\Gamma_0, \Gamma_1, \ldots, \Gamma_{nest(\Gamma)})$ by the recurrence

- $\Gamma_0 := \Gamma$,
- $\Gamma_{d+1} := \{\psi : \Diamond\psi \in \Gamma_d\} \cup \{\psi : \Box\psi \in \Gamma_d\}$.

The *family of modal levels* of Γ is the set $\Gamma^\circ := \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{nest(\Gamma)}\}$.

Note that, for every $d \in \{0, \ldots nest(\Gamma)\}$, $nest(\Gamma_d) \leq nest(\Gamma) - d$. In particular $nest(\Gamma_{nest(\Gamma)}) = 0$.

Definition 4.2.4. Let Γ be a closed set of formulas. A *Hintikka function* over some $\Gamma_d \in \Gamma^\circ$ is a mapping $H : \Gamma_d \longrightarrow L_n$ such that

1. H is a homomorphism of non modal connectives (which includes the Delta operator and truth constants),
2. $H(\Diamond\psi) = \neg H(\Box\neg\psi)$, for each $\Diamond\psi \in \Gamma_d$,
3. $H(\Box\psi) = \neg H(\Diamond\neg\psi)$, for each $\Box\psi \in \Gamma_d$.

It is said that H is an *atom* if there exists a Kripke model $\mathfrak{M} = \langle W, R, V\rangle$ and a world $w \in W$ such that, for each formula $\psi \in \Gamma$, it holds that $H(\psi) = V(\psi, w)$.

Lemma 4.2.5. *Let $H : \Gamma_d \longrightarrow L_n$ and $H' : \Gamma_{d+1} \longrightarrow L_n$ be two Hintikka functions, then:*

$$\begin{aligned} & \min\{H'(\psi) \Rightarrow H(\Diamond\psi) : \Diamond\psi \in \Gamma_d\} = \\ = \; & \min\{H(\Box\vartheta) \Rightarrow H'(\vartheta) : \Box\vartheta \in \Gamma_d\}. \end{aligned}$$

Proof. For every formula $\Diamond\psi \in \Gamma_d$ it is obvious that

$$\begin{aligned} & H'(\psi) \Rightarrow H(\Diamond\psi) = \\ = \; & \neg H(\Diamond\psi) \Rightarrow \neg H'(\psi) = \\ = \; & H(\neg\Diamond\psi) \Rightarrow H'(\neg\psi) = \\ = \; & H(\Box\neg\psi) \Rightarrow H'(\neg\psi). \end{aligned}$$

Then, using that $\Diamond\psi \in \Gamma_d$ iff $\Box\neg\psi \in \Gamma_d$ (by Definition 4.2.2), we get that

$$\min\{H'(\psi) \Rightarrow H(\Diamond\psi) : \Diamond\psi \in \Gamma_d\} = \min\{H(\Box\neg\psi) \Rightarrow H'(\neg\psi) : \Box\neg\psi \in \Gamma_d\}.$$

From this fact, it easily follows that

$$\min\{H'(\psi) \Rightarrow H(\Diamond\psi) : \Diamond\psi \in \Gamma_d\} = \min\{H(\Box\vartheta) \Rightarrow H'(\vartheta) : \Box\vartheta \in \Gamma_d\}.$$ □

Definition 4.2.6. Let $H : \Gamma_d \longrightarrow L_n$ be a Hintikka function, $k \in L_n$ and $\Diamond\psi \in \Gamma_d$. We say that a Hintikka function $H' : \Gamma_{d+1} \longrightarrow L_n$ is *induced by* $\Diamond\psi$ and *r-related* to H (in symbols, $H' \in H_{\Diamond\psi,r}$) if the following conditions hold:

- $H(\Diamond\psi) = r * H'(\psi)$,
- for each $\Box\vartheta \in \Gamma_d$, it holds that $H(\Box\vartheta) \leq r \Rightarrow H'(\vartheta)$.

Lemma 4.2.7. *Let Γ be a closed set of formulas, $\Gamma_d \in \Gamma^\circ$ and H a Hintikka function over Γ_d. If H is an atom, then for every $\Diamond\psi \in \Gamma_d$, there is some $r \in L_n$ and some $H' \in H_{\Diamond\psi,r}$ such that H' is an atom.*

Proof. Let H be an atom over Γ_d and $\Diamond\psi \in \Gamma_d$. Then, by Definition 4.2.4, there exist a Kripke model $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$ such that

$$V(\Diamond\psi, w) = H(\Diamond\psi).$$

Hence there exists $w' \in W$ such that

$$V(\Diamond\psi, w) = R(w, w') * V(\psi, w').$$

Let $H' : \Gamma_{d+1} \longrightarrow L_n$ be the Hintikka function defined by $H'(\varphi) = V(\varphi, w')$, for every formula $\varphi \in \Gamma_{d+1}$. It is obvious that H' is an atom. Take $r = R(w, w')$, then

$$H(\Diamond\psi) = V(\Diamond\psi, w) = R(w, w') * V(\psi, w') = r * H'(\psi)$$

i.e., H and H' satisfy the first condition of Definition 4.2.6. On the other hand, for each $\Box\vartheta \in \Gamma_d$, we have that

$$V(\Box\vartheta, w) = \min\{R(w, w'') \Rightarrow V(\vartheta, w'') : w'' \in W\},$$

and hence

$$H(\Box\vartheta) = V(\Box\vartheta, w) \leq R(w, w') \Rightarrow V(\vartheta, w') = r \Rightarrow H'(\vartheta).$$

So, there is $r \in L_n$ such that $H' \in H_{\Diamond\psi,r}$. □

Definition 4.2.8. Let Γ be a finite closed set of formulas, H be a Hintikka function over Γ_0, and $\mathcal{H}$ be a family of Hintikka functions with domains (denoted by *dom*) belonging to Γ°. We say that $\mathcal{H}$ is a *witness set generated by* H *on* Γ when

1. $H \in \mathcal{H}$,
2. if $I \in \mathcal{H}$ and $\Diamond\psi \in dom(I)$, then there is some $r \in L_n$ and some $J \in I_{\Diamond\psi,r}$ such that $J \in \mathcal{H}$,
3. if $J \in \mathcal{H}$ and $J \neq H$, then there are $I^0, \ldots, I^r \in \mathcal{H}$ satisfying:
   - $I^0 = H$,
   - $I^r = J$,
   - for each $0 \leq i < r$, there are a formula $\Diamond\psi \in dom(I^i)$ and an element $r \in L_n$ such that $I^{i+1} \in I^i_{\Diamond\psi,r}$.

Lemma 4.2.9. *Let* Γ *be a finite closed set of formulas, and* H *be a Hintikka function over* Γ_0 *(i.e.,* Γ*). Then,* H *is an atom iff there is a witness set generated by* H *on* Γ*.*

Proof. Let Γ be a finite closed set of formulas, and H a Hintikka function over Γ_0.

($\Rightarrow$) We proceed by induction on the nesting degree of the set $dom(H)$.

(0) If $nest(\Gamma_d) = 0$ and H is an atom, then $\mathcal{H} = \{H\}$ is a witness set generated by H on Γ_0.

(d) Let $nest(\Gamma_d) = d$ and H be an atom over Γ_d. Suppose, by inductive hypothesis, that, for each $\Gamma_s \in \Gamma^\circ$ such that $nest(\Gamma_s) < d$ and each Hintikka function H' over Γ_s, it holds that, if H' is an atom, then there is a witness set generated by H' on Γ_s. Since H is an atom over Γ_d, then, by Lemma 4.2.7, for each $\Diamond\psi \in \Gamma_d$ there exist $r \in L_n$ and an atom $I^\psi \in H_{\Diamond\psi,r}$ over Γ_{d+1}. Since the degree of $\Gamma_{d+1} < d$, then, by inductive hypothesis, each atom I^ψ generates a witness set $\mathcal{I}^\psi$ on Γ_{d+1}. So, the set

$$\mathcal{H} = \{H\} \cup \bigcup_{\Diamond\psi \in \Gamma_d} \mathcal{I}^\psi$$

is a witness set generated by H on Γ.

(⇐) Suppose now that there is a witness set $\mathcal{H}$ generated by H on Γ, then we have to show that there exists a model which satisfies H. So, define the model $\mathfrak{M} = \langle W, R, V \rangle$, where:

- $W = \mathcal{H}$,
- $R(I, I') = \begin{cases} \min\{I'(\chi) \Rightarrow I(\Diamond\chi) : \Diamond\chi \in dom(I)\}, \\ \text{ if } I' \in I_{\Diamond\psi,r} \text{ for some } r \in L_n \text{ and} \\ \text{some } \Diamond\psi \in dom(I) \\ \\ 0, \text{ otherwise,} \end{cases}$
- for each variable $p \in At$ and $I \in \mathcal{H}$, let $V(p, I) = I(p)$.

On the one hand, since for each $I \in \mathcal{H}$, $dom(I)$ contains a finite number of formulas of the form $\Diamond\psi$, then, by Definition 4.2.8, each element of the model has a finite number of R-successors. On the other hand, whenever $I' \in I_{\Diamond\psi,r}$, then $nest(dom(I')) < nest(dom(I))$ and, therefore, the depth of the model is finite as well (it is indeed equal to $nest(\Gamma)$).

To end the proof, we have to show that, for every formula $\varphi \in \Gamma$, it holds that $V(\varphi, H) = H(\varphi)$. In order to obtain this result we will prove by induction that for each $I \in W$, it holds that $V(\varphi, I) = I(\varphi)$. So, let $I \in W$ and $\varphi \in dom(I)$, then:

- If $\varphi = p$ is a propositional variable, then, by definition of V, we have that $V(p, I) = I(p)$.
- If φ is a propositional combination of variables or modal formulas, since H is a Hintikka function, by Definition 4.2.4 it holds that $V(\varphi, H) = H(\varphi)$.
- Let $\varphi = \Diamond\psi$ and suppose, by inductive hypothesis, that for each $J \in W$ such that $nest(dom(J)) < nest(dom(I))$ and for each formula χ, it holds that $V(\chi, J) = J(\chi)$. By Definitions 4.2.6 and 4.2.8, we have that there exists $J \in I_{\Diamond\psi,r}$, for a $r \in L_n$, such that, for each $\Box\vartheta \in dom(I)$, we have that $I(\Box\vartheta) \leq r \Rightarrow J(\vartheta)$, then, by residuation, $r \leq I(\Box\vartheta) \Rightarrow J(\vartheta)$, for each $\Box\vartheta \in dom(I)$ and, therefore, by Lemma 4.2.5 and the construction of $\mathfrak{M}$,

$$\begin{aligned} & r \leq \\ \leq \ & \min\{I(\Box\vartheta) \Rightarrow J(\vartheta) : \Box\vartheta \in dom(I)\} = \\ = \ & \min\{J(\chi) \Rightarrow I(\Diamond\chi) : \Diamond\chi \in dom(I)\} = \\ = \ & R(I, J). \end{aligned}$$

So, by Definition 4.2.6 and the inductive hypothesis,

$$
\begin{aligned}
& I(\Diamond\psi) = \\
= \quad & r * J(\psi) \leq \\
\leq \quad & R(I,J) * J(\psi) = \\
= \quad & R(I,J) * V(\psi, J) \leq \\
\leq \quad & \max\{R(I,I') * V(\psi, I') : I' \in W\} = \\
= \quad & V(\Diamond\psi, I).
\end{aligned}
$$

On the other hand, let $I' \in W$ be such that $I' \in I_{\Diamond\chi,r'}$ for a $\Diamond\chi \in dom(I)$ and $r' \in L_n$, then, by the construction of $\mathfrak{M}$ and inductive hypothesis,

$$
\begin{aligned}
& I(\Diamond\psi) \geq \\
\geq \quad & I(\Diamond\psi) \wedge I'(\psi) = \\
= \quad & (I'(\psi) \Rightarrow I(\Diamond\psi)) * I'(\psi) \geq \\
\geq \quad & \min\{I'(\vartheta) \Rightarrow I(\Diamond\vartheta) : \Diamond\vartheta \in dom(I)\} * I'(\psi) = \\
= \quad & R(I,I') * V(\psi, I').
\end{aligned}
$$

Hence

$$
\begin{aligned}
& I(\Diamond\psi) \geq \\
\geq \quad & \max\{R(I,I') * V(\psi, I') : I' \in W\} = \\
= \quad & V(\Diamond\psi, I).
\end{aligned}
$$

So, $V(\Diamond\psi, I) = I(\psi)$.

So, for each formula φ, $V(\varphi, H) = H(\varphi)$ and, then, H is an atom over Γ. □

Next we consider the algorithm $Witness(H,\Gamma)$ given in Figure 4.1. This algorithm returns a boolean, and is very close to the one given in [BdRV01] for the minimal classical modal logic.

Lemma 4.2.10. *Let Γ be a finite closed set of formulas, and $H : \Gamma \longrightarrow L_n$. Then, $Witness(H,\Gamma)$ returns* **true** *if and only if H is a Hintikka function over Γ that generates a witness set in Γ. Otherwize it returns* **false**.

Proof. Let Γ be a finite closed set of formulas, and $H : \Gamma \longrightarrow L_n$.

(⇒) Suppose that $Witness(H,\Gamma)$ returns **true**, we proceed by induction on the degree of Γ.

```
if H is a Hintikka function and Γ = dom(H)
  and for each subformula ◇ψ ∈ dom(H) there are
  r ∈ L_n and a Hintikka function I ∈ H_{◇ψ,r} such that
  Witness(I, dom(I))
 then
     return true
else
     return false
end if
```

Figure 4.1: The Algorithm $Witness(H, \Gamma)$

(0) If $nest(\Gamma) = 0$ and $Witness(H, \Gamma)$ returns **true** then, H is a Hintikka function over Γ, and hence $\mathcal{H} = \{H\}$ is a witness set generated by H on Γ.

(d) Let $nest(\Gamma) = d$ and suppose, by inductive hypothesis, that for each set Γ' of formulas such that $\Gamma' \subseteq \Gamma$ and $nest(\Gamma') < d$ and each function $H' : \Gamma' \longrightarrow L_n$, it holds that, if $Witness(H', \Gamma')$ returns **true**, then H' is a Hintikka function over Γ' that generates a witness set in Γ'. If $Witness(H, \Gamma)$ returns **true** then, on the one hand, H is a Hintikka function over Γ. On the other hand, for each formula $\Diamond\psi \in \Gamma$, there are $r \in L_n$ and $I \in H_{\Diamond\psi,r}$ such that $Witness(I, \Gamma')$, where $\Gamma' \in \Gamma^\circ$ is such that $nest(\Gamma') = d - 1$. Since $nest(\Gamma') < d$, and $Witness(I, \Gamma')$ returns **true**, then, by inductive hypothesis, I is a Hintikka function over Γ' that generates a witness set $\mathcal{I}^\psi$ in Γ'. So, the set

$$\mathcal{H} = \{H\} \cup \bigcup_{\Diamond\psi \in \Gamma} \mathcal{I}^\psi$$

is a witness set generated by H on Γ.

($\Leftarrow$) Suppose that H is a Hintikka function over Γ that generates a witness set in Γ, we proceed by induction on the degree of Γ.

(0) If $nest(\Gamma) = 0$ then it is enough that H is a Hintikka function over Γ for $Witness(H, \Gamma)$ to return **true**.

(d) Let $nest(\Gamma) = d$ and suppose, by inductive hypothesis, that for each set Γ' of formulas such that $\Gamma' \subsetneq \Gamma$ and $nest(\Gamma') < d$ and each function $H' : \Gamma' \longrightarrow L_n$, it holds that, if H' is a Hintikka function over Γ' that generates a witness set in Γ', then $Witness(H', \Gamma')$ returns **true**. So, if H is a Hintikka function over Γ that generates a witness set $\mathcal{H}$ in Γ, then, by Definition 4.2.8, we have that, for each formula $\Diamond\psi \in \Gamma$ there are $r \in L_n$ and $I \in H_{\Diamond\psi,r} \cap \mathcal{H}$. Then we have that I is a Hintikka function over Γ' that generates a witness set in Γ', where $\Gamma' \in \Gamma^\circ$ is such that $nest(\Gamma') = d - 1$. Hence, since $nest(\Gamma') <$

d, then, by inductive hypothesis, $Witness(I, \Gamma')$ returns **true**. So, $Witness(H, \Gamma)$ returns **true**. $\square$

Theorem 4.2.11. $\mathbf{Sat_1}(\mathtt{Fr}, L^{L_n}_{n,\triangle})$ *is in PSPACE.*

Proof. Let φ be a modal formula. By Lemmas 4.2.9 and 4.2.10, we have that φ is r-satisfiable iff there is a Hintikka function $H : Cl(\varphi) \longrightarrow L_n$ such that $H(\varphi) = r$ and $Witness(H, Cl(\varphi))$ returns **true**. Thus we need to provide a PSPACE implementation of $Witness$. Consider a non-deterministic Turing machine that guesses a Hintikka function H over $Cl(\varphi)$ and runs $Witness(H, Cl(\varphi))$. Then, using Savitch's Theorem it is enough to provide a machine that runs in NPSPACE in order to obtain the desired result. The key points of the implementation are the following:

1. As pointed out in [BdRV01], encoding a subset Γ of $Cl(\varphi)$ requires space $\mathcal{O}(|\varphi|)$ (here $|\varphi|$ refers to the length of the encoding of φ). On the one hand, each element of a function $H : \Gamma \longrightarrow L_n$ can be represented as an ordered pair $\langle \psi, i \rangle \in \Gamma \times L_n$ and, on the other hand, $|H| = |\Gamma|$. Hence, if $j = \max\{|\overline{r}| : r \in L_n\}$, then encoding a Hintikka function requires space bounded above by $|\varphi| + j \cdot |\varphi|$, that is space $\mathcal{O}(|\varphi|)$.

2. For each subformula $\Diamond\psi \in dom(H)$, whether there are $r \in L_n$ and a Hintikka function $I \in H_{\Diamond\psi,r}$, can be checked separately. Given a subformula $\Diamond\psi \in dom(H)$, the value $r \in L_n$ and the Hintikka function $I \in H_{\Diamond\psi,r}$ to be checked can be selected by non-deterministic choice. Note that, although the size of the set $H_{\Diamond\psi,r}$ can be in $\mathcal{O}(n^{|\Diamond\psi|})$, for a given function I, we do not need to check every element of $H_{\Diamond\psi,r}$ to see whether $I \in H_{\Diamond\psi,r}$, since we only need to test if I satisfies the conditions of Definition 4.2.6 and this can be done within space linear on the size of I.

Hence, by the previous points, every time that algorithm $Witness$ is applied to a function H and its domain $Cl(\varphi)$, a subformula $\Diamond\psi \in dom(H)$ is selected and a $r \in L_n$ and $I \in H_{\Diamond\psi,r}$ are non-deterministically chosen, the space needed is in $\mathcal{O}(|\varphi|)$. So, since $nest(\varphi)$ recursive calls are needed until we meet a Hintikka function I whose domain contains no modal formula and $nest(\varphi) \leq |\varphi|$, the amount of space required to run the algorithm is $\mathcal{O}(|\varphi|^2)$. Moreover, to keep track of the subformulas that have been checked by the algorithm, it is enough to implement two kinds of pointers to the modal operators occurring in the representation of φ: one pointer to indicate that, for a given subformula $\Diamond\psi \in dom(H)$ it has been fully checked whether there is $r \in L_n$ and a Hintikka function $I \in H_{\Diamond\psi,r}$ such that $Witness(I, dom(I))$, and the other pointer when the same has not yet been fully checked. $\square$

Theorem 4.2.12. $\mathrm{Sat}_1(\mathtt{CFr}, L^{L_n}_{n,\triangle})$ *is in PSPACE.*

Proof. It is easy to see that the same algorithm given in Figure 4.1, but replacing $k \in L_n$ with $k \in \{0, 1\}$, computes $\mathrm{Sat}_1(\mathtt{CFr}, Ł^{L_n}_{n,\triangle})$. $\square$

4.2.2 PSPACE hardness

Here we will prove that the concept r-satisfiability is PSPACE-hard. Since the case of the crisp frames is more simple than the case of the unrestricted ones, we will explain firstly how to obtain hardness for the modal logic of crisp frames.

The PSPACE-hardness of the set $\mathrm{Sat}_1(\mathtt{CFr}, \text{Ł}_n)$ is proved by using a polynomial reduction into the problem of satisifiability for classical Kripre models.

Theorem 4.2.13. $\mathrm{Sat}_1(\mathtt{CFr}, L_n)$ *is PSPACE-hard.*

Proof. Let us consider the mapping tr from classical modal formulas into our modal formulas defined by

- $tr(p) = p \otimes \overset{(n-1)}{\cdots} \otimes p$, if p is a propositional variable,
- $tr(\bot) = \bot$,
- $tr(\varphi_1 \wedge \varphi_2) = tr(\varphi_1) \wedge tr(\varphi_2)$,
- $tr(\varphi_1 \to \varphi_2) = tr(\varphi_1) \to tr(\varphi_2)$,
- $tr(\Diamond\varphi) = \Diamond\ tr(\varphi)$.

This translation is clearly polynomial (because essentially we are only replacing variables and because L_n is fixed), and by induction on formulas it is easy to check that for all modal formulas φ, it holds that

- φ is modally satisfiable in a classical Kripke model, iff
- $tr(\varphi)$ is modally satisfiable in a crisp Kripke model.

By the PSPACE-hardness of classical modal logic ([Lad77]) the proof finishes. □

Unfortunately, for the case of the set $\mathrm{Sat}_1(\mathtt{Fr}, \text{Ł}_n)$, the authors do not know how to get the PSPACE-hardness by a reduction from the classical case. One reduction can be obtained by the mapping tr' defined like tr except for the condition

- $tr'(\Diamond\varphi) = (\Diamond\ tr'(\varphi))^{n-1}$,

but this reduction is not polynomial. Thus, in order to prove our next theorem we need to go into the details of codifying Quantified Boolean Formulas QBF (it is well known that validity of QBF is PSPACE-complete). Since we essentially use the same ideas that are used in the classical modal case (see the proof given in [BdRV01, Theorem 6.50]), we will not go into all the details of the proof.

Theorem 4.2.14. $\mathrm{Sat}_1(\mathtt{Fr}, L_n)$ *is PSPACE-hard.*

Proof. Let us consider β a QBF formula. We define a modal formula $f(\beta)$ in the same way as in [BdRV01, p. 390]. It is well known from [BdRV01, Theorem 6.50] that

- β is valid, iff
- $f(\beta)$ is modally satisfiable in a classical Kripke model.

It is quite straightforward to check that for the formulas of the form $f(\beta)$ it happens that

- $f(\beta)$ is modally satisfiable in a classical Kripke model, iff
- $f(\beta)$ is modally satisfiable in a Kripke Ł$_n$-model.

This fact is based on the properties stated at the end of Section 1.1.2. □

To finish this section let us point out that when our language has the Delta operator, this last proof can be simplified quite a lot just by realizing that the reduction tr' can be somehow converted into one that is polynomial; this is so because

$$\triangle\varphi \leftrightarrow \varphi^{n-1}$$

is a valid formula. In this simplification it is crucial that the length of $\triangle\varphi$ is much shorter than the length of φ^{n-1}.

4.3 Concept satisfiability in the general case of finite-valued FDLs

The proof given in Section 4.2 can not be straightforwardly generalized to the case of every finite-valued $\mathfrak{I}\mathcal{ALCED}^T$, because some steps relies on the good behavior of Łukasiewicz negation with respect to the quantifiers. So, in this section we will consider directly the satisfiability problem in the general setting of finite-valued $\mathfrak{I}\mathcal{ALCED}^T$. This gives us also the possibility of proposing a new procedure based on the one presented in Definition 3.1.2.

In the rest of the present section, instead of talking about r-satisfiability we will use the terminology "modally r-satisfiable" (see Definition 4.3.1) because we will keep the terminology r-satisfiable for the case that we consider propositional assignations. We just introduce this notion in a FDL environment, since, in the previous section, its meaning was clear from the context.

Definition 4.3.1. (Modal r-satisfiability in $\mathfrak{I}\mathcal{ALCED}^T$) A concept C is said to be *modally r-satisfiable* in case that there is an interpretation $\mathcal{I}$ and an object $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = r$.

Next we define the main computational problem we deal with in this section, together with its parametrized versions. It is worth saying that we are not only considering a different computational problem for every finite MTL-chain **T**, we also consider one computational problem which can be understood as the uniform version of the ones parametrized by **T**.

Definition 4.3.2. The computational problem **Satisf** is the following one:

INPUT: $(\mathbf{T}, C, r)$ where $\mathbf{T}$ is a finite MTL-chain, C is a concept of $\mathfrak{I}\mathcal{ALCED}^T$ and $r \in T$.

OUTPUT: Yes/No depending whether C is modally r-satisfiable or not.

Moreover, for every finite MTL-chain $\mathbf{T}$, the computational problem $\mathbf{Satisf_T}$ is the one obtained by fixing the finite MTL-chain in the previous problem.

We can think on the elements of T as truth values. Our interest in the present section is on finite MTL chains, and so we will always assume that the lattice part of $\mathbf{T}$ is fixed in the sense that T is the set $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ (for some natural number $n \geq 2$) and that $0 < \frac{1}{n-1} < \ldots < \frac{n-2}{n-1} < 1$. In particular, n will always refer to the cardinal of T.

The assumption on the lattice part of $\mathbf{T}$ that we have adopted above makes that the input $\mathbf{T}$, in the uniform problem, is codified by its cardinal n and the tables of the t-norm and its residuum.

Notice that the **Satisf** problem we deal with in this section is a more general problem than $\mathbf{Satisf_T}$. The main statement about **Satisf** on the present section is the following theorem.

Theorem 4.3.3. **Satisf** *is PSPACE-complete.*

The rest of the section is devoted to give the proof of the membership in PSPACE. The proof that concept modal r-satisfiability for finite-valued $\mathfrak{I}\mathcal{ALCED}^T$ is PSPACE hard is the same proof we provided for Theorem 4.2.13 for the case of $\mathfrak{I}\mathcal{ALCED}^T$ based on $Ł_n$. So, we will not repeat it here. In other words, we only need to prove the PSPACE membership.

Notation. For the sake of clarification sometimes we will use $\cdot$ to mean the concatenation of strings, but in most cases we will just juxtapose the symbols we want to concatenate.

4.3.1 PSPACE upper bound

In this section we are going to prove that **Satisf** is in PSPACE. In particular, this implies that each one of the parametrized satisfiability problems by a finite MTL-chain $\mathbf{T}$ also belongs to PSPACE. In order to achieve this result, we will give a PSPACE algorithm inspired by [Háj05]. Our proof follows the same pattern as the proofs in [BdRV01] and [BCE11], but here we do not make use of Hintikka sets or functions, like in the cited papers or in Section 4.2.

Preliminary definitions

Several technical definitions will be needed later to prove PSPACE membership. We state these definitions now.

Definition 4.3.4.

An *occurrence* of a subconcept D in C is determined by the occurrence of a constructor or of an atomic concept. We will use _ to mark the occurrence considered.

It is worth noticing that every concept is equivalent to a propositional combination of atoms (i.e., generalized atoms and atomic concepts). Here by propositional combination we allow the use of all constructors except $\forall R$ and $\exists R$.

The labeling function provided in Definition 3.2.2 does not bring enough information in order to cut up the propositional theory P into polynomial slices. For this reason we have to enhance that first version of the labeling function by means of a more general version.

In the next definition we provide a labeling system that is a modification of the one given in Definition 3.2.2. It is crucial for the proof to give such a modification because this modification allows to recursively define the domain of the interpretation that possibly satisfies a given concept. Given a concept C, our labeling system assigns each occurrence of a subconcept of C a number that gives an account of the syntactic structure of C. It is closely related to what in [SSS91] is called the *skeleton* of a constraint system, and it is worth emphasizing that the labelling is defined on occurrences (not on subconcepts).

Definition 4.3.5 (Labeling (general version))**.** Let C be a concept. A *labelling function* (*label* for short) $l_C(\cdot)$ is the function which associates to every occurrence D of a subconcept in C a string of symbols in $N_R \cup \mathbb{N}$ defined by the conditions:

1. $l_C(C)$ is the empty sequence ε,
2. if D is a propositional combination of concepts $D_1, \dots, D_j$, then $l_C(D_i) := l_C(D)$ for every $i \leq j$.
3. if D is $\forall R.D'$ or $\exists R.D'$, then $l_C(D')$ is the concatenated sequence $l_C(D) \cdot Ri$, where i is the minimum non-zero number j such that the sequence $l_C(D) \cdot Rj$ has not been used to label any occurrence in C.

We will denote by Λ_C the set of labels of all occurrences in C. Given $\lambda \in \Lambda_C$, we define $\text{path}(\lambda)$ as the finite sequence of symbols in N_R obtained by deleting in the sequence λ the symbols from $\mathbb{N}$, and we will refer to it as the *role path* of λ. We define the *length* of λ, in symbols $|\lambda|$, as the number of symbols in the sequence $\text{path}(\lambda)$.
For every atomic role $R \in N_R$, we introduce the binary relation $\prec_R$ among labels by the condition:

$$\lambda \prec_R \lambda' \text{ if and only if } \text{path}(\lambda') = \text{path}(\lambda) \cdot R.$$

And the relation $\prec$ is defined as $\bigcup\{\prec_R : R \in N_R\}$.

It is worth saying that for every concept C, there are labellings l_C. For the sake of simplicity, whenever in the future we have a fixed concept C, we will write l instead of l_C.

Example 4.3.6. *Let us consider, as an example, the concept*

$$\exists S.\big(\exists R.A \to \exists R.(\forall R.A \boxtimes \exists S.A)\big).$$

Then, we have the following labels:

$$l\Big(\underline{\exists S}.\big(\exists R.A \to \exists R.(\forall R.A \boxtimes \exists S.A)\big)\Big) = \varepsilon$$
$$l\Big(\exists S.\big(\underline{\exists R}.A \to \underline{\exists R}.(\forall R.A \boxtimes \exists S.A)\big)\Big) = S0$$
$$l\Big(\exists S.\big(\exists R.\underline{A} \to \exists R.(\forall R.A \boxtimes \exists S.A)\big)\Big) = S0R0$$
$$l\Big(\exists S.\big(\exists R.A \to \exists R.(\underline{\forall R}.A \boxtimes \underline{\exists S}.A)\big)\Big) = S0R1$$
$$l\Big(\exists S.\big(\exists R.A \to \exists R.(\forall R.\underline{A} \boxtimes \exists S.A)\big)\Big) = S0R1R0$$
$$l\Big(\exists S.\big(\exists R.A \to \exists R.(\forall R.A \boxtimes \exists S.\underline{A})\big)\Big) = S0R1S0.$$

We remind the reader that we follow the above convention to use _ to denote occurrences.

From the above introduced labeling system, we are going to define the set of individuals employed to build an interpretation for a given concept C. Thus, from now on we assume that C and a labelling l are fixed. The individuals we have talked about are the ones introduced in the following definition.

Definition 4.3.7. The set Σ_C is defined as the set

$$\Sigma_C := \{\lambda_1 \cdot \ldots \cdot \lambda_s : s \in \mathbb{N}, \lambda_1 \prec \lambda_2 \prec \ldots \prec \lambda_s\}$$

formed by sequences of labels. For the case $s = 0$ we have that $\varepsilon \in \Sigma_C$. Given $\sigma = \lambda_1 \cdot \ldots \cdot \lambda_s \in \Sigma_C$, we define the *length* of σ, in symbols $|\sigma|$, as the number s. And we define its *role path* $\text{path}(\sigma)$ as $\text{path}(\lambda_s)$. Following the same pattern, we will write $\sigma \prec_R \sigma'$ and $\sigma \prec \sigma'$ when the corresponding relation holds between λ_s and $\lambda'_{s'}$.

It is straightforward that $|\sigma| = |\lambda_s|$. In the rest of this section we will sometimes refer to the elements of Σ_C as constants. The underlying idea is that the set Σ_C of constants is indeed the domain of an interpretation that modally satisfies the concept C. Unfortunately, the cardinality of Σ_C might be not polynomial on the size of concept C.

The next definition is very similar to one stated in [Háj05] and reported here as Definition 3.1.2. It gives an account of how to make a partition of the theory obtained by applying Hájek's reduction to a given concept. The theory we consider now is a propositional one (non-modal) over the set At of variables and is defined as

$$\{B(\sigma) : B \text{ occurrence of an atom in } C \text{ and } \sigma \in \Sigma_C\} \cup$$
$$\{R(\sigma, \sigma') : R \in N_R, \sigma, \sigma' \in \Sigma_C \text{ and } \sigma \prec_R \sigma'\}.$$

We will use the name of *assertion* to denote expressions $B(\sigma)$ where B is an occurrence of an atom in C and $\sigma \in \Sigma_C$. For every formula φ obtained from the set At of variables using propositional operators (i.e., non-modal) we define At_φ as the set of variables appearing in φ. Analogously, we can consider At_Φ for any set of such formulas.

Definition 4.3.8. Let $B(\sigma)$ be an assertion such that B is the occurrence of a generalized atom in C. Then the *Hájek set* $H_C(B(\sigma))$ is defined distinguishing the following two cases.

($\forall$) if $B = \forall R.D$, then $H_C(\forall R.D(\sigma))$ is the following set of formulas:

- $\forall R.D(\sigma) \equiv \big(R(\sigma, \sigma \cdot l(D)) \sqsupset D(\sigma \cdot l(D))\big)$,
- $\forall R.D(\sigma) \sqsupset \big(R(\sigma, \sigma \cdot l(E)) \sqsupset D(\sigma \cdot l(E))\big)$, for each occurrence E of a generalized atom occurring in C such that $\text{path}\,(l(E)) = \text{path}\,(l(D))$;

($\exists$) if $B = \exists R.D$, then $H_C(\exists R.D(\sigma))$ is the following set of formulas:

- $\exists R.D(\sigma) \equiv \big(R(\sigma, \sigma \cdot l(D)) \boxtimes D(\sigma \cdot l(D))\big)$,
- $\big(R(\sigma, \sigma \cdot l(E)) \boxtimes D(\sigma \cdot l(E))\big) \sqsupset \exists R.D(\sigma)$, for each occurrence E of a generalized atom occurring in C such that $\text{path}\,(l(E)) = \text{path}\,(l(D))$.

The formula in $H_C(B(\sigma))$ having the connective $\equiv$ as main connective will be called the *main formula* of $H_C(B(\sigma))$. The formula $B(\sigma)$ will be called the *head* of each one of the elements in $H_C(B(\sigma))$; and we will give the name of *body* to the formula lying on the opposite side of the head.

Definition 4.3.9. Let C be a concept and $\sigma \in \Sigma_C$. Then, the *Hájek theory* $\mathcal{H}_C(\sigma)$ of σ is the set:

$$\mathcal{H}_C(\sigma) := \bigcup \{H_C(D(\sigma)) : \text{path}\,(\sigma) = \text{path}\,(l(D)),$$

$$D \text{ occurrence of a generalized atom}\}$$

Recall Definition 3.2.6, for the sake of simplicity, for every subconcept D and $\sigma \in \Sigma_C$, throughout this section, we will denote $pr(D(\sigma))$ by $D(\sigma)$ and the set $\{p\colon p \text{ occurs in } pr(\mathcal{H}_C(\sigma))\}$ by $At_{\mathcal{H}_C(\sigma)}$.

The next definition will be heavily used in the future in order to give to each Hájek theory of a given concept C a self-standing status as well as to make a bridge between the model that is claimed to satisfy concept C and the algorithm which says that there exists one.

Definition 4.3.10. Let $e : At_{\mathcal{H}_C(\sigma)} \longrightarrow T$ and $e' : At_{\mathcal{H}_C(\sigma')} \longrightarrow T$ be mappings for some $\sigma, \sigma' \in \Sigma_C$. Then, we say that e and e' are *mutually consistent* if they assign the same value to common elements, that is, for every $p \in At_{\mathcal{H}_C(\sigma)} \cap At_{\mathcal{H}_C(\sigma')}$, it holds that

$$e(p) = e'(p).$$

Witness sets and satisfiability

We now define what is a *Witness set* in the new framework. Following [BdRV01], this structure is used as a bridge structure between a model that is supposed to satisfy a given concept C and a procedure that decides whether such a model exists. We will use again the name used in Definition 4.2.8, but adapting the notion to the new framework.

Definition 4.3.11. Let C be a concept, let $\sigma \in \Sigma_C$, let $e : At_{\mathcal{H}_C(\sigma)} \longrightarrow T$ be a mapping such that $e(\mathcal{H}_C(\sigma)) = 1$ and let

$$\mathcal{W} \subseteq \bigcup \big\{ Func\big(At_{\mathcal{H}_C(\sigma')},\, \mathbf{T}\big) : \sigma' \in \Sigma_C \big\}$$

where $Func(A, B)$ refers to the mappings from A into B. We say that $\mathcal{W}$ is a *witness set generated by* e if:

1. $e \in \mathcal{W}$,
2. for every $e' \in \mathcal{W}$ with $e' : At_{\mathcal{H}_C(\sigma')} \longrightarrow T$, if a generalized atom E appears in the body of a formula in $\mathcal{H}_C(\sigma')$, then there is a mapping $e'' \in \mathcal{W}$ such that
   - $e'' : At(\mathcal{H}_C(\sigma' \cdot l(E))) \longrightarrow T$,
   - $e''\mathcal{H}_C(\sigma' \cdot l(E))) = 1$,
   - e' and e'' are mutually consistent.
3. for every $e' \in \mathcal{W}$ with $e' : At_{\mathcal{H}_C(\sigma')} \longrightarrow T$, there are $e_0, \ldots, e_k \in \mathcal{W}$ such that:
   - $e_0 = e$,
   - $e_k = e'$ and $k = |\sigma'|$,
   - for every $0 < i \leq k$, e_i is mutually consistent with e_{i-1},
   - for every $0 < i \leq n$, there is $\sigma_i \in \Sigma_C$, such that
     - $e_i : At_{\mathcal{H}_C(\sigma_i)} \to T, \quad \sigma_{i-1} \prec \sigma_i$,
     - $e_i(\mathcal{H}_C(\sigma_i)) = 1$ (here we consider $\sigma_0 := \varepsilon$).

We will say that e *generates a witness set* in the case that there is some $\mathcal{W}$ such that $\mathcal{W}$ is a witness set generated by e.

The next lemma will allow us to show that the concept C is satisfiable if and only if there exists a mapping e which generates a witness set on it.

Lemma 4.3.12. *Let C be a concept. For every $\sigma \in \Sigma_C$, every $D_1, \ldots, D_i$ occurrences in C, and every $r_1, \ldots, r_i \in T$, if* $\text{path}(\sigma) = \text{path}(l(D_1)) = \ldots = \text{path}(l(D_i))$ *then the following statements are equivalent:*

1. *there is an interpretation $\mathcal{I}$ and an individual $a \in \Delta^{\mathcal{I}}$ such that*

$$D_1^{\mathcal{I}}(a) = r_1,\ \ldots,\ D_i^{\mathcal{I}}(a) = r_i,$$

2. *there is a mapping* $e\colon At_{\mathcal{H}_C(\sigma)\cup\{D_1(\sigma),\ldots,D_i(\sigma)\}} \longrightarrow T$*, such that*
   - $e(D_1(\sigma)) = r_1,\ \ldots,\ e(D_i(\sigma)) = r_i$ *and*
   - e *generates a witness set.*

Proof. $(1 \Rightarrow 2)$: Suppose that there is an interpretation $\mathcal{I}$ and an individual $a \in \Delta^{\mathcal{I}}$ such that

$$D_1^{\mathcal{I}}(a) = r_1,\ \ldots,\ D_i^{\mathcal{I}}(a) = r_i.$$

Define the propositional evaluation $e\colon At_{\mathcal{H}_C(\sigma)\cup\{D_1(\sigma),\ldots,D_i(\sigma)\}} \longrightarrow T$ such that, for every $D(\sigma)$ appearing in $\mathcal{H}_C(\sigma) \cup \{D_1(\sigma), \ldots, D_i(\sigma)\}$:

$$e(D(\sigma)) = D^{\mathcal{I}}(a).$$

It is clear that the propositional evaluation e satisfies the first condition of the statement. Now we have to prove that e generates a witness set. In order to do this, we inductively define the witness set $\mathcal{W}$.

- The first step is obviously obtained by setting $e := a$.
- Suppose that $e' : At_{\mathcal{H}_C(\sigma')} \longrightarrow T$ has been already defined as

  $$e'(D(\sigma')) := D^{\mathcal{I}}(b)$$

  for $b \in \Delta^{\mathcal{I}}$ and $D(\sigma')$ appearing in $At_{\mathcal{H}_C(\sigma')}$. Since $\mathcal{I}$ is a model of C, we have that for every generalized atom $QR.E$ appearing in the head of a formula in $\mathcal{H}_C(\sigma')$, there is $c \in \Delta^{\mathcal{I}}$ such that

  $$QR.E^{\mathcal{I}}(b) \equiv (R^{\mathcal{I}}(b,c) \Box E^{\mathcal{I}}(c)) = 1$$

  for $\Box \in \{\boxtimes, \sqsupset\}$. Hence, by setting

  - $e'(F(\sigma' \cdot l(F))) = e''(F(\sigma' \cdot l(F))) := F^{\mathcal{I}}(c)$
  - $e'(R(\sigma', \sigma' \cdot l(F))) := R^{\mathcal{I}}(b,c)$

  for every $F(\sigma' \cdot l(F))$ and $R(\sigma', \sigma' \cdot l(F))$ appearing in $At_{\mathcal{H}_C(\sigma')}$, we obtain that

  - $e'(\mathcal{H}_C(\sigma')) = 1$,
  - e' and e'' are mutually consistent.

  Repeating the process for every $\sigma' \in \Sigma_C$, we obtain that for every $e' \in \mathcal{W}$ with $e' : At_{\mathcal{H}_C(\sigma')} \longrightarrow T$, there are $e_0, \ldots, e_k \in \mathcal{W}$ such that:

  - $e_0 = e$,

- $e_k = e'$ and $k = |\sigma'|$,
- for every $0 < i \leq k$, e_i is mutually consistent with e_{i-1},
- for every $0 < i \leq n$, there is $\sigma_i \in \Sigma_C$, such that
  - $e_i : At_{\mathcal{H}_C(\sigma_i)} \to T$, $\sigma_{i-1} \prec \sigma_i$,
  - $e_i(\mathcal{H}_C(\sigma_i)) = 1$.

$(2 \Rightarrow 1)$: Suppose that there is a mapping $e : At_{\mathcal{H}_C(\sigma) \cup \{D_1(\sigma), \ldots, D_i(\sigma)\}} \longrightarrow T$, such that $e(D_1(\sigma)) = r_1, \ldots, e(D_i(\sigma)) = r_i$, and e generates a witness set. Then we have to show that there exists an interpretation $\mathcal{I}$ and an individual $a \in \Delta^{\mathcal{I}}$ such that $D_1^{\mathcal{I}}(a) = r_1, \ldots, D_i^{\mathcal{I}}(a) = r_i$. So, define the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where:

- $\Delta^{\mathcal{I}} = \Sigma_C$,
- $R^{\mathcal{I}}(\sigma, \sigma') = \begin{cases} e_\sigma(R(\sigma, \sigma')), & \text{if } R(\sigma, \sigma') \text{ occurs in } \mathcal{H}_C(\sigma) \\ 0, & \text{otherwise} \end{cases}$
- for every atomic concept A and $\sigma \in \Sigma_C$, define $A^{\mathcal{I}}(\sigma) = e_\sigma(A(\sigma))$, if $A(\sigma)$ occurs in $\mathcal{H}_C(\sigma)$ and $A^{\mathcal{I}}(\sigma) = 0$, otherwise.

Now we have to show by induction on concepts, that, for every occurrence E of a subconcept of C and every $\sigma \in \Delta^{\mathcal{I}}$, $E^{\mathcal{I}}(\sigma) = e_\sigma(E(\sigma))$.

- If E is either an atomic or a constant concept, it holds by definition of $\mathcal{I}$.
- If E is a propositional combination of concepts, this is trivial.
- Let $E = \forall R.F$ and suppose that $F^{\mathcal{I}}(\sigma) = e_\sigma(F(\sigma))$ and $R^{\mathcal{I}}(\sigma, \sigma \cdot l(F)) = e_\sigma(R(\sigma, \sigma \cdot l(F))$. On the one hand, by Definition 4.2.8 and Definition 4.3.8, it holds that $e_\sigma(\mathcal{H}_C(\sigma)) = 1$ and

$$\begin{aligned} & e_\sigma(\forall R.F(\sigma)) = \\ = \quad & e_\sigma(R(\sigma, \sigma \cdot l(F))) \Rightarrow e_\sigma(F(\sigma \cdot l(F))) = \\ = \quad & e_\sigma(R(\sigma, \sigma \cdot l(F))) \Rightarrow e_{\sigma \cdot l(F)}(F(\sigma \cdot l(F))) = \\ = \quad & R^{\mathcal{I}}(\sigma, \sigma \cdot l(F)) \Rightarrow F^{\mathcal{I}}(\sigma \cdot l(F)). \end{aligned}$$

  On the other hand, again by Definition 4.2.8 and Definition 4.3.8, it holds that, for every generalized atom G such that $\text{path}\,(l(G)) = \text{path}\,(l(F))$,

$$\begin{aligned} & e_\sigma(\forall R.F(\sigma)) \leq \\ \leq \quad & e_\sigma(R(\sigma, \sigma \cdot l(G))) \Rightarrow e_\sigma(F(\sigma \cdot l(G))) = \\ = \quad & e_\sigma(R(\sigma, \sigma \cdot l(G))) \Rightarrow e_{\sigma \cdot l(G)}(F(\sigma \cdot l(G))) = \\ = \quad & R^{\mathcal{I}}(\sigma, \sigma \cdot l(G)) \Rightarrow F^{\mathcal{I}}(\sigma \cdot l(G)) \end{aligned}$$

  So, $e_\sigma(\forall R.F(\sigma)) = \min_{x \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(\sigma, x) \Rightarrow F^{\mathcal{I}}(x)\} = (\forall R.F)^{\mathcal{I}}(\sigma)$.

- Let $E = \exists R.F$ and suppose that $F^{\mathcal{I}}(\sigma) = e_\sigma(F(\sigma))$ and $R^{\mathcal{I}}(\sigma, \sigma \cdot l(F)) = e_\sigma(R(\sigma, \sigma \cdot l(F))$. On the one hand, by Definition 4.2.8 and Definition 4.3.8, it holds that $e_\sigma(\mathcal{H}_D(\sigma)) = 1$ and

$$\begin{aligned} & e_\sigma(\exists R.F(\sigma)) = \\ = \;& e_\sigma(R(\sigma, \sigma \cdot l(F)))) * e_\sigma(F(\sigma \cdot l(F))) = \\ = \;& e_\sigma(R(\sigma, \sigma \cdot l(F))) * e_{\sigma \cdot l(F)}(F(\sigma \cdot l(F))) = \\ = \;& R^{\mathcal{I}}(\sigma, \sigma \cdot l(F)) * F^{\mathcal{I}}(\sigma \cdot l(F)) \end{aligned}$$

  On the other hand, again by Definition 4.2.8 and Definition 4.3.8, it holds that, for every generalized atom G such that $\text{path}\,(l(G)) = \text{path}\,(l(F))$,

$$\begin{aligned} & e_\sigma(\exists R.F(\sigma)) \geq \\ \geq \;& e_\sigma(R(\sigma, \sigma \cdot l(G))) * e_\sigma(F(\sigma \cdot l(G))) = \\ = \;& e_\sigma(R(\sigma, \sigma \cdot l(G))) * e_{\sigma \cdot l(G)}(F(\sigma \cdot l(G))) = \\ = \;& R^{\mathcal{I}}(\sigma, \sigma \cdot l(G)) * F^{\mathcal{I}}(\sigma \cdot l(G)) \end{aligned}$$

  So, $e_\sigma(\exists R.F(\sigma)) = \max_{x \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(\sigma, x) * F^{\mathcal{I}}(x)\} = (\exists R.F)^{\mathcal{I}}(\sigma)$.

Hence, for every concept E and every $\sigma \in \Delta^{\mathcal{I}}$, it holds that $E^{\mathcal{I}}(\sigma) = e_\sigma(E(\sigma))$. □

Using the occurrence C itself and the constant ε we get the following corollary.

Corollary 4.3.13. *Let C be a concept and $r \in T$. The following statements are equivalent.*

1. *C is modally r-satisfiable,*
2. *there is a mapping $e : At_{\mathcal{H}_C(\varepsilon)} \longrightarrow T$ such that $e(C(\varepsilon)) = r$ and e generates a witness set on C.*

Witness sets and procedures

Let us now consider the procedure $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ given in Figure 4.2. This procedure takes as *admissible inputs* strings made by

- an element $\sigma \in \Sigma_C$ and
- a set of pairs $\langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle$, where, for $1 \leq j \leq i$, D_j is the occurrence of a concept in C and r_j is a truth value,

and has three possible outputs:

- true,

Write down $\mathcal{H}_C(\sigma) \cup \{D_1(\sigma), \ldots, D_i(\sigma)\}$.
if there is a mapping $e_\sigma : At_{\mathcal{H}_C(\sigma)\cup\{D_j(\sigma):\, 1\leq j\leq i\}} \longrightarrow T$ such that $e_\sigma(D_j(\sigma)) = r_j$, for $1 \leq j \leq i$ and $e_\sigma(\mathcal{H}_C(\sigma)) = 1$ **then**
  if for every $j \leq i$ it holds that $nest(D_j) = 0$ **then**
    return **true**
  else
    return the following list of strings
    $(\sigma \cdot l(E_1), \langle E_1, r_{1_1}\rangle, \ldots, \langle E_k, r_{1_k}\rangle)$,
    $\vdots$
    $(\sigma \cdot l(E_k), \langle E_1, r_{k_1}\rangle, \ldots, \langle E_k, r_{k_k}\rangle)$,
    where $\{E_1, \ldots, E_k\}$ are the occurrences in the body of $\mathcal{H}_C(\sigma)$ and, for every $1 \leq h, m \leq k$, $e_\sigma(E_m(\sigma \cdot l(E_h))) = r_{h_m}$.
  end if
else
  return **false**
end if

Figure 4.2: Algorithm $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$

- false
- a list of strings
  $(\sigma \cdot l(E_1), \langle E_1, r_{1_1}\rangle, \ldots, \langle E_k, r_{1_k}\rangle)$,
  $\vdots$
  $(\sigma \cdot l(E_k), \langle E_1, r_{k_1}\rangle, \ldots, \langle E_k, r_{k_k}\rangle)$,
  each one of these strings being an admissible input[2].

What it is crucial is that in case that $\text{path}\,(\sigma) = \text{path}\,(l(D_1)) = \ldots = \text{path}\,(l(D_i))$, also the strings obtained as output satisfy this equality requirement.

This procedure will be later used as a subroutine by the algorithm $Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ (see Figure 4.3) in order to check for the r-satisfiability of a given concept C. For this reason it is parametrized with a concept C which does not appear within the input string.

Now we check that the time needed by algorithm $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ to reach an answer is non-deterministically polynomial on the length of the input.

Lemma 4.3.14. *Algorithm $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ is in NPTIME.*

Proof. Let C be a concept, $\sigma \in \Sigma_C$, $D_1 \ldots, D_i$ occurrences of subconcepts of C such that $path(l(D_j)) = path(l(D_{j+1})) = path(\sigma)$, for $1 \leq j < i$ and let us

[2]Note that here, with the notation $(\langle E_1, r_{1_1}\rangle, \ldots, \langle E_k, r_{1_k}\rangle)$ we mean that the pairs of concept and truth values appearing in the output are *among* $\langle E_1, r_{1_1}\rangle, \ldots, \langle E_k, r_{1_k}\rangle$, even if they are not all indeed present.

denote, for short, the input $\sigma, \langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle$ by φ. First of all we need to see which is the size of $\mathcal{H}_C(\sigma)$ with respect to the size of the input. As we can see from Definition 4.3.8, given a generalized atom $E(a)$, the size of a single formula appearing in a Hájek theory $H_C(E(a))$ is at most $2 \cdot |E|$ (here $|E|$ refers to the length of the encoding of E). Since, for every generalized atom E appearing in the input φ, it holds that $|E| \leq |\varphi|$, then the time needed to write down a single formula appearing in $H_C(E(a))$ is in $\mathcal{O}(|\varphi|)$. Again Definition 4.3.8 says us that, for every generalized atom $QR.E$, with $Q \in \{\forall, \exists\}$, appearing in the input φ, such that $|l(QR.E)| = |\sigma|$, the number of formulas in $H_C(QR.E(\sigma))$, is the number of all the generalized atoms $QP.F$ appearing in the input φ, such that $path(l(QP.F)) = path(\sigma)$ and $P = R$, which is less than $|\varphi|$. Hence, for every generalized atom $QR.E$ appearing in the input φ, such that $path(l(QR.E)) = path(\sigma)$, the number of formulas in $H_C(QR.E(\sigma))$, the time needed to write $H_C(QR.E(\sigma))$ is in $\mathcal{O}(|\varphi|^2)$. By Definition 4.3.9, we have that, in order to calculate the size of $\mathcal{H}_C(\sigma)$, we need to sum the sizes of theories $H_C(E(\sigma))$, of generalized atoms E appearing in the input such that $|l(E)| = |\sigma|$. Since the number of such generalized atoms is less that $|\varphi|$, then the time needed to write down $\mathcal{H}_C(\sigma)$ is in $\mathcal{O}(|\varphi|^3)$.

Furthermore, it is easy to see that the size of $e_\sigma(\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \colon 1 \leq j \leq i\})$ is constant on the size of $\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \colon 1 \leq j \leq i\}$ (the constant factor depending on the encoding of the mapping $e_\sigma(\cdot)$). Since, as we have seen, the size of $\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \colon 1 \leq j \leq i\}$ is in $\mathcal{O}(|\varphi|^3)$, so is the size of $e_\sigma(\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \colon 1 \leq j \leq i\})$.

It is well-known (see [Häh01]) that satisfiability for propositional finite-valued logics is an NP-complete problem. Hence, answering whether for a given mapping e_σ from $At_{\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \colon 1 \leq j \leq i\}}$ to T it holds that $e_\sigma(D_j(\sigma)) = r_j$, for $1 \leq j \leq i$ and $e_\sigma(\mathcal{H}_C(\sigma)) = 1$ is a task that can be accomplished in an amount of time that is polynomial on the cardinality of the set $At_{\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \colon 1 \leq j \leq i\}}$. Therefore, the time needed to accomplish this task is still polynomial on the size of φ. Moreover, we will need to write down a possible solution to the above problem in the form $e_\sigma(E_1(\sigma_1)) = r_1, \dots, e_\sigma(E_m(\sigma_m)) = r_m$, where $E_1(\sigma_1), \dots, E_m(\sigma_m) \in At_{\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \colon 1 \leq j \leq i\}}$ and $r_1, \dots, r_m \in T$. It is easy to see that the time needed to write down such a solution is constant in the size of $At_{\mathcal{H}_C(\sigma) \cup \{D_j(\sigma) \colon 1 \leq j \leq i\}}$ (the constant factor depending on the encoding of the truth values).

Finally, when the output is not simply a boolean, it is just part of the above mentioned solution re-written in a different form. Since the size of $e_\sigma(E_1(\sigma_1)) = r_1, \dots, e_\sigma(E_m(\sigma_m)) = r_m$, coincides with the size of $(\sigma_1, \langle E_1, r_1\rangle, \dots, \langle E_m, r_m\rangle), \dots, (\sigma_m, \langle E_1, r_1\rangle, \dots, \langle E_m, r_m\rangle)$, then the time needed to write down the output of algorithm $Node_C$ is also polynomial on the size of φ. $\square$

Next we consider the algorithm $Witness_C(\sigma, \langle D_1, r_1\rangle, \dots, \langle D_i, r_i\rangle)$ given in Figure 4.3.

Lemma 4.3.15. *Let C be a concept. For every $\sigma \in \Sigma_C$, every $D_1, \dots, D_i$*

if $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns **true then**
    return **true**
end if
if $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns a list of strings **and** for each string $\sigma \cdot l(E_m), \langle E_1, r_{m_1}\rangle, \ldots, \langle E_k, r_{m_k}\rangle$ in this list, it holds that $Witness_C(\sigma \cdot l(E_m), \langle E_1, r_{m_1}\rangle, \ldots, \langle E_k, r_{m_k}\rangle)$ returns **true** **then**
    return **true**
else
    return **false**
end if

Figure 4.3: Algorithm $Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$

occurrences in C, and every $r_1, \ldots, r_i \in T$, if $\text{path}(\sigma) = \text{path}(l(D_1)) = \ldots = \text{path}(l(D_i))$ *then the following statements are equivalent.*

1. *$Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns true,*

2. *there is a mapping $e : At_{\mathcal{H}_C(\sigma)\cup\{D_1(\sigma),\ldots,D_i(\sigma)\}} \longrightarrow T$, such that $e(D_1(\sigma)) = r_1, \ldots, e(D_i(\sigma)) = r_i$ and e generates a witness set.*

Proof. The proof of each one of the directions is done by induction (but decreasing the step): first of all we consider the case that $|\sigma| = nest(C)$, and then we show that if we know the statement for all σ' with $|\sigma'| = d + 1$ then we also know it for the case that $|\sigma| = d$.

(1 ⇒ 2): Suppose that $Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns **true**, we proceed by induction on the degree of C.

(0) If $nest(D_j) = 0$, for $1 \leq j \leq i$, then $\mathcal{H}_C(\sigma)$ is empty. Since $Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns **true**, then e is a mapping over $\{D_j(\sigma) : 1 \leq j \leq i\}$ such that $e(D_j(\sigma)) = r_j$ and $\mathcal{W} = \{e\}$ is a witness set generated by e on $\{D_j(\sigma) : 1 \leq j \leq i\}$.

(d) Let $nest(D_j) > 0$, for at least one j such that $1 \leq j \leq i$ and suppose, by inductive hypothesis, that,

- for each occurrences $E_1, \ldots, E_k$ of concepts occurring in C such that there exist $R \in N_R$ and $n \in \mathbb{N}$ with $l(E_1) = \ldots = l(E_k) = l(D_j) \cdot Rn$,
- for each $\sigma' \in \Sigma_C$, such that there exists $1 \leq m \leq k$ with $\sigma' = \sigma \cdot l(E_m)$,
- for each $r_1, \ldots, r_k \in T$,

it holds that, if

$$Witness_C(\sigma', \langle E_1, r_1\rangle, \ldots, \langle E_k, r_k\rangle) \text{ returns } \textbf{true},$$

then there is a mapping $e_{\sigma'} : At_{\mathcal{H}_C(\sigma')\cup\{E_h(\sigma')\colon 1\leq m\leq k\}} \longrightarrow T$ such that

- $e_{\sigma'}(E_m(\sigma')) = r_m$, for $1 \leq m \leq k$,
- $e_{\sigma'}(\mathcal{H}_C(\sigma')) = 1$,
- $e_{\sigma'}$ generates a witness set $\mathcal{W}_{\sigma'}$ on $E_1, \ldots, E_k$.

Now, suppose that $Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns **true**, then:

1. $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ returns a list of strings $\{\sigma \cdot l(E_m), \langle E_1, r_{m_1}\rangle, \ldots, \langle E_k, r_{m_k}\rangle\colon 1 \leq m \leq k\}$ (remember that $nest(D_j) > 0$ by hypothesis) and, therefore (see Figure 4.2), there is a mapping $e : At_{\mathcal{H}_C(\sigma)\cup\{D_1(\sigma),\ldots,D_i(\sigma)\}} \longrightarrow T$, such that $e(D_1(\sigma)) = r_1, \ldots, e(D_i(\sigma)) = r_i$
2. For each string $\sigma \cdot l(E_m), \langle E_1, r_{m_1}\rangle, \ldots, \langle E_k, r_{m_k}\rangle$ in the output of $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$, it holds that $Witness_C(\sigma \cdot l(E_m), \langle E_1, r_{m_1}\rangle, \ldots, \langle E_k, r_{m_k}\rangle)$ returns **true**.

Hence, by inductive hypothesis, for each occurrence E_m appearing in the output of $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$, there is a mapping $e_{\sigma'} : At_{\mathcal{H}_C(\sigma')\cup\{E_h(\sigma')\colon 1\leq m\leq k\}} \longrightarrow T$ such that

- $e_{\sigma'}(E_m(\sigma')) = r_m$, for $1 \leq m \leq k$,
- $e_{\sigma'}(\mathcal{H}_C(\sigma')) = 1$,
- $e_{\sigma'}$ generates a witness set $\mathcal{W}_{\sigma'}$ on $E_1, \ldots, E_k$.

Moreover, since the truth values $r_{1_1}, \ldots, r_{k_k}$ are those appearing in the output of $Node_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$, then e_σ overlaps with each $e_{\sigma'}$. So, the set:

$$\mathcal{W} = \{e_\sigma\} \cup \bigcup\{\mathcal{W}_{e_{\sigma'}} : \sigma' = \sigma \cdot l(E_m) \text{ with } 1 \leq m \leq k\}$$

is a witness set generated by e_σ.

(2 $\Leftarrow$ 1): Suppose that $e_\sigma : At_{\mathcal{H}_C(\sigma)\cup\{D_1(\sigma),\ldots,D_i(\sigma)\}} \longrightarrow T$ is a mapping such that $e_\sigma(D_1(\sigma)) = r_1, \ldots, e_\sigma(D_i(\sigma)) = r_i$ and e_σ generates a witness set.

(0) If $nest(D_j) = 0$, for every $1 \leq j \leq i$ then it is enough that e_σ be a mapping such that $e_\sigma(D_j(\sigma)) = r_j$, for $Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$ to return **true**.

(d) Let $nest(D_j) > 0$ for at least one $1 \leq j \leq i$ and suppose, by inductive hypothesis, that

- for each occurrences $E_1, \ldots, E_k$ of concepts occurring in C such that there exist $R \in N_R$ and $n \in \mathbb{N}$ with $l(E_1) = \ldots = l(E_k) = l(D_j) \cdot Rn$,
- for each $\sigma' \in \Sigma_C$, such that there exists $1 \leq m \leq k$ with $\sigma' = \sigma \cdot l(E_m)$,
- for each $r_1, \ldots, r_k \in T$,

it holds that, if there is a mapping $e_{\sigma'} : At_{\mathcal{H}_C(\sigma')\cup\{E_h(\sigma')\colon 1\le m\le k\}} \longrightarrow T$ such that

- $e_{\sigma'}(E_m(\sigma')) = r_m$, for $1 \le m \le k$,
- $e_{\sigma'}(\mathcal{H}_C(\sigma')) = 1$,
- $e_{\sigma'}$ generates a witness set $\mathcal{W}_{\sigma'}$ on $E_1, \ldots, E_k$.

then

$$Witness_C(\sigma', \langle E_1, r_1\rangle, \ldots, \langle E_k, r_k\rangle) \text{ returns } \mathbf{true},$$

Now, if e_σ is a mapping which generates a witness set $\mathcal{W}_{e_\sigma}$ on $\{D_1, \ldots, D_i\}$, then, by Definition 4.2.8, for each occurrence of a generalized atom E appearing in the body of a formula in $\mathcal{H}_C(\sigma)$ there is a mapping $e_{\sigma'} \in \mathcal{W}$ such that

- $e_{\sigma'}(\mathcal{H}_C(\sigma')) = 1$,
- e_σ and $e_{\sigma'}$ are mutually consistent.

We consider the overlapping relation *ol* to be the transitive closure of the mutually consistent relation between mappings. Then the set

$$\{e' \in \mathcal{W}\colon e_\sigma \; ol \; e'\}$$

is a witness set generated by e_σ on $\{D_1, \ldots, D_i\}$. Hence, by inductive hypothesis,

- for each occurrences $E_1, \ldots, E_k$ of concepts occurring in C such that there exist $R \in N_R$ and $n \in \mathbb{N}$ with $l(E_1) = \ldots = l(E_k) = l(D_j) \cdot Rn$,
- for each $\sigma' \in \Sigma_C$, such that there exists $1 \le m \le k$ with $\sigma' = \sigma \cdot l(E_m)$,
- for each $r_1, \ldots, r_k \in T$,

there is a mapping $e_{\sigma'} : At_{\mathcal{H}_C(\sigma')} \longrightarrow T$ such that

$$Witness_C(\sigma', \langle E_1, r_1\rangle, \ldots, \langle E_k, r_k\rangle) \text{ returns } \mathbf{true}.$$

Hence (see Figure 4.3), it holds that

$$Witness_C(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle) \text{ returns } \mathbf{true},$$

In particular, we have that the statement holds for ε, C and r. □

Corollary 4.3.16. *Let C be a concept and $r \in T$. Then $Witness_C(\varepsilon, \langle C, r\rangle)$ returns* **true** *if and only if there is a mapping $e : \mathcal{H}_C(\varepsilon) \cup \{C(\varepsilon)\} \longrightarrow T$ such that $e(C(\varepsilon)) = r$ that generates a witness set.*

Main result

Combining Corollaries 4.3.13 and 4.3.16 we can now prove the main result.

Theorem 4.3.3 Satisf *is in PSPACE.*

Proof. Corollaries 4.3.13 and 4.3.16 tell us that the algorithm in Figure 4.3 does what we want. It only remains to see that this algorithm belongs to PSPACE.

Let C be an $\mathfrak{I}\mathcal{ALCED}^T$ concept. By Lemma 4.3.12 and Lemma 4.3.15 we have that C is r-satisfiable if and only if there is a partial propositional evaluation $e : pr(\mathcal{H}_C(\varepsilon)) \longrightarrow T$ such that $Witness(\varepsilon, \langle C, r\rangle)$ returns **true**. Hence we need to prove that $Witness$ can be given a PSPACE implementation. Consider a non-deterministic Turing machine that guesses a strings $\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle$ and runs $Witness(\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle)$, then we need to prove that this machine runs in NPSPACE and, by an appeal to Savitch's Theorem, we will achieve the desired result.

Algorithm $Witness$ is a recursive algorithm and, at every recursive call, subroutine $Node_C$ is triggered over one of the strings $\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle$ obtained from a previous triggering of $Node_C$. The choice of the string to be processed by $Node_C$ at every successive step can be done by non-deterministic guess.

Due to the overlapping of mappings e_σ, for $\sigma \in \Sigma_C$, at every application of subroutine $Node_C$ on a string $\sigma, \langle D_1, r_1\rangle, \ldots, \langle D_i, r_i\rangle$, the only information needed is the output obtained by subroutine $Node_C$ on strings $\sigma', \langle D_1', r_1'\rangle, \ldots, \langle D_i', r_i'\rangle$, for every σ' that is a prefix of σ. So, at each step, the remaining information can be deleted.

Intuitively, Σ_C can be represented as a tree and the only information that is needed at each step is the one lying in the path from the root to the present step. At every successive recursive call the modal degrees of concepts $D_1, \ldots, D_i$ is strictly less than the modal degrees of concepts processed at the previous call and at most $nest(C)$ recursive calls are needed until we meet a Hájek set without generalized atoms in the bodies of formulas. So, the maximum amount of information to be retained in memory is the output of subroutine $Node_C$ multiplied by $nest(C)$. On the one hand, $nest(C)$ is at most lineal on the size of C and, therefore, of the input. On the other hand, by Lemma 4.3.14 the space needed to run subroutine $Node_C$ and to write down its input is polynomial on the size of the input. Hence the amount of space needed by algorithm $Witness$ is polynomial on the size of the input. □

4.4 Related work

Computational complexity is a problem that has been addressed only in recent years by researchers on FDL. Besides the results reported in this chapter, some work has been undertaken on problems that consider the presence of a knowledge base.

In [BP11d] it is proved that concept witnessed r-satisfiability with respect to a general TBox for language $\mathcal{ALC}$ over (not necessarily linear) finite De Morgan lattices is EXPTIME-complete. The result is proved by means of a reduction to automata theory. Notice that in [BP11d] the truth function considered for concept constructors are min and max for conjunction and disjunction respectively, a unary function that satisfies the De Morgan laws for the negation and Kleene-Dienes implication. This implies that the implication constructor and the existential quantifier are definable like in Zadeh's semantics.

In [BP11f] the result of [BP11d] is enhanced by adding a finite t-norm to the operations of the De Morgan lattice and using its residuum in the semantics of the value restriction and of inclusion axioms. With such semantics, the existential quantifier is no more definable as in the De Morgan lattices considered in [BP11d]. So, the language considered in [BP11f] is $\mathcal{ALCE}$. Nevertheless, concept satisfiability with respect to general knowledge bases is proved to be EXPTIME-complete. Again, the result is proved by means of a reduction to automata theory.

The semantics considered in [BP11e] is the same as in [BP11f] and the language is $\mathcal{ALCE}$ with inverse roles, that are not been considered in this dissertation. By means of a reduction to automata theory it is proved that entailment of lower bound inclusion axioms[3] by general knowledge bases is EXPTIME-complete. The same problem is proved to be PSPACE-complete if the TBox is acyclic. Moreover it is proved that concept satisfiability with respect to acyclic TBoxes is PSPACE-complete as well. Again, the proof is based on a recursive reduction to automata theory. The result in [BP11e] generalizes the one in [BP11f], not only because the language considered is more expressive, but also because the concept subsumption problem is considered, that was not considered in [BP11f]. Moreover the problem of concept satisfiability with respect to acyclic TBoxes is considered.

[3]In [BP11e] this problem is indeed called *concept subsumption with respect to knowledge bases*.

Chapter 5

Conclusions and future work

In the first part of this chapter we summarize the main contributions of the dissertation. In a second part we sketch future lines of research taking into account the framework of FDLs already presented in the introduction.

5.1 Main contributions

The idea of this dissertation is a re-thinking FDL strongly relying on MFL. The main contributions of the present dissertation to the realization of this idea are the following:

Fuzzy Description Languages In this dissertation we have considered a general framework to deal with Fuzzy Description Logic over a semantics based on Mathematical Fuzzy Logic. Following Hájek we have proposed FDL languages having two constructors for conjunction, two constructors for disjunction (called in both cases strong and weak), one or two constructors for complementation (depending whether the residuated negation of the underlying logic is involutive or not), a constructor for implication (not definable as in the classical case except for the Łukasiewicz case) and finally having some optional constructors like truth constants and Delta operator. This implies the use of a novel notation for constructors that expands the one traditionally used in classical DL and in the earlier works in FDL. As part of this task, we have proposed a description language enriched with novel concept constructors that are the FDL versions of the logical operators used in the framework of MFL. So, we have considered a language enriched with concept constructors for strong conjunction $\boxtimes$, strong disjunction $\boxplus$, implication $\sqsupset$, complementation $\sim$, Monteiro-Baaz Delta operator $\triangle$ and a suitable set of truth constants $\overline{r}$. The semantics is defined using t-norms and their residua either defined over $[0, 1]$ or over a finite chain.

Once set up the new languages, we have considered the consequences of assuming a semantics based on t-norms. So, we have systematically studied the hierarchies of FDL languages depending both the constructors allowed in each language and the t-norm used and their differences with respect to the classical $\mathcal{ALC}$ languages hierarchy. Finally in this FDL languages we have defined fuzzy KBs following the tradition of FDL. About fuzzy axioms, we have studied the simplifications that can be performed on the types of axioms in order to narrow the symbology used to build up fuzzy knowledge bases. About reasoning tasks, we have studied the reductions that can be performed between them, depending on the languages and t-norms considered.

Relations to other formalisms The fact of relying on MFL allows to describe the relations of FDL to other logical formalisms such as Fuzzy First Order Logic and Fuzzy Multi-modal Logic. Our proposal tries to relate FDL and MFL so, in the dissertation these relations are studied in deep giving the translations from FDL concepts and axioms to first order and multi-modal formulas (in the case of multi-modal we have also provided a translation from multi-modal to FDL). In both cases we have proved that the translations are meaningful by means of a translation between the respective semantics. This is a very important topic that has been well studied in the classical DLs, but that has not yet been systematically developed in traditional FDL. Roughly speaking we can say that these translations justify the fact that these FDLs defined in this dissertation are the description languages associated to Fuzzy Logics studied in MFL. We point out that the study of the relations of FDL to other formalisms allows to easily transfer the results on decidability and complexity obtained from one formalism to others, as we have already seen in Section 4.2.

(Un)decidability and complexity This dissertation also provides some (un)decidability results that have been proved by the author during his doctoral studies. The decidability result obtained is limited to concept satisfiability and subsumption without a knowledge base for language $\mathfrak{I}\mathcal{ALE}$ over product t-norm. The prove is rather interesting because it considers quasi-witnessed interpretations. For this reasons it has to take into account the problem of dealing with infinite interpretations, differently from previous results that limits to witnessed interpretations which provides finite model property for the concept satisfiability problem in $\mathfrak{I}\mathcal{ALCE}$ language. The decidability results are restricted to validity and positive satisfiability but it is still an open problem for 1-satisfiability.

The undecidability result is really general because, having been proved for a rather basic language ($\mathcal{ALC}$), it ensures undecidability for a quite wide family of description languages over the infinite Łukasiewicz t-norm.

Under a computational complexity point of view the dissertations provides the following results:

1. Satisfiability and validity of formulas are PSPACE-complete for the Multi-modal Logic of all Kripke $Ł_n$-frames.

2. Concept r-satisfiability and 1-subsumption are PSPACE-complete for language $\mathfrak{I}\mathcal{ALCED}^S$, over any finite MTL-chain.

Study of algorithms Besides the results achieved under a computational point of view, the author thinks that a valuable contribution of this dissertation is the study of the algorithms applied to reasoning tasks in FDLs. The practical aspects of FDL, in fact, involves that they have to be thought also with the aim of designing future real programs and not only as plain results on the possibility of their application. For this reason, the proved results as well as the procedures used to prove them are explained in full (and often boring) details.

In Section 4.2 a generalization of the procedure based on Hintikka sets to the case of Multi-modal Logic of all Kripke Ł$_n$-frames is studied. The result proved through this procedure is a particular case of the result in Section 4.3. The aim of the investigation on procedures is the reason for which this procedure is investigated even though a more general result is provided in the dissertation: the former section has indeed to be seen as an investigation on a different procedure used, more than on the result provided.

The novel PSPACE procedure provided in Section 4.3 to prove PSPACE upper bound for the case of the FDL language $\mathfrak{I}\mathcal{ALCED}^S$, over any finite MTL-chain is based on Hájek's reduction for Łukasiewicz. In fact the algorithm reducing concept satisfiability to satisfiability of a set of propositional formulas remains valid for any FDL language $\mathfrak{I}\mathcal{ALCE}$ over a finite MTL-chain since all models over a finite chain are witnessed. Moreover, this general uniform reduction provided in [Háj05] is investigated, throughout the dissertation, under other points of view such as

1. its generalization to the quasi-witnessed satisfiability problem in Section 3.2,

2. the fact that it is not polynomial in Section 4.1.

5.2 Open problems and future work

Besides the results achieved in this dissertation, some problems have been left open. In particular, we outline the following:

- The decidability of the concept 1-satisfiability problem for language $[0,1]_\Pi$-$\mathfrak{I}\mathcal{ALE}$ with respect to unrestricted (that is, not only quasi-witnessed) interpretations (see Appendix A, for details).

- Characterizing computational complexity of these algorithms is still an open problem in the case of an infinite set of truth values.

As future work we could give a long list of problems to deal with, either on the theoretical side or on the applications. As we have said in the Preface, we consider FDLs proposed here as a kernel of what has to be a Fuzzy Description

Logic. The agenda of FDLs contains many concepts and subjects that are not addressed here, like fuzzy modifiers, fuzzy quantifiers and problems like how to deal with uncertainty in this framework. We know that these are hard topics, very important. They have been already addressed in several papers in fuzzy logic in the wider sense and need to be studied deeper in order to be incorporated in the framework of FDLs presented here. Nevertheless we restrict the list to some interesting problems to be studied in the framework proposed in this dissertation because there is a lot of work to do in this setting.

- A study of more expressive FDLs languages over a finite (MTL) BL-chains.
- a systematic study of the intractability sources for $\mathfrak{I}\mathcal{ALCE}$ FDL languages,
- the study of the computational complexity of more expressive FDLs based on finite t-norms,
- the design of novel procedures for more expressive FDLs based on finite t-norms,
- a confrontation between the algorithms for FDLs based on finite t-norms, under the point of view of the execution speed; in particular the ones based on Hintikka sets, on Hájek sets, on Automata Theory, on completion forests and on a reduction to classical DLs,
- the design of working programs based on the algorithms provided in this dissertation.

This could be an interesting work-program both since we can study how complexity changes when going from two valued to n-valued logics and since, from the applications point of view, it seems that the FDLs that are applicable in a near future are those valued on a finite chain.

Moreover, there are further issues to be faced in the case of an infinite set of truth values, such as the decidability of the satisfiability problem in languages richer than $\mathfrak{I}\mathcal{ALCE}$ and of the KB consistency problem with restricted forms of knowledge bases (like e.g. acyclic ones).

Appendix A

Quasi-witnessed completeness of first order Product Logic with standard semantics

In this appendix we prove that the first order $[0,1]_\Pi$-tautologies coincides with the first order tautologies with respect to the one generated subalgebra. In particular this imply that first order $[0,1]_\Pi$-tautologies and $[0,1]_\Pi$-positive satisfiable formulas coincide with the same sets of formulas restricted to quasi-witnessed models over $[0,1]_\Pi$, a result needed in the first part of Chapter 3 to prove decidability of validity and positive satisfiability for concepts in FDLs over product logic.

Recall that an *one-generated subalgebra* of $[0,1]_\Pi$ is the subalgebra of $[0,1]_\Pi$ whose domain is $\{a^0, a^1, a^2, \ldots\} \cup \{0\}$, for $a \in (0,1)$.

In [Háj98c, Theorem 5.4.30] the author proves that $[0,1]_Ł$-tautologies coincide with the common $Ł_n$-tautologies for $n \geq 2$, i.e., coincide with the common tautologies of the finite subalgebras of $[0,1]_Ł$. In [EGN10] the authors prove that the result is not valid for a logic of a *t*-norm different from Łukasiewicz. But Hájek's result can be read in another way since $Ł_n$ are the one-generated subalgebras of $[0,1]_Ł$ whose generator is a rational number. What we prove in this appendix is that this reading of Hájek's result can be generalized to First Order Product Logic.

In order to prove this result we first prove some lemmas and provide some definitions. Firstly we prove the following lemma that uses only residuation condition, and thus it is also true for any MTL-chain (prelinear residuated chain).

Lemma A.0.1. *In any* Π*-chain the following inequalities hold:*

1. $(x \Leftrightarrow x') * (y \Leftrightarrow y') \leq (x \Rightarrow y) \Leftrightarrow (x' \Rightarrow y')$,

2. $(x \Leftrightarrow x') * (y \Leftrightarrow y') \leq (x * y) \Leftrightarrow (x' * y')$,

3. $\inf_{i\in I}\{x_i \Leftrightarrow y_i\} \leq \inf_{i\in I}\{x_i\} \Leftrightarrow \inf_{i\in I}\{y_i\}$,

4. $\inf_{i\in I}\{x_i \Leftrightarrow y_i\} \leq \sup_{i\in I}\{x_i\} \Leftrightarrow \sup_{i\in I}\{y_i\}$.

Proof. The proofs are easy consequences of residuation property

$$x * y \leq z \qquad \text{iff} \qquad x \leq y \Rightarrow z. \tag{res}$$

In particular we point out that $x * (x \Rightarrow y) \leq y$. Next we prove each one of the items.

1. By symmetry it is enough to prove that $(x' \Rightarrow x) * (y \Rightarrow y') \leq (x \Rightarrow y) \Rightarrow (x' \Rightarrow y')$; and this is a consequence of residuation.

2. By symmetry it is enough to prove that $(x \Rightarrow x') * (y \Rightarrow y') \leq (x * y) \Rightarrow (x' * y')$; and this is a consequence of residuation.

3. Since we are considering a chain, we can suppose, without loss of generality, that $\inf_{i\in I}\{y_i\} \leq \inf_{i\in I}\{x_i\}$. Thus, $\inf_{i\in I}\{x_i\} \Leftrightarrow \inf_{i\in I}\{y_i\} = \inf_{i\in I}\{x_i\} \Rightarrow \inf_{i\in I}\{y_i\}$. It is obvious that it is enough to prove that

$$\inf_{i\in I}\{x_i \Rightarrow y_i\} \leq \inf_{i\in I}\{x_i\} \Rightarrow \inf_{i\in I}\{y_i\},$$

   and this is an easy consequence of residuation because for every $i \in I$,

$$\inf_{i\in I}\{x_i \Rightarrow y_i\} * \inf_{i\in I}\{x_i\} \leq (x_i \Rightarrow y_i) * x_i \leq y_i.$$

4. Without loss of generality we can assume that $\sup_{i\in I}\{y_i\} \leq \sup_{i\in I}\{x_i\}$. Thus, $\sup_{i\in I}\{x_i\} \Leftrightarrow \sup_{i\in I}\{y_i\} = \sup_{i\in I}\{x_i\} \Rightarrow \sup_{i\in I}\{y_i\}$. It is obvious that it is enough to prove that

$$\inf_{i\in I}\{x_i \Rightarrow y_i\} \leq \sup_{i\in I}\{x_i\} \Rightarrow \sup_{i\in I}\{y_i\}.$$

   This is true because if $a = \inf_{i\in I}\{x_i \Rightarrow y_i\}$, then for every $i \in I$,

$$a * x_i \leq y_i;$$

   and hence,

$$a * \sup_{i\in I}\{x_i\} = \sup_{i\in I}\{a * x_i\} \leq \sup_{i\in I}\{y_i\}. \qquad \square$$

The proof we give for Theorem A.0.5 is based on a continuity argument, and resembles the one given in [Háj98c, Theorem 5.4.30]. The main difference is that while Hájek introduces a *distance* between models on the same domain, in this argument we consider a dual notion, which we call *similarity* and denote by S. In the case of Łukasiewicz, since the duality, there is no essential difference between considering a distance or a similarity, but this is not the case for Product Logic, where it is crucial to consider a similarity.

Definition A.0.2 (Similarity)**.** Let Γ be a predicate language with a finite number of predicate symbols $P_1, \ldots, P_n$, and let $\mathbf{M}, \mathbf{M}'$ be two models over $[0,1]_\Pi$ on the same domain M such that r_{P_i} and r'_{P_i} are the interpretations of the predicate symbols in $\mathbf{M}$ and $\mathbf{M}'$ respectively.

1. For each predicate symbol $P \in \Gamma$ with arity $ar(P)$, we define
$$S(r_P, r'_P) := \inf_{a \in M^{ar(P)}} \{r_P(a) \Leftrightarrow r'_P(a)\} =$$
$$= \inf_{a \in M^{ar(P)}} \left\{ \frac{\min\{r_P(a), r'_P(a)\}}{\max\{r_P(a), r'_P(a)\}} \right\}$$
2. Moreover, we define
$$S(\mathbf{M}, \mathbf{M}') := S(r_{P_1}, r'_{P_1}) * \ldots * S(r_{P_n}, r'_{P_n}).$$

Definition A.0.3. We define the complexity $\tau(\varphi)$ of a formula φ as follows:

1. $\tau(\varphi) = 0$, if φ is atomic or $\bot$,
2. $\tau(\varphi * \psi) = 1 + max\{\tau(\varphi), \tau(\psi)\}$, if $\star \in \{\to, \otimes\}$,
3. $\tau(Qx\, \varphi) = \tau(\varphi)$, if $Q \in \{\forall, \exists\}$.

This complexity captures the number of nested propositional connectives in the formula.

Lemma A.0.4. *Assume Γ is a predicate language with n predicate symbols. Let $\mathbf{M}$ and $\mathbf{M}'$ be two first order structures over $[0,1]_\Pi$ on the same domain M, and let φ be a first order formula. Then, for all $\varepsilon \in [0,1)$,*

$$\text{if } S(\mathbf{M}, \mathbf{M}') > \sqrt[n \cdot 2^{\tau(\varphi)}]{\varepsilon}, \text{ then,}$$
$$\text{for each evaluation } v,\ (\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon.$$

Proof. It is enough to prove that if $\mathbf{M}$ differs from $\mathbf{M}'$ only by the interpretation of one predicate symbol P, then

(C_φ) for all $\varepsilon \in [0,1)$, if $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$, then,
for each evaluation v, $(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon$.

We show that this condition (C_φ) holds by induction on the length of the formula φ.

- If φ is either atomic or $\bot$, then it is obvious.
- Let us suppose $\varphi = \psi \star \chi$ with $\star \in \{\to, \otimes\}$, and $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$. Then, $S(\mathbf{M}, \mathbf{M}') > \max\{\sqrt[2^{\tau(\psi)}]{\sqrt{\varepsilon}}, \sqrt[2^{\tau(\chi)}]{\sqrt{\varepsilon}}\}$. Using the inductive hypothesis for $\sqrt{\varepsilon}$, we get that
$$(\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon},$$
$$(\|\chi\|_{\mathbf{M},v} \Leftrightarrow \|\chi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon}.$$

  Hence, by the first two items in Lemma A.0.1 we get that

$$(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon} * \sqrt{\varepsilon} = \varepsilon.$$

- Let us suppose that $\varphi = Qx\,\psi$, with $Q \in \{\forall, \exists\}$, and $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$. Then, $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\psi)}]{\varepsilon}$. By the inductive hypothesis we get that $(\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}) \geq \varepsilon$ for each evaluation v. Hence,

$$\inf_v \{\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}\} \geq \varepsilon.$$

  By the last two items in Lemma A.0.1 it follows that

$$(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon.$$

Hence, the lemma is proved. □

We are now ready to prove the main result of the present Appendix.

Theorem A.0.5. *A first-order formula φ is a $[0,1]_\Pi$-tautology if and only if it is a tautology in any one-generated subalgebra of $[0,1]_\Pi$.*

Proof. The result is an obvious consequence of the previous lemma. Suppose that φ is not a $[0,1]_\Pi$-tautology, then there is a structure $\mathbf{M}$ and an evaluation v such that $\|\varphi\|_{\mathbf{M},v} < \varepsilon$ for some $\varepsilon < 1$. Take $s \in (0,1)$ such that $s^n > \sqrt[n \cdot 2^{\tau(\varphi)}]{\varepsilon}$, and denote by $\langle s \rangle$ the subalgebra of $[0,1]$ generated by s. For every predicate symbol P, let $r'_P(a)$ be $\min\{t \in \langle s \rangle : t \geq r_P(a)\}$. Now we define the structure $\mathbf{M}' = (M, r'_{P_1}, \ldots, r'_{P_n})$ over the algebra $\langle s \rangle$. An easy computation shows that $S(r_P, r'_P) \geq s$ for every predicate symbol P; hence, $S(\mathbf{M}, \mathbf{M}') \geq s^n > \sqrt[n \cdot 2^{\tau(\varphi)}]{\varepsilon}$. By Lemma A.0.4, $(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon$. This together with the fact that $\|\varphi\|_{\mathbf{M},v} < \varepsilon$ implies that $\|\varphi\|_{\mathbf{M}',v} \neq 1$. This finishes the proof. □

Corollary A.0.6. *A first-order formula φ is positively satisfiable w.r.t models over $[0,1]_\Pi$ iff it is a positively satisfiable w.r.t models in any one-generated subalgebra of $[0,1]_\Pi$ iff it is positively satisfiable w.r.t quasi-witnessed models over $[0,1]_\Pi$.*

But the theorem is unknown to be true for 1-satisfiability since if a formula φ is 1-satisfiable the similarity between models does not allow us to prove that there is a model over the one generated algebra (therefore a quasi-witnessed model) where φ is evaluated 1 as well. In general this means that our argument does not allow to prove that 1-satisfiability and 1-satisfiability$_Q$ coincide.

As a consequence, the following question arises:

Open problem Is positive satisfiability equivalent to 1-satisfiability in product logic?

Two are the known facts about this open problem:

1. The equivalence is true for propositional logic since if φ is positively satisfiable by the evaluation v, then take the evaluation e defined by $e(p) = \neg\neg v(p)$ and an easy computation shows that $e(\varphi) = 1$.

2. The conjecture is equivalent to the validity of the following restricted deduction-detachment theorem (restricted DDT) for standard semantics:

$$\varphi \models \bar{0} \text{ iff } \models \varphi \to \bar{0},$$

The proof of the point 2 is obvious. Suppose first that formulas 1-satisfiable coincide with positively satisfiable formulas. Since $\varphi \models \bar{0}$ means that φ is not a 1-satisfiable formula and thus it is not a positively satisfiable formula. Then in all first order models φ is evaluated as 0 and the result is obvious.

Suppose now that the restricted DDT is true. If φ is not 1-satisfiable, then $\varphi \models \bar{0}$ and by the supposition, $\models \varphi \to \bar{0}$ which implies that φ is evaluated as 0 in all first order models. Therefore φ is not positively satisfiable.

Remarks about expansions with truth-constants and/or an involutive negation

Expansions with truth constants Let $\Pi\forall^c$ the expansion of $\Pi\forall$ with rational truth constants, i.e., with adding one truth constant $\bar{r}$ for each $r \in (0,1) \cup Q$ and the canonical interpretation of them (each truth constants $\bar{r}$ is interpreted in $[0,1]_\Pi$ as its value r). In this setting, obviously, the one generated subalgebra has to contain at least the rationals and thus is very different of the one generated subalgebra of $\Pi\forall$. Nevertheless the following results hold:

1. In $\Pi\forall^c$ with general semantics, formulas $(\mathcal{C}\exists)$ and $(\Pi\mathcal{C}\forall)$ are tautologies (the proof is completely analogous to the one given in Lemma 27, of [CE11], see Lemma B.4.3 of Appendix B) and thus $\Pi\forall^c$ with general semantics is complete w.r.t. quasi-witnessed models. But for standard semantics we can not generalize results in this appendix to the expansions with truth constants and the problem if $\Pi\forall^c$ with standard semanticsis complete w.r.t. quasi-witnessed models remains open.

2. In $\Pi\forall^c$ it is clear that positive satisfiability is different from 1-satisfiability since for any $r \in (0,1) \cup Q$ $\bar{r}$ is always positive satisfiability and not 1-satisfiable. Therefore decidability for 1-satisfiability problem on FDLs over product logic with truth constants is not settled by the method used by us and it is still an open problem.

Expansions with an involutive negation Let $\Pi\forall_\sim$ the expansion of $\Pi\forall$ with adding a unary connective $\sim$ with the interpretaton of the new connective given by $\|\sim\varphi\|_{\mathbf{M},\mathbf{v}} = 1 - \|\varphi\|_{\mathbf{M},\mathbf{v}}$. In this setting the one-generated subalgebra (subalgebra generated by an element $a \in (0,1)$) contains not only the set $\{a^n \mid n \in N\}$ but also their negations $\{\sim a^n \mid n \in N\}$, their products and, so on.

Proposition A.0.7. *The support of the subalgebra of* $[0,1]_{\Pi,\sim}$ *generated by an element* $a \in (0,1)$ *is a dense set on the real unit interval.*

Proof. Let $\langle a \rangle$ be the subalgebra of $[0,1]_{\Pi,\sim}$ generated by $a \in (0,1)$ and suppose that $x, y \in \langle a \rangle$ with $x > y$. Obviously $\langle a \rangle$ has to contain the elements of the sequence $a > a^2 > ... > a^n > ...$ (with limit 0) and of the sequence $\sim a <\sim a^2 < ... <\sim a^n < ...$ (with limit 1). Thus $\langle a \rangle$ has to contain also the elements of the sequence $x\cdot \sim a < x\cdot \sim a^2 < ... < x\cdot \sim a^n < ...$ (with limit x). Thus there is an n such that $x > x\cdot \sim a^n > y$ and therefore the support of $\langle a \rangle$ is dense in the real unit interval. □

To finish this remark we will prove the following result

Theorem A.0.8. *In $\Pi\forall_\sim$, $[0,1]_*$-tautologies do not coincide with $[0,1]_*$-tautologies restricted to quasi-witnessed models.*

To prove the result we will use the following lemmas.

Lemma A.0.9. *In $\Pi\forall_\sim$ with standard semantics the formula*

$$((\exists x)P(x)) \leftrightarrow (\sim (\forall x) \sim P(x))$$

is a tautology.

Proof. The result is an easy consequence of the fact that any involutive negation is strictly decreasing from $[0,1]$ to itself and thus satisfying $\inf_{i\in I} a_i = \sup_{i\in I} \sim a_i$ where a_i is any sequence of elements of $[0,1]$. □

Lemma A.0.10. *If $a \in [0,1]$ and $f : I \to [0,1]$ is a function, then*

$$\sup_{i\in I}(f(i) \to a) = 1 \quad \textit{iff} \quad \sup_{i\in I}(\sim a \to\sim f(i)) = 1$$

Proof. Suppose $\sup_{i\in I}(f(i) \to a) = 1$, then there are two possibilities:

- There exists $i \in I$ such that $f(i) \leq a$, then, being $\sim$ involutive, $\sim a \leq\sim f(i)$ and thus the result is obvious. In fact if $a = 0$ this is the only possibility.
- For all $i \in I$, $f(i) < a$. Then $1 = \sup_{i\in I}(f(i) \to a) = \sup_{i\in I} \frac{a}{f(i)}$ and so $\inf_{i\in I} f(i) = a$ wich is equivalent to $\sup i \in I(\sim f(i)) =\sim a$. Therefore $\sup_{i\in I}(\sim a \to\sim f(i)) =\sim a \to \sup_{i\in I} \sim f(i) = 1$.

The other direction is proved analogously. □

Lemma A.0.11. *In $\Pi\forall_\sim$ with standard semantics, quasi-witnessed models coincide with witnessed models.*

Proof. By definition the formula

$$(\exists x)((\exists y)P(y) \to P(x))$$

is a $[0,1]_*$-tautology for quasi-witnessed models. By lemma A.0.9 this implies that the formula

$$(\exists x)((\sim (\forall y) \sim P(y)) \to P(x))$$

is also a $[0,1]_*$-tautology for quasi-witnessed models. And being $\|\sim (\forall y) \sim P(y)\|_{\mathbf{M},\mathbf{v}}$ a fixed element of $[0,1]$, applying the previous lemma, we have that

$$(\exists x)(\sim P(x) \to (\forall y) \sim P(y))$$

is again a $[0,1]_*$-tautology for quasi-witnessed models.. And being $\sim$ involutive, the formula is equivalent to the axiom $(\mathcal{C}\forall)$ of witnessed models and thus the proposition is proved. □

From the results of the previous lemmata, we can prove the theorem. We have proved that $\Pi(\mathcal{C}\forall)$ is a $[0,1]_*$-tautology for quasi-witnessed models. But this formula is not a $[0,1]_*$-tautology for all first order models. Take the standard model such that $M = \mathbb{N}$ and $P^{\mathcal{I}}(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\|(\exists x)(P(x) \to (\forall y)P(y))\|_{\mathbf{M},\mathbf{v}} = \sup_{n\in\mathbb{N}}\{\tfrac{1}{n} \to \inf_{n\in\mathbb{N}} \tfrac{1}{n}\} = \sup_{n\in\mathbb{N}}\{\tfrac{1}{n} \to 0\} = 0.$$

Appendix B

Strict core fuzzy logics and quasi-witnessed models

This appendix contains a copy of the contents published in the paper *Strict core fuzzy logics and quasi-witnessed models*- It is included for the purpose of showing the work of the author in quasi-witnessed models under general semantics.

B.1 Introduction

Fuzzy Logics (both propositional and first-order) as many-valued residuated logics were defined by Petr Hájek in his celebrated book [Háj98c]. He defined, on the one hand, propositional fuzzy logics as extensions of the Basic Fuzzy Logic BL and, on the other hand, their algebraic counterpart, the variety of BL-algebras. Moreover he proved that BL and all its axiomatic extensions are complete with respect to evaluations over the BL-chains belonging to the corresponding variety. The fact that for each axiomatic extension of BL there is a corresponding subvariety of BL-algebras is a consequence of the fact that BL and its extensions are logics algebraizable in the sense of Blok and Pigozzi (see [GNE05]). Special interest have the results in [Háj98b] and in [CEGT00], where it is proved that BL is the logic of continuous t-norms and their residua. Well known axiomatic extensions of BL are Łukasiewicz, Gödel and Product Logics (denoted as Ł, G and Π respectively). In his book, Hájek also defined the predicate logic corresponding to BL and its axiomatic extensions (denoted adding $\forall$ after the name of the propositional logic). Moreover he defined their semantics as first-order safe structures taking values on BL-chains of the corresponding variety and proved their completeness with respect to these models. Taking into account that a t-norm has residuum if and only if it is left-continuous, Esteva and Godo in [EG01] defined both propositional and first-order MTL (for Monoidal t-norm based Logic) whose propositional logic is proved to be the logic of left-continuous t-norms in [JM02]. In [EG01] it is also defined their algebraic counterpart, the variety of MTL-algebras. The first-order versions of MTL and its axiomatic extensions are proved to be complete with respect to first-order structures evaluated over MTL-chains belonging to the corresponding variety. In recent times first-order Fuzzy Logic has been deeply studied. Recall that generalizing the classical case, the value of a universally (existentially) quantified formula is defined as the infimum (supremum) of the values of the results of replacing the quantified variable by the interpretation of a term of the language in a first-order model. Notice that in the context of Classical Logic, as well as every finitely valued logic, infima and suprema turn out to be minima and maxima, respectively. However, when we move to infinitely valued logics, this is not the case, the infimum or supremum of a set of values C may be an element $c \notin C$, i.e., a quantified formula may have no *witness*. Following these ideas, Hájek introduced in [Háj07a], [Háj07b] the notion of *witnessed model*, i.e., a model in which each quantified formula has a witness and proved that this is an important property because it implies a limited form of finite model property for certain fragments of predicate fuzzy logic (see [Háj05]). Moreover, Cintula and Hájek introduce in [HC06] the so-called witnessed axioms that, added to any first-order core fuzzy logic, give a logic complete with respect to witnessed models. Subsequently they prove that these axioms are derivable in Łukasiewicz first-order Logic, showing that Ł∀ is complete with respect to witnessed models (we will say that Ł∀ has the witnessed model property), but also that neither Gödel, nor Product first-order Logic share this property because witnessed axioms are not theorems of these logics. In fact no other first-order logic of a continuous t-norm enjoys this property, since it is related to continuity of the truth functions, a property that only Łukasiewicz logic has. Nevertheless, in [LM07] it is proved that Product Predicate Logic enjoys a weaker property, what we call *quasi-witnessed model property*. Quasi-witnessed models[1] are models in which, whenever the value of a universally quantified formula is strictly greater than 0, then it has a witness, while existentially quantified formulas are always witnessed.

In this paper we introduce both the so-called *strict core* fuzzy logics and *quasi-witnessed* axioms (generalizations of the witnessed axioms of Hájek-Cintula to cope with quasi-witnessed models) and prove, following the style of [HC06] that, if we add quasi-witnessed axioms to any first-order strict core fuzzy logic, the resulting logic enjoys the quasi-witnessed model property. From this result, the one in [LM07] about the completeness of Product first-order Logic with respect to quasi-witnessed models, will follow as a corollary. Moreover, we prove that quasi-witnessed axioms are theorems in no logic of a continuous t-norm but Product and Łukasiewicz predicate logics. Finally we study the expansion of first-order strict core fuzzy logics by Δ operator. We give the so-called Δ-*quasi-witnessed axioms* and prove that adding these axioms to any strict Δ-core fuzzy logic, we obtain a first-order fuzzy logic which is complete with respect to quasi-witnessed models.

[1]These models are called "closed models" in [LM07] but we decided, after some discussions with colleagues, to use the more informative name of "quasi-witnessed models". We take into account the fact that the name "closed" is used in mathematics and logic in different contexts with different meanings and could induce some confusion.

B.2 Preliminaries

B.2.1 Propositional logic

The logic MTL has been defined in [EG01] and has, as primitive binary connectives, a strong conjunction $\odot$, a weak conjunction $\wedge$ and an implication $\rightarrow$ and, as primitive 0-ary connective, the constant symbol $\bot$. This logic has been axiomatized with the following set of axioms:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,

(A2) $(\varphi \odot \psi) \rightarrow \varphi$,

(A3) $(\varphi \odot \psi) \rightarrow (\psi \odot \varphi)$,

(A4) $\varphi \odot (\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \psi)$,

(A5) $(\varphi \wedge \psi) \rightarrow \varphi$,

(A6) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$,

(A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \odot \psi) \rightarrow \chi)$,

(A7b) $((\varphi \odot \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,

(A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$,

(A9) $\bot \rightarrow \varphi$.

And its unique rule of inference is Modus Ponens (MP).

From the primitive connectives it is possible to define more, in particular:

$$\begin{array}{rcl} \varphi \vee \psi & := & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ \varphi \equiv \psi & := & (\varphi \rightarrow \psi) \odot (\psi \rightarrow \varphi) \\ \neg\varphi & := & \varphi \rightarrow \bot \\ \top & := & \bot \rightarrow \bot \end{array}$$

The logic SMTL[2] is defined in the literature as the axiomatic extension of MTL by the axiom:

(S) $\quad \varphi \wedge \neg\varphi \rightarrow \bot \quad$ (strictness)

In this paper we are going to deal with other important axiomatic extensions of MTL. The logic BL is the axiomatic extension of MTL by the following axiom,

(D) $\quad \varphi \wedge \psi \rightarrow \varphi \odot (\varphi \rightarrow \psi) \quad$ (divisibility)

The logic SBL is the axiomatic extension of BL by axiom (S), or, equivalently, it is the axiomatic extension of SMTL by axiom (D).

Product logic has been defined in [HGE96] and it can be seen as the axiomatic extension of SBL by the following axiom,

(Π) $\quad \neg\neg\chi \rightarrow (((\varphi \odot \chi) \rightarrow (\psi \odot \chi)) \rightarrow (\varphi \rightarrow \psi)) \quad$ (simplification)

[2]SMTL means *strict MTL* in the sense that $(\varphi \wedge \neg\varphi) \leftrightarrow 0$ is a theorem. Algebraically this property is called "pseudo-complementation" and denoted as (PC) in some more algebraic works like [GJKO07].

Hence Product Logic is the axiomatic extension of SMTL by axioms (D) and (Π).

Gödel logic is the axiomatic extension of BL (or either SBL or SMTL) by the following axiom:

(Id) $\varphi \to (\varphi \odot \varphi)$ (idempotence)

Finally, Łukasiewicz logic is the axiomatic extension of BL by the following axiom:

(Inv) $\neg\neg\varphi \to \varphi$ (involutive negation)

Definition B.2.1. 1. An *MTL-algebra* $\mathbf{A}= \langle A, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is a bounded commutative integral residuated lattice which satisfies the equation:

(PL) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$ (pre-linearity)

2. An *SMTL-algebra* $\mathbf{A}= \langle A, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is a MTL-algebra which satisfies the equation:

   (S) $x \cap (x \Rightarrow 0) = 0$ (strictness)

3. A *BL-algebra* $\mathbf{A}= \langle A, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is an MTL-algebra which satisfies the equation:

   (D) $x \cap y = x * (x \Rightarrow y)$ (divisibility)

4. A Π*-algebra* $\mathbf{A}= \langle A, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is an SMTL-algebra which satisfies the equations (D) and:

   (Π) $((z \Rightarrow 0) \Rightarrow 0) \Rightarrow (((x * z) \Rightarrow (y * z)) \Rightarrow (x \Rightarrow y)) = 1$ (simplification)

5. A *Gödel-algebra* $\mathbf{A}= \langle A, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is a BL-algebra which satisfies the equation:

   (Id) $x = x * x$ (idempotence)

6. An *MV-algebra* $\mathbf{A}= \langle A, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is a BL-algebra which satisfies the equation:

   (Inv) $x = (x \Rightarrow 0) \Rightarrow 0$ (involutive negation)

Moreover, if any of them is linearly ordered, we say that it is an *MTL-chain* (respectively *SMTL-chain*, Π*-chain* and so on).

All the logics defined in these preliminaries are algebraizable in the sense of Blok and Pigozzi (see [GNE05]) and its algebraic semantics is the variety of the corresponding MTL-algebras. Moreover all of these logics are chain-complete (what is called "semilinear" in [CN10]) in the sense that they are strong complete for evaluations over the chains of the corresponding variety.

A natural semantics for the MTL logic and their axiomatic extensions are the evaluations over the real unit interval, i.e. over the MTL-chains whose lattice reduct is $[0, 1]$ with the usual order. These chains, called standard chains are related to a special kind of operation called "t-norms".

Definition B.2.2. A *t-norm* is a binary operation $*$ on the real unit interval $[0, 1]$ that is associative, commutative, non-decreasing in both arguments and having 1 as neutral (unit) element.

Left continuity of a t-norm is characterized by the existence of an unique binary operation $\Rightarrow$ satisfying for all $a, b, c \in [0, 1]$ the following condition (called *residuation*):

$$a * b \leq c \text{ if and only if } a \leq b \Rightarrow c$$

The operator $\Rightarrow$ is called the *residuum* of the t-norm $*$ and it is defined as

$$x \Rightarrow y = \max\{z \in [0, 1] \mid x * z \leq y\}$$

Using this residuum, the following result characterize standard chains.

Proposition B.2.3. *A structure $\langle [0, 1], \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is a standard MTL-chain if and only if $*$ is a left-continuous t-norm and $\Rightarrow$ is its residuum. This structure will be denoted from now on as $[0, 1]_*$. Moreover a standard chain satisfies divisibility (Hence it is a BL-chain) if and only if the t-norm is continuous.*

In [JM02] it is proved that MTL are *strong standard complete* (strong complete for evaluations over the standard chains), i.e. for any set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over a standard chain,

$$\Gamma \vdash_{MTL} \varphi \text{ iff } e(\varphi) = 1 \text{ for any evaluation } e \text{ such that } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma.$$

This result is not automatically translatable to axiomatic extensions of MTL. It is easily extended to SMTL and the standard SMTL-chains but not to BL and the standard BL-chains (hence neither to its axiomatic extensions). If $\mathcal{L}$ is either BL or SBL or Łukasiewicz or Product or Gödel logic only the finite strong standard completeness results are valid, i.e. for any *finite* set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over a standard $\mathcal{L}$-chain,

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ iff } e(\varphi) = 1 \text{ for any evaluation } e \text{ such that } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma,$$

An interesting result for Łukasiewicz Product and Gödel logics is that the corresponding standard-chains are all isomorphic[3]. The most used representative of standard chains of these three logics (unique up to isomorphisms), are the ones defined by the so-called Łukasiewicz, product and minimum t-norms and their residua (collected in Table 1).

From the previous results seems natural the definition of the logic of a (continuous) t-norm.

Definition B.2.4. We say that a logic (called $\mathcal{L}(*)$) is the logic of a continuous t-norm $*$ if it is an axiomatic extension of BL which is finite strong standard complete with respect to evaluations over the standard chain $[0, 1]_*$, i.e. for any *finite* set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over $[0, 1]_*$,

$$\Gamma \vdash_{\mathcal{L}(*)} \varphi \text{ iff } e(\varphi) = 1 \text{ for any evaluation } e \text{ such that } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma.$$

[3] In fact for Gödel logic there is only one standard chain while for Łukasiewicz and Product there are infinite different but isomorphic ones.

$*$	Minimum (Gödel)	Product (of real numbers)	Łukasiewicz
$x * y$	$\min(x, y)$	$x \cdot y$	$\max(0, x + y - 1)$
$x \to_* y$	$\begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise} \end{cases}$	$\min(1, 1 - x + y)$
n_*	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$	$1 - x$

Table B.1: The three main continuous t-norms.

All the logics considered so far enjoy two important properties we need to define the class of logics we are interested in.

Definition B.2.5. 1. We say that a logic $\mathcal{L}$ enjoys the *Local Deduction Theorem* (*LDT*, for short) if for each theory T and formulas φ, ψ, it holds that $T, \varphi \vdash \psi$ iff there exists a natural number n such that $T \vdash \varphi^n \to \psi$, where $\varphi^n = \varphi \odot \ldots \odot \varphi$, n times.

2. We say that a logic $\mathcal{L}$ enjoys *Invariance under Substitution* (*Sub*, for short) if, for every formulas φ, ψ, χ it holds that $\varphi \equiv \psi \vdash \chi(\varphi) \equiv \chi(\psi)$.

Next we recall the definition of *core fuzzy logic* given in [HC06] (a family of logics that encompasses all logics considered so far) and we introduce the *strict core fuzzy logic* we will deal with in this paper.

Definition B.2.6. 1. We say that a logic $\mathcal{L}$ is a *core fuzzy logic* if it is finitary, enjoys *LDT*, *Sub* and expands MTL.

2. We say that a logic $\mathcal{L}$ is a *strict core fuzzy logic* if it is finitary, enjoys *LDT*, *Sub* and expands SMTL.

Throughout this preliminary section, we will denote by $\mathcal{L}$ any core fuzzy logic.

B.2.2 Predicate logic

In order to define what a predicate logic is, we have, previously, to define what a *predicate language* is.

Definition B.2.7. A *predicate language* Γ is compound by a set of relation symbols $P_1, \ldots, P_n, \ldots$, each one with arity ≥ 1, a set of function symbols $f_1, \ldots, f_n, \ldots$, each one with its arity, and a set of constant symbols $c_1, \ldots, c_n, \ldots$, that are 0-ary function symbols.

Terms and *formulas* of a predicate language are defined as usual in the literature.

Following [Háj98c], given a propositional residuated logic $\mathcal{L}$, we define the first-order logic associated with $\mathcal{L}$ (denoted by $\mathcal{L}\forall$), as follows:

Definition B.2.8. $\mathcal{L}\forall$ is the first-order logic such that:

1. its language is compound by a predicate language Γ and a set of logical symbols obtained by adding, to the set of logical symbols of $\mathcal{L}$, the two "classical" quantifiers $\forall$ and $\exists$ and,

2. it is axiomatized by means of the following set of axiom schemata:

   (P) the axioms resulting from the axioms of $\mathcal{L}$ after the substitution of propositional variables by formulas of the new predicate language.

   ($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ.

   ($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ.

   ($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi(x))$, where x is not free in χ.

   ($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi(x) \to \chi)$, where x is not free in χ.

   ($\forall$3) $(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi(x))$, where x is not free in χ.

3. its rules of inference are Modus Ponens (MP) and generalization (G): From φ infer $(\forall x)\varphi(x)$.

The following definitions are required to prove the main results given in Section B.3. They are typical within the framework of Classical first-order Logic. Their presentation in our context follows the generalization, due to [HC06], necessary to adapt them to a many-valued framework.

Definition B.2.9. We say that a theory T' in a predicate language Γ' is an *expansion* of a theory T in a predicate language Γ, if $\Gamma \subseteq \Gamma'$ and, each formula provable in T is provable in T'. We say that T' is a *conservative expansion* of T if T' is an expansion of T and each formula in the language of T, provable in T', is provable in T.

Definition B.2.10. A theory T is *linear* if, for each pair of sentences φ, ψ, we have $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$.

Definition B.2.11. Let Γ and Γ' be predicate languages such that $\Gamma \subseteq \Gamma'$ and T a Γ'-theory. We say that T is $\forall$-Γ-*Henkin* if, for each Γ-sentence $\varphi = (\forall x)\psi(x)$ such that $T \nvdash \varphi$, there is a constant c in Γ' such that $T \nvdash \psi(c)$.
We say that T is $\exists$-Γ-*Henkin* if, for each Γ-sentence $\varphi = (\exists x)\psi(x)$ such that $T \vdash \varphi$, there is a constant c in Γ' such that $T \vdash \psi(c)$.
A theory is called Γ-*Henkin* if it is both $\forall$-Γ-Henkin and $\exists$-Γ-Henkin.
If $\Gamma = \Gamma'$, we say that T is $\forall$-Henkin ($\exists$-Henkin, Henkin).

From a semantic point of view first-order models are compound of a set of elements, an algebra of truth values and an assignation function.

Definition B.2.12. A *first-order structure* for a given predicate language Γ is a pair $(\mathbf{A}, \mathbf{M})$, where $\mathbf{A}$ is an $\mathcal{L}$-chain and $\mathbf{M}$=$(M, (P_{\mathbf{M}})_{P\in\Gamma}, (f_{\mathbf{M}})_{f\in\Gamma}, (c_{\mathbf{M}})_{c\in\Gamma})$, where:

1. The set M, called *domain*, is a non-empty set,

2. for each predicate symbol $P \in \Gamma$ of arity n, $P_{\mathbf{M}}$ is an n-ary $\mathbf{A}$-fuzzy relation on M,

3. for each function symbol $f \in \Gamma$ of arity n, $f_{\mathbf{M}}$ is an n-ary (crisp) function on M and

4. for each constant symbol $c \in \Gamma$, $c_{\mathbf{M}}$ is an element of M.

The truth value $\|\varphi\|_v^{\mathbf{A},\mathbf{M}}$ of a predicate formula φ in a given model v is defined as follows.

Definition B.2.13. Let Γ be a predicate language, $\mathbf{A}$ an $\mathcal{L}$-chain and $(\mathbf{A},\mathbf{M})$ a first-order structure, then a first-order assignation v is a homomorphism $v : Var \to M$. As usual each assignation, defined on the set of individual variables, extends univocally to a first-order assignation (that we will denote by v as well) satisfying, for every terms $t_1, \ldots, t_n$ and each n-ary function $f \in \Gamma$, that $v(f(t_1, \ldots, t_n)) = f_{\mathbf{M}}(v(t_1), \ldots, v(t_n))$.
Moreover, each assignation v, defined on the set of individual variables yields a first-order model $\|\cdot\|_v^{(\mathbf{A},\mathbf{M})} : Fm_{\mathcal{L}\forall} \to \mathbf{A}$ such that:

1. for each n-tuple of terms $t_1, \ldots, t_n$ and each n-ary relation $P \in \Gamma$, it holds that $\|P(t_1, \ldots, t_n)\|_v^{(\mathbf{A},\mathbf{M})} = P_{\mathbf{M}}(v(t_1), \ldots, v(t_n)) \in A$,

2. if φ, ψ are formulas, $\star_{\mathcal{L}}$ a binary logical connective and $\star_{\mathbf{A}}$ its truth function, then $\|\varphi \star_{\mathcal{L}} \psi\|_v^{(\mathbf{A},\mathbf{M})} = \|\varphi\|_v^{(\mathbf{A},\mathbf{M})} \star_{\mathbf{A}} \|\psi\|_v^{(\mathbf{A},\mathbf{M})}$.

3. if $\varphi(x_1, \ldots, x_n)$ is a formula with n free variables and v is a first-order assignation such that $v(x_i) = a_i$ and $a_i \in M$, for $1 < i \leq n$, then we have that $\|(\forall x_1)\varphi(x_1, x_2, \ldots, x_n)\|_v^{(\mathbf{A},\mathbf{M})} = \inf_{a \in M}\{\|\varphi(a, a_2, \ldots, a_n)\|^{(\mathbf{A},\mathbf{M})}\}$,

4. if $\varphi(x_1, \ldots, x_n)$ is a formula with n free variables and v is a first-order assignation such that $v(x_i) = a_i$ and $a_i \in M$, for $1 < i \leq n$, then we have that $\|(\exists x_1)\varphi(x_1, x_2, \ldots, x_n)\|_v^{(\mathbf{A},\mathbf{M})} = \sup_{a \in M}\{\|\varphi(a, a_2, \ldots, a_n)\|^{(\mathbf{A},\mathbf{M})}\}$.

Clearly, depending on the model, the infimum and supremum of a set of values of formulas do not necessarily exist and, in this case we will say that a given quantified formula has an undefined truth value. Following [Háj98c], we will say that if, for a given model v, both infima and suprema of sets of values are defined for every formula, then v is a *safe* model. Moreover, if, for a given first-order structure $(\mathbf{A},\mathbf{M})$, each assignation v defined in it is safe, we will say that $(\mathbf{A},\mathbf{M})$ is a safe structure.

From now on and for simplicity, we will omit the name "safe" before the first-order structures, i.e., when we speak about a first-order structure $(\mathbf{A},\mathbf{M})$, we implicitly mean a *safe* first-order structure $(\mathbf{A},\mathbf{M})$.

The concepts of *satisfiability* and *validity* are defined in the usual way.

In [HC06], we find the following useful definitions and result, which we report without proof. In what follows, we will denote by $\mathbf{A}$ any $\mathcal{L}$-chain.

Definition B.2.14. Let $(\mathbf{A_1}, \mathbf{M_1})$ and $(\mathbf{A_2}, \mathbf{M_2})$ be structures in the languages Γ_1 and Γ_2 respectively and let $\Gamma_1 \subseteq \Gamma_2$. We say that a pair (f, g) is an *elementary embedding* if:

1. the mapping f is an injection of M_1 into M_2,
2. the mapping g is an embedding of $\mathbf{A_1}$ into $\mathbf{A_2}$,
3. for each Γ_1-formula $\varphi(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in M_1$, it holds that $g(\|\varphi(a_1, \ldots, a_n)\|^{(\mathbf{A_1},\mathbf{M_1})}) = \|\varphi(f(a_1), \ldots, f(a_n))\|^{(\mathbf{A_2},\mathbf{M_2})}$.

Definition B.2.15. Let T be a theory. We define $[\varphi]_T = \{\psi \mid T \vdash \varphi \equiv \psi\}$ and $L_T = \{[\varphi]_T \mid \varphi \text{ a formula }\}$. The *Lindenbaum algebra* of the theory T ($\mathbf{Lind}_T$, in symbols) has domain L_T and operations $c_{\mathbf{Lind}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [c(\varphi_1, \ldots, \varphi_n)]_T$, for every n-ary propositional connective c.

Definition B.2.16. Let T be a linear Henkin theory, then the *canonical model* of T is the structure $(\mathbf{Lind}_T, \mathbf{CM}(T))$, where $\mathbf{Lind}_T$ is the Lindenbaum algebra of theory T, the domain of $\mathbf{CM}(T)$ consists of object constants $c_{\mathbf{CM}(T)} = c$ and terms built without variables. Moreover for every predicate n-ary symbol $P \in \Gamma$, $P_{\mathbf{CM}(T)}(t_1, \ldots, t_n) = [P(t_1, \ldots, t_n)]_T$.

From here on, for simplicity, we will write $\mathbf{CM}(T)$ to denote $(\mathbf{Lind}_T, \mathbf{CM}(T))$.

Definition B.2.17. For each structure $(\mathbf{A}, \mathbf{M})$, let $\mathbf{Alg}((\mathbf{A}, \mathbf{M}))$ be the subalgebra of $\mathbf{A}$ whose domain is the set $\{\|\varphi\|_v^{\mathbf{A},\mathbf{M}} \mid \varphi, v\}$ of truth degrees of formulas under all $\mathbf{M}$-assignation v of variables. Call $(\mathbf{A}, \mathbf{M})$ *exhaustive* if $\mathbf{A} = \mathbf{Alg}((\mathbf{A}, \mathbf{M}))$.

The next lemma is a direct consequence of Lemma 4 in [HC06] and we will not prove it here.

Lemma B.2.18. *Let T_1, T_2 be $\mathcal{L}\forall$-theories. If T_2 is a conservative expansion of T_1, then, for each exhaustive model $(\mathbf{A}, \mathbf{M})$ of T_1, there exists a linear Henkin $\mathcal{L}\forall$-theory T extending T_2 such that $(\mathbf{A}, \mathbf{M})$ can be elementarily embedded into $\mathbf{CM}(T)$.*

B.2.3 The witnessed model property

Witnessed models have been firstly defined in [Háj05] in the following way:

Definition B.2.19. For any structure $(\mathbf{A}, \mathbf{M})$, a formula $(\forall y)\varphi(y, x_1, \ldots, x_n)$ is $\mathbf{A}$-witnessed in $\mathbf{M}$ if, for each assignation $c_1, \ldots, c_n \in M$, to $x_1, \ldots, x_n$, there is $c \in M$ such that $\|(\forall y)\varphi(y, c_1, \ldots, c_n)\|^{\mathbf{A},\mathbf{M}} = \|\varphi(c, c_1, \ldots, c_n)\|^{\mathbf{A},\mathbf{M}}$. Similarly for $(\exists y)\varphi(y, x_1, \ldots, x_n)$. $\mathbf{M}$ is $\mathbf{A}$-witnessed if all quantified formulas are $\mathbf{A}$-witnessed in $\mathbf{M}$.

As said above, within the framework of classical predicate logic, where the first-order structures are evaluated on a two element chain, there is no need of making a difference between witnessed and non witnessed models, because every model is indeed witnessed, and the same holds for every finite-valued logic. The need of speaking about witnessed models arises when we move to infinite-valued logics, since we can meet sets of truth values whose infima (resp. suprema) is not an element of the set. Later on, in [HC06], Hájek and Cintula consider the following couple of axioms (called witnessed axioms) already given by Baaz in [Baa96]:

(C∃) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,

(C∀) $((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

They prove that each first-order core fuzzy logic $\mathcal{L}\forall$, extended with this couple of axioms (denoted $\mathcal{L}\forall^w$), is complete with respect to the witnessed models evaluated over $\mathcal{L}$-chains. Moreover, in [Háj07a] it is proved that Łukasiewicz predicate logic is the only logic of a continuous t-norm equivalent with its witnessed axiomatic extension, i.e., (C∃) and (C∀) are theorems of Łukasiewicz predicate Logic. As a consequence of this fact Łukasiewicz is the only logic of a continuous t-norm which is complete with respect to witnessed models, i.e. it satisfies the *witnessed model property.*

B.3 Completeness with respect to quasi-witnessed models

In this section we will give the definitions of quasi-witnessed axioms and quasi-witnessed models, which are a generalization of witnessed axioms and models. We stress that in this paper the starting point are strict core fuzzy logics, because the result is related with the behavior of Gödel negation. Subsequently we will state and prove the main result of this paper, i.e., that if we add quasi-witnessed axioms to any predicate strict core fuzzy logic, we obtain a logic that is complete with respect to quasi-witnessed models. In what follows $\mathcal{L}$ will denote a strict core fuzzy logic.

Definition B.3.1. Let Γ be a predicate language and $(\mathbf{A}, \mathbf{M})$ a first-order structure, then we say that a Γ-formula $\varphi(x, y_1, \ldots, y_n)$ is $\mathbf{A}$-*quasi-witnessed* in $\mathbf{M}$ if:

1. For each tuple $c_1, \ldots, c_n$ of elements in M there exists an element $a \in M$ such that $\|(\exists x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = \|\varphi(a, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})}$.

2. For each tuple $c_1, \ldots, c_n$ of elements in M either $\|(\forall x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = 0$, or there exists an element $b \in M$ such that $\|(\forall x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = \|\varphi(b, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})}$.

We say that a first-order structure $(\mathbf{A}, \mathbf{M})$ is quasi-witnessed if for each formula and for every assignation v of the variables on $\mathbf{M}$ the formula is quasi-witnessed.

Definition B.3.2. Let $\mathcal{L}\forall$ be any strict core first-order logic, we denote by $\mathcal{L}\forall^{qw}$ the axiomatic extension of $\mathcal{L}\forall$ by the following axiom schemata called, from now on, "quasi-witnessed axioms":

(C∃) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,

(ΠC∀) $\neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

These quasi-witnessed axioms are a modification of the witnessed axioms given above. The first one, $(C\exists)$, is a witnessed axiom and the second one says that the witnessed axiom $(C\forall)(\exists y)(\varphi(y) \to (\forall x)\varphi(x))$ is valid in a structure $(\mathbf{A}, \mathbf{M})$ only when the truth value of $(\forall x)\varphi(x)$ is different from 0, i.e., when $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 1$.

Next lemma proves the soundness of quasi-witnessed axioms with respect to the above defined quasi-witnessed models.

Lemma B.3.3. *If an $\mathcal{L}\forall$-structure $(\mathbf{A}, \mathbf{M})$ is quasi-witnessed, then it satisfies $(C\exists)$ and $(\Pi C\forall)$.*

Proof. Let $(\mathbf{A},\mathbf{M})$ be a quasi-witnessed $\mathcal{L}\forall$-structure and $\varphi(x)$ a Γ formula with one free variable, then:

1. Since, by the first condition of Definition B.3.1, there exists an element $a \in M$ such that $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} = \|(\exists x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, then $(\mathbf{A},\mathbf{M}) \models (\exists x)\varphi(x) \to \varphi(a)$. So, by axiom (∃1) and (MP), $(\mathbf{A},\mathbf{M}) \models (\exists y)((\exists x)\varphi(x) \to \varphi(y))$.

2. By the second condition of Definition B.3.1, there exists $b \in M$ such that either $\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})} = \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, or $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$. If $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$, then, $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$ and, trivially we have $(\mathbf{A},\mathbf{M}) \models \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$. If, on the other hand, $\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})} = \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, then $(\mathbf{A},\mathbf{M}) \models \varphi(b) \to (\forall x)\varphi(x)$, and, by axiom (∃1) and (MP), $(\mathbf{A},\mathbf{M}) \models (\exists y)(\varphi(y) \to (\forall x)\varphi(x))$. So, $(\mathbf{A},\mathbf{M}) \models \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

□

As for witnessed models, the converse of the last lemma does not hold as we will see in Example B.4.4.

However, as in [HC06], it is possible to prove the next result.

Lemma B.3.4. *Let Γ be a predicate language, and $(\mathbf{A}, \mathbf{M})$ an exhaustive model of a Γ-theory T. Then $(\mathbf{A}, \mathbf{M})$ is an $\mathcal{L}\forall^{qw}$-model of T iff it can be elementarily embedded into a quasi-witnessed model of T.*

Proof. **(⇒)** Let (**A**,**M**) be an exhaustive $\mathcal{L}\forall^{\mathrm{qw}}$-model of T. By Lemma B.2.18, there is a linear Henkin theory T' extending T, such that (**A**,**M**) can be elementarily embedded into $\mathbf{CM}(T')$. Hence $\mathbf{CM}(T')$ is an $\mathcal{L}\forall^{\mathrm{qw}}$-model of T and we have to show that $\mathbf{CM}(T')$ is quasi-witnessed.

Due to the construction of the canonical model, each element of the domain of $\mathbf{CM}(T')$ is a constant. Let $\varphi(x)$ be a formula with one free variable and suppose that $\|(\forall x)\varphi(x)\|^{\mathbf{CM}(T')} > 0$, then we have that $\|\neg\neg(\forall x)\varphi(x)\|^{\mathbf{CM}(T')} = 1$. Hence $T' \vdash \neg\neg(\forall x)\varphi(x)$. By axiom $(\Pi C\forall)$, we have that $T' \vdash \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$, then, by (MP), $T' \vdash (\exists y)(\varphi(y) \to (\forall x)\varphi(x))$. Since T' is $\exists$-Henkin, then there exists some c such that $T' \vdash \varphi(c) \to (\forall x)\varphi(x)$. So, by axiom $(\forall 1)$, we obtain that $\|\varphi(c)\|^{\mathbf{CM}(T')} = \|(\forall x)\varphi(x)\|^{\mathbf{CM}(T')}$. The proof of the other condition is similar to Hájek's and Cintula's proof of Lemma 5 in [HC06] and we will not repeat it here.

(⇐) Suppose now that $(\mathbf{A},\mathbf{M})$ can be elementarily embedded into a quasi-witnessed model of T, hence, $(\mathbf{A},\mathbf{M})$ is an $\mathcal{L}\forall$-model of T. By Lemma B.3.3, we have that $(\mathbf{A},\mathbf{M})$ is an $\mathcal{L}\forall$-model of $T \cup \{(C\exists), (\Pi C\forall)\}$, which is equivalent to say that $(\mathbf{A},\mathbf{M})$ is an $\mathcal{L}\forall^{\mathrm{qw}}$ model of T.

□

Theorem B.3.5. *Let T be a theory and φ a formula in a given predicate language, then $T \vdash_{\mathcal{L}\forall^{qw}} \varphi$ iff $(\mathbf{A},\mathbf{M}) \models \varphi$ for every quasi-witnessed model $(\mathbf{A},\mathbf{M})$ of the theory T.*

Proof. The completeness of $\mathcal{L}\forall$ with respect to all (not only quasi-witnessed) $(\mathbf{A},\mathbf{M})$-models is ensured by Theorem 5 of [HC06], so we will restrict ourselves to the *quasi-witnessed* part.

(⇒) As a consequence of Theorem 5 of [HC06], we only have to check whether a quasi-witnessed model satisfies axioms $(C\exists)$ and $(\Pi C\forall)$, but this result has been already shown in Lemma B.3.3.

(⇐) Suppose that $T \nvdash_{\mathcal{L}\forall^{\mathrm{qw}}} \varphi$, then there exists an $\mathcal{L}\forall^{\mathrm{qw}}$-model $(\mathbf{A},\mathbf{M})$ of T, such that $(\mathbf{A},\mathbf{M}) \not\models \varphi$. Hence, by Lemma B.3.4, there exists a quasi-witnessed model $(\mathbf{A}',\mathbf{M}')$ of T such that $(\mathbf{A}',\mathbf{M}') \not\models \varphi$.

□

B.4 The case of predicate Product Logic

In this section we will show that the axioms $(C\exists)$ and $(\Pi C\forall)$ are provable in $\Pi\forall$, i.e., that the logics $\Pi\forall$ and $\Pi\forall^{qw}$ are equivalent. In order to do that, let us recall that $\Pi\forall$ is complete with respect to all models over a product chain and any product chain is isomorphic to the negative cone of a linearly ordered abelian group with an added bottom (See Theorem 2.5 in [CT00]).

Definition B.4.1. Let $\mathbf{G} = \langle \mathbf{G}, +, -, \mathbf{0} \rangle$ be a totally ordered abelian group, then we denote by G^- the negative part of G, i.e., $G^- = \{x \in G \mid x \leq 0\}$. Moreover, we denote by $\mathfrak{P}(\mathbf{G})$ the structure $\langle G^- \cup \{\bot\}, \otimes, \Rightarrow, \bot \rangle$, where $\bot$ is an element which does not belong to G, and $\otimes$, $\Rightarrow$ are two binary operations defined as follows:

$$x \otimes y = \begin{cases} x + y & \text{if } x, y \in G^-, \\ \bot & \text{otherwise,} \end{cases}$$

and

$$x \Rightarrow y = \begin{cases} 0 \wedge (y - x) & \text{if } x, y \in G^-, \\ 0 & \text{if } x = \bot, \\ \bot & \text{if } x \in G^- \text{ and } y = \bot. \end{cases}$$

As a consequence of Theorem 2.5 and Remark 2.2 of [CT00] we have the following useful result. pr Let A be a non-trivial Π-chain. There exists a linearly ordered abelian group $\mathbf{G}$, such that $A \cong \mathfrak{P}(\mathbf{G})$. Moreover, $\mathbf{G}$ is univocally determined up to isomorphism. Notice that the isomorphism of the above proposition maps the neutral element of the group onto the maximum element of the product chain and the added bottom $\bot$ to the minimum element of the product chain.

Let $\mathbf{G}$ be a linearly ordered abelian group and $a, \{a_i\}_{i\in\omega} \in G$: it is well known that, on the one hand, if $\{a_i\}_{i\in\omega}$ is an increasing sequence and has limit a, then $\{a - a_i\}_{i\in\omega}$ is a decreasing sequence and has limit 0. On the other hand, if $\{a_i\}_{i\in\omega}$ is a decreasing sequence and has limit a, then $\{a_i - a\}_{i\in\omega}$ is a decreasing sequence and has limit 0. So, since, by Definition B.4.1, the truncated subtraction of the group is the interpretation of product implication and the constant 0 of the group is the isomorphic image of the maximum element 1 of the product chain, then, by means of Proposition B.4, we can infer the following corollary.

Corollary B.4.2. *Let $\mathbf{A}$ be a product chain and $a, \{a_i\}_{i\in\omega} \in A$, then if $\{a_i\}_{i\in\omega}$ is either an increasing or decreasing sequence with limit a, then $\{a \Rightarrow a_i\}_{i\in\omega}$ is an increasing sequence with limit* 1*.*

With the help of the last corollary, we can prove the main result of this section.

Lemma B.4.3. *The quasi-witnessed axioms* ($C\exists$) *and* ($\Pi C\forall$) *are theorems of* $\Pi\forall$*.*

Proof. We will show it semantically. Since $\Pi\forall$ is complete w.r.t. models over linearly ordered product algebras, we have to prove that the quasi-witnessed axioms are tautologies for these models. Let $\mathbf{A}$ be a product chain, then:

($C\exists$) Since $\|(\exists y)((\exists x)\varphi(x) \to \varphi(y))\|^{(\mathbf{A},\mathbf{M})} = \sup_{y\in M}\{\sup_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} \Rightarrow \|\varphi(y)\|^{(\mathbf{A},\mathbf{M})}\}$ and variables x and y range over the same values, then, by Corollary B.4.2, $\|(\exists y)((\exists x)\varphi(x) \to \varphi(y))\|^{(\mathbf{A},\mathbf{M})} = 1$. So, axiom ($C\exists$) is a theorem of $\Pi\forall$.

($\Pi C\forall$) We know by definition, that $\|\neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))\|^{(\mathbf{A},\mathbf{M})} = \neg\neg \inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} \Rightarrow \sup_{y\in M}\{\|\varphi(y)\|^{(\mathbf{A},\mathbf{M})} \Rightarrow \inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\}\}$. If $\inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} = 0$, the result is obvious. Otherwise (being a Gödel negation) $\neg\neg \inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} = 1$ and, therefore, the value of the whole formula will be equal to 1 iff $\|(\exists y)(\varphi(y) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = \sup_{y\in M}\{\|\varphi(y)\|^{(\mathbf{A},\mathbf{M})} \Rightarrow \inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\}\} = 1$, but this is a direct consequence of Corollary B.4.2. So, axiom ($\Pi C\forall$) is a theorem of $\Pi\forall$.

□

Next example shows that validity of quasi-witnessed axioms does not guarantee that models are quasi-witnessed (notice that last lemma ensures that all models of first-order Product Logic satisfy the quasi-witnessed axioms).

Example B.4.4. *Consider the first-order language with only one unary predicate symbol P and a model over the standard product chain $([0,1]_\Pi, (\omega, r_P))$, where $r_P(n) = \frac{1}{m} + \frac{1}{n+2}$, for a fixed but arbitrary positive integer $m > 1$. By Lemma B.4.3, this model satisfies the quasi-witnessed axioms but it is not a quasi-witnessed model because, on the one hand, $\|(\forall x)P(x)\|^{([0,1]_\Pi,(\omega,r_P))} = \frac{1}{m} > 0$ and, on the other hand, for each $n \in N$, $\|P(n)\|^{([0,1]_\Pi,(\omega,r_P))} > \frac{1}{m} = \|(\forall x)P(x)\|^{([0,1]_\Pi,(\omega,r_P))}$. So, it does not respect condition 2 of Definition B.3.1.*

This last result, together with Theorem B.3.5, is an alternative way to prove the result in [LM07].

Corollary B.4.5. *Let T be a theory and φ a formula in a given predicate language, then $T \vdash_{\Pi\forall} \varphi$ iff $(\mathbf{A},\mathbf{M}) \models \varphi$ for every quasi-witnessed model $(\mathbf{A},\mathbf{M})$ of the theory T.*

However, we can not generalize the above result to the logic defined by an arbitrary left-continuous t-norm. In order to prove this result we adapt and generalize the result in [Háj07a]. Actually we can show that there is no other logic of a continuous t-norm that is complete with respect to quasi-witnessed models, but Product and Łukasiewicz.

Lemma B.4.6. *Let $*$ be a continuous t-norm. If $\mathcal{L}(*)\forall$ proves both ($C\exists$) and ($\Pi C\forall$), then $*$ is isomorphic to either Łukasiewicz or product t-norm.*

Proof. In [Háj07a] it is already proved that ($C\exists$) is only valid in $\mathcal{L}(*)\forall$ if $*$ is isomorphic to either the Lukasiewicz or the product t-norm. Here we give a unified proof. If the standard algebra induced by a continuous t-norm $*$ is not isomorphic to either $[0,1]_Ł$ or $[0,1]_\Pi$, then it has at least one element $a \in (0,1)$ which is idempotent. Let $([0,1]_*, (\omega, r_P))$ be a model of $\mathcal{L}(*)\forall$ and $\{a_n\}_{n\in\omega}$ a sequence of elements of $[0,1]$, different from a. Let $\{a_n\}_{n\in\omega}$ be a strictly increasing sequence of elements of $[0,1]$ such that $\sup\{a_n\}_{n\in\omega} = a$. Consider the above given structure in which, for each $n \in \omega$, $\|r_P(n))\|^{([0,1]_*,(\omega,r_P))} = a_n$. In this structure, when $\varphi(x) = P(x)$, we have that $\|(\exists y)((\exists x)\varphi(x) \to \varphi(y))\|^{([0,1]_*,(\omega,r_P))} =$

$\sup_{m\in\omega}\{\sup_{n\in\omega}\{a_n\} \Rightarrow a_m\} = \sup_{m\in\omega}\{a \Rightarrow a_m\} = \sup_{m\in\omega}\{a_m\} = a \neq 1$. So, $(C\exists)$ is not a theorem of $\mathcal{L}(*)\forall$.

□

It is interesting to notice that $(\Pi C\forall)$ is only valid in $\mathcal{L}(*)\forall$ if $[0,1]_*$ is isomorphic to either $[0,1]_{Ł}$, $[0,1]_\Pi$ or the ordinal sum of two copies of Łukasiewicz t-norms $[0,1]_{Ł} \oplus [0,1]_{Ł}$. Let $\{a_n\}_{n\in\omega}$ be a strictly decreasing sequence of elements of $[0,1]$ such that $\inf\{a_n\}_{n\in\omega} = a$, being a either the bottom of a non-Łukasiewicz component or of a Łukasiewicz component whose top element is not 1. In both cases consider the above given structure in which, for each $n \in \omega$, $\|r_P(n))\|^{([0,1]_*,(\omega,r_P))} = a_n$. In this structure, when we take $\varphi(x) = P(x)$, an easy computation shows that axiom $(\Pi C\forall)$ is not sound. Moreover it is not difficult to prove that $(\Pi C\forall)$ is valid when $*$ is isomorphic to either Łukasiewicz or the ordinal sum of two copies of Łukasiewicz t-norm.

Last Lemma allows us to prove the next general result.

pr Let $*$ be a continuous t-norm. Then $\mathcal{L}(*)\forall$ proves both $(C\exists)$ and $(\Pi C\forall)$ iff $[0,1]_*$ is isomorphic to either $[0,1]_{Ł}$ or $[0,1]_\Pi$.

Proof. One direction is proven in Corollary B.4.5 for Product Logic and is a consequence of witnessed completeness for Łukasiewicz. The other direction is a direct consequence of Lemma B.4.6. □

B.5 Δ-strict fuzzy logics

In this section we deal with the expansion of a logic $\mathcal{L}$ with the new unary connective Δ (denoted $\mathcal{L}_\Delta$) and quasi-witnessed models. Logics $\mathcal{L}_\Delta$, were introduced in [Háj98c] as the expansions of $\mathcal{L}$ with the unary connective Δ, satisfying the necessitation inference rule (from φ deduce $\Delta\varphi$) and the following axioms, introduced in [Baa96] in the framework of Gödel Logic:

$(A_\Delta 1)$ $\Delta\varphi \vee \neg\Delta\varphi$,

$(A_\Delta 2)$ $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$,

$(A_\Delta 3)$ $\Delta\varphi \to \varphi$,

$(A_\Delta 4)$ $\Delta\varphi \to \Delta\Delta\varphi$,

$(A_\Delta 5)$ $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$.

Semantically, the main feature of these expansions is that, in $\mathcal{L}_\Delta$-chains, it holds that for each formula φ and each propositional evaluation e, $e(\Delta\varphi) = 1$, if $e(\varphi) = 1$ and $e(\Delta\varphi) = 0$, if $e(\varphi) < 1$. Moreover, expansions of a logic by Δ connective enjoy the following property.

Definition B.5.1. We say that a logic $\mathcal{L}$ enjoys *Delta Deduction Theorem* (ΔDT, for short) if, for each theory T and formulas φ, ψ, it holds that $T, \varphi \vdash \psi$ iff $T \vdash \Delta\varphi \to \psi$.

The last definition gives a way to define the class of logics we are interested in throughout this section.

From [CEG+09], we report the next useful definition.

Definition B.5.2. We say that a logic $\mathcal{L}_\Delta$ is a Δ-*core fuzzy logic* if it enjoys ΔDT, Sub and expands MTL_Δ.

Throughout this section $\mathcal{L}_\Delta$ will denote the extension of a Δ-core fuzzy logic by the strictness axiom (S).

As in [HC06], here also the failure of Lemma B.2.18 does not allow us to prove a similar result as Theorem B.3.5 for a logic $\mathcal{L}_\Delta\forall$. Nevertheless, it is possible, also in this context, to prove a simpler result.

Definition B.5.3. We denote by $\mathcal{L}_\Delta\forall^{\Delta qw}$ the axiomatic extension of $\mathcal{L}_\Delta\forall$ by the following axiom schemata called, from now on, "Δ-quasi-witnessed axioms":

($\mathbf{C}_\Delta\exists$) $(\exists y)\Delta((\exists x)\varphi(x) \to \varphi(y))$,

($\Pi\mathbf{C}_\Delta\forall$) $\neg\neg(\forall x)\varphi(x) \to ((\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x)))$.

We can prove, as in [HC06], that the extension of a logic $\mathcal{L}_\Delta\forall$ by means of these axioms, is complete with respect to quasi-witnessed models, but not with respect to models that are embeddable into a quasi-witnessed model (like the extension of a strict core fuzzy logic by the usual quasi-witnessed axioms). So, it makes sense to say that these extensions are the logics of quasi-witnessed models. The main result will follow easily after a couple of simple lemmas.

Lemma B.5.4. *Axioms* $(C_\Delta\exists)$ *and* $(\Pi C_\Delta\forall)$ *are true in every quasi-witnessed model.*

Proof. Let $\mathbf{A}$ be an $\mathcal{L}_\Delta$-chain, $(\mathbf{A}, \mathbf{M})$ be a first-order quasi-witnessed structure, then:

1. Since $(\mathbf{A}, \mathbf{M})$ is a quasi-witnessed structure, then there exists $a \in M$ such that $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} = \sup_{b\in M}\{\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})}\} = \|(\exists x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$ and, therefore, we have that $\|(\exists y)\Delta((\exists x)\varphi(x) \to \varphi(y))\|^{(\mathbf{A},\mathbf{M})} = \sup_{b\in M}\{\|\Delta((\exists x)\varphi(x) \to \varphi(b))\|^{(\mathbf{A},\mathbf{M})}\} = \|\Delta((\exists x)\varphi(x) \to \varphi(a))\|^{(\mathbf{A},\mathbf{M})} = \|\Delta(1)\|^{(\mathbf{A},\mathbf{M})} = 1$.

2. Since $(\mathbf{A}, \mathbf{M})$ is a quasi-witnessed structure, then either $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$ or there exists $a \in M$ such that $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} = \inf_{b\in M}\{\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})}\} = \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$. In the first case, trivially, $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$ and, therefore $\|\neg\neg(\forall x)\varphi(x) \to ((\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x)))\|^{(\mathbf{A},\mathbf{M})} = 1$. In the second case, by strictness, we have that $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 1$ and the axiom is then valid since $\|(\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = \sup_{b\in M}\{\|\Delta(\varphi(b) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})}\} = \|\Delta(\varphi(a) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = \|\Delta(1)\|^{(\mathbf{A},\mathbf{M})} = 1$.

□

Lemma B.5.5. *Axioms* ($C_\Delta\exists$) *and* ($\Pi C_\Delta\forall$) *are false in every model that is not quasi-witnessed.*

Proof. We will prove it only for the second axiom, the proof for the first one is almost the same. Let $\mathbf{A}$ be an $\mathcal{L}_\Delta$-chain, $(\mathbf{A}, \mathbf{M})$ a first-order structure that is not quasi-witnessed. Then there exists a formula $\varphi(x)$ such that both for each $a \in M$, $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} \neq \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$ and $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} \neq 0$. Hence $\|(\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = \sup_{b\in M}\{\|\Delta(\varphi(b) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})}\} = \sup_{b\in M}\{0\} = 0$. □

We are, now, ready to prove the main result of this section.

Theorem B.5.6. *Let T be a theory and φ a formula in a given predicate language, then $T \vdash_{\mathcal{L}_\Delta\forall^{\Delta qw}} \varphi$ iff $(\mathbf{A}, \mathbf{M}) \models \varphi$ for every quasi-witnessed model $(\mathbf{A}, \mathbf{M})$ of the theory T.*

Proof. The completeness of $\mathcal{L}_\Delta\forall$ with respect to all (not only quasi-witnessed) $(\mathbf{A}, \mathbf{M})$-models is ensured by Theorem 10 of [HC06], so we will restrict ourselves to the quasi-witnessed part.

($\Rightarrow$) As a consequence of Theorem 10 of [HC06], we only have to check whether a quasi-witnessed model satisfies axioms ($C_\Delta\exists$) and ($\Pi C_\Delta\forall$), but this has been proven in Lemma B.5.4.

($\Leftarrow$) Suppose that $T \nvdash_{\mathcal{L}_\Delta\forall^{\Delta\mathrm{qw}}} \varphi$, then there exists an $\mathcal{L}_\Delta\forall^{\Delta\mathrm{qw}}$-structure $(\mathbf{A}, \mathbf{M})$ of T, such that $(\mathbf{A}, \mathbf{M}) \not\models \varphi$. By Lemma B.5.5, structure $(\mathbf{A}, \mathbf{M})$ is quasi-witnessed and, moreover, $\|\varphi\|^{(\mathbf{A},\mathbf{M})} < 1$.

□

Unlike quasi-witnessed axioms of previous section, Δ-quasi-witnessed axioms are not derivable in any logic. The argument to prove this result is the same as in Lemma B.5.5 or in Example B.4.4.

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