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ALGEBRAS ASSIGNED TO TERNARY RELATIONS

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Abstract. We show that to every centred ternary relation T on a set A there can be assigned (in a non-unique way) a ternary operation t on A such that the identities satisfied by $(A;t)$ reflect relational properties of T. We classify ternary operations assigned to centred ternary relations and we show how the concepts of relational subsystems and homomorphisms are connected with subalgebras and homomorphisms of the assigned algebra $(A;t)$. We show that for ternary relations having a non-void median can be derived so-called median-like algebras $(A;t)$ which become median algebras if the median $M_T(a,b,c)$ is a singleton for all $a,b,c \in A$. Finally, we introduce certain algebras assigned to cyclically ordered sets.

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In [2] and [3], the first and the third author showed that to certain relational systems $\mathcal{A} = (A;R)$, where $A \neq \varnothing$ and R is a binary relation on A, there can be assigned a certain groupoid $\mathcal{G}(A) = (A;\circ)$ which captures the properties of R. Namely, we have $x \circ y = y$ if and only if $(x,y) \in R$. In these papers we worked with so-called directed relational systems, i. e. for all $x,y \in A$ we have

$$U_R(x,y) := \{z \in A \mid (x,z),(y,z) \in R\} \neq \varnothing.$$

We are inspired by the idea of assigning a groupoid (called directoid) to a directed poset. This idea has its origin in the paper [6] by J. Ježek and R. Quackenbush. Then some structural properties of the assigned groupoid $\mathcal{G}(A)$ can be used for introducing certain structural properties of $\mathcal{A} = (A;R)$; in particular, we introduced congruences, quotient relational systems and homomorphisms which are in accordance with the corresponding concepts in $\mathcal{G}(A)$.

Hence, there arises the natural question if a similar way can be used for ternary relational systems and algebras with one ternary relation. In a particular case, such a

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correspondence exists. It is for the ternary relation "betweenness" and the so-called median algebras, see e.g. [1,5] or [11].

However, there exist also other useful ternary relations for which a similar construction is not already derived, in particular the so-called cyclic orders, see e.g. [4,7,8] and [9].

Moreover, more general ternary relations were already investigated in [10] and [11] and hence our problem can be extended to a more general case than betweenness. However, to get a construction of a ternary operation, a certain restriction on the ternary relation is necessary.

In the following let A denote a fixed arbitrary non-empty set.

1. Ternary operations assigned to ternary relations

We introduce the following concepts:

Definition 1. Let T be a ternary relation on A and $a,b\in A$. The set

$$Z_T(a,b):=\{x\in A\,|\,(a,x,b)\in T\}$$

is called the **centre of** (a,b) **with respect to** T. The ternary relation T on A is called **centred** if $Z_T(a,b)\neq\varnothing$ for all elements $a,b\in A$.

Definition 2. Let T be a ternary relation on A and $a,b,c\in A$. The set

$$M_T(a,b,c):=Z_T(a,b)\cap Z_T(b,c)\cap Z_T(c,a)$$

will be called the **median of** (a,b,c) **with respect to** T.

The concept of a median was originally introduced in lattices and structures derived from lattices. In particular, two sorts of medians are usually considered: $m(x,y,z)=(x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$ and $M(x,y,z)=(x\vee y)\wedge(y\vee z)\wedge(z\vee x)$.

Now we show that to every centred ternary relation there can be assigned ternary operations.

Definition 3. Let T be a centred ternary relation on A and t a ternary operation on A satisfying

$$t(a,b,c)\begin{cases}=b & \text{if } (a,b,c)\in T\\ \in Z_T(a,c) & \text{otherwise.}\end{cases}$$

Such an operation t is called **assigned to** T.

Remark 1. By definition, if T is a centred ternary relation on A and t assigned to T then $(a,t(a,b,c),c)\in T$ for all $a,b,c\in A$.

Lemma 1. *Let T be a centred ternary relation on A and t an assigned operation. Let $a,b,c\in A$. Then $(a,b,c)\in T$ if and only if $t(a,b,c)=b$.*

Proof. By Definition 3, if $(a,b,c)\in T$ then $t(a,b,c)=b$. Conversely, assume $(a,b,c)\notin T$. Then $t(a,b,c)\in Z_T(a,c)$. Now $t(a,b,c)=b$ would imply $(a,b,c)=(a,t(a,b,c),c)\in T$ contradicting $(a,b,c)\notin T$. Hence $t(a,b,c)\neq b$. ☐

To illuminate the role of the median, let us consider the following example:

Example 1. Let $\mathcal{L} = (L;\vee,\wedge)$ be a lattice. Define a ternary operation T on L as follows:

$$(a,b,c) \in T \quad \text{if and only if} \quad a \wedge c \leq b \leq a \vee c.$$

Put $m(x,y,z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $M(x,y,z) := (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. If $p \in M_T(a,b,c)$ then $p \in Z_T(a,b)$, $p \in Z_T(b,c)$ and $p \in Z_T(c,a)$, i. e. $a \wedge b \leq p \leq a \vee b$, $b \wedge c \leq p \leq b \vee c$ and $c \wedge a \leq p \leq c \vee a$ whence $m(a,b,c) \leq p \leq M(a,b,c)$. This yields

$$M_T(a,b,c) = [m(a,b,c), M(a,b,c)],$$

the interval in $\mathcal{L}$. It is well-known that $m(x,y,z) = M(x,y,z)$ if and only if $\mathcal{L}$ is distributive. Hence, $\mathcal{L}$ is distributive if and only if $|M_T(a,b,c)| = 1$ for all $a,b,c \in L$.

The previous example was used in [5] for the definition of a median algebra. If $\mathcal{L}$ is a distributive lattice then the algebra $(L;m)$ is called the median algebra derived from $\mathcal{L}$. Of course, there exist median algebras which are not derived from a lattice, see [1] for details, but in every median algebra there can be introduced a ternary relation "between" by putting

$$(a,b,c) \in T_m \quad \text{if and only if} \quad m(a,b,c) = b.$$

In what follows, we show how this construction can be generalized and we get a characterization of some important properties of ternary relations by means of identities of their assigned operations.

Theorem 1. *A ternary operation t on A is assigned to some centred ternary relation T on A if and only if it satisfies the identity*

$$t(x,t(x,y,z),z) = t(x,y,z). \tag{1.1}$$

Proof. Let $a,b,c \in A$.
Assume that T is a ternary relation on A and t an assigned operation. If $(a,b,c) \in T$ then $t(a,b,c) = b$ and hence $t(a,t(a,b,c),c) = t(a,b,c)$. If $(a,b,c) \notin T$ then $t(a,b,c) \in Z_T(a,c)$ and hence $(a,t(a,b,c),c) \in T$ which yields $t(a,t(a,b,c),c) = t(a,b,c)$. Thus t satisfies identity (1.1).

Conversely, assume $t : A^3 \to A$ satisfies (1.1) and define $T := \{(x,y,z) \in A^3 \mid t(x,y,z) = y\}$. If $(a,b,c) \in T$ then $t(a,b,c) = b$ and, if $(a,b,c) \notin T$ then $(a,t(a,b,c),c) \in T$ whence $t(a,b,c) \in Z_T(a,c)$, i. e. t is assigned to T. □

We can consider a number of properties of ternary relations which are used in [1–11] for "betweenness" and for "cyclic orders".

Definition 4. Let T be a ternary relation on A. We call T
– **reflexive** if $|\{a,b,c\}| \leq 2$ implies $(a,b,c) \in T$;
– **symmetric** if $(a,b,c) \in T$ implies $(c,b,a) \in T$;

– **antisymmetric** if $(a,b,a) \in T$ implies $a = b$;
– **cyclic** if $(a,b,c) \in T$ implies $(b,c,a) \in T$;
– R**-transitive** if $(a,b,c),(b,d,e) \in T$ implies $(a,d,e) \in T$;
– t_1**-transitive** if $(a,b,c),(a,d,b) \in T$ implies $(d,b,c) \in T$;
– t_2**-transitive** if $(a,b,c),(a,d,b) \in T$ implies $(a,d,c) \in T$;
– R**-symmetric** if $(a,b,c) \in T$ implies $(b,a,c) \in T$;
– R**-antisymmetric** if $(a,b,c),(b,a,c) \in T$ implies $a = b$;
– **non-sharp** if $(a,a,b) \in T$ for all $a,b \in A$;
– **cyclically transitive** if $(a,b,c),(a,c,d) \in T$ implies $(a,b,d) \in T$.

Theorem 2. *Let T be a centred ternary relation on A and t an assigned operation. Then (i) – (xi) hold:*
(i) T is reflexive if and only if t satisfies the identities

$$t(x,x,y) = t(y,x,x) = t(y,x,y) = x.$$

(ii) T is symmetric if and only if t satisfies the identity

$$t(z,t(x,y,z),x) = t(x,y,z).$$

(iii) T is antisymmetric if and only if t satisfies the identity

$$t(x,y,x) = x.$$

(iv) T is cyclic if and only if t satisfies the identity

$$t(t(x,y,z),z,x) = z.$$

(v) T is R-transitive if and only if t satisfies the identity

$$t(x,t(t(x,y,z),u,v),v) = t(t(x,y,z),u,v).$$

(vi) T is t_1-transitive if and only if t satisfies the identity

$$t(t(x,u,t(x,y,z)),t(x,y,z),z) = t(x,y,z).$$

(vii) T is t_2-transitive if and only if t satisfies the identity

$$t(x,t(x,u,t(x,y,z)),z) = t(x,u,t(x,y,z)).$$

(viii) T is R-symmetric if and only if t satisfies the identity

$$t(t(x,y,z),x,z) = x.$$

(ix) If t satisfies the identity

$$t(t(x,y,z),x,z) = t(x,y,z)$$

then T is R-antisymmetric.
(x) T is non-sharp if and only if t satisfies the identity

$$t(x,x,y) = x.$$

(xi) T is cyclically transitive if and only if t satisfies the identity

$$t(x,t(x,y,t(x,z,u)),u) = t(x,y,t(x,z,u)).$$

Proof. Let $a,b,c,d,e \in A$.
(i) is clear.
(ii) t satisfies $t(z,t(x,y,z),x) = t(x,y,z)$ if and only if $(z,t(x,y,z),x) \in T$ for all $x,y,z \in A$.
"$\Rightarrow$": $(a,t(a,b,c),c) \in T$ and hence $(c,t(a,b,c),a) \in T$.
"$\Leftarrow$": If $(a,b,c) \in T$ then $(c,b,a) = (c,t(a,b,c),a) \in T$.
(iii) "$\Rightarrow$": $(a,t(a,b,a),a) \in T$ and hence $t(a,b,a) = a$.
"$\Leftarrow$": If $(a,b,a) \in T$ then $a = t(a,b,a) = b$.
(iv) t satisfies $t(t(x,y,z),z,x) = z$ if and only if $(t(x,y,z),z,x) \in T$ for all $x,y,z \in A$.
"$\Rightarrow$": $(a,t(a,b,c),c) \in T$ and hence $(t(a,b,c),c,a) \in T$.
"$\Leftarrow$": If $(a,b,c) \in T$ then $(b,c,a) = (t(a,b,c),c,a) \in T$.
(v) t satisfies $t(x,t(t(x,y,z),u,v),v) = t(t(x,y,z),u,v)$ if and only if $(x,t(t(x,y,z),u,v),v) \in T$ for all $x,y,z,u,v \in A$.
"$\Rightarrow$": $(a,t(a,b,c),c),(t(a,b,c),t(t(a,b,c),d,e),e) \in T$ and hence $(a,t(t(a,b,c),d,e),e) \in T$.
"$\Leftarrow$": If $(a,b,c),(b,d,e) \in T$ then $(a,d,e) = (a,t(t(a,b,c),d,e),e) \in T$.
(vi) t satisfies $t(t(x,u,t(x,y,z)),t(x,y,z),z) = t(x,y,z)$ if and only if $(t(x,u,t(x,y,z)),t(x,y,z),z) \in T$ for all $x,y,z,u \in A$.
"$\Rightarrow$": $(a,t(a,b,c),c),(a,t(a,d,t(a,b,c)),t(a,b,c)) \in T$ and hence $(t(a,d,t(a,b,c)),t(a,b,c),c) \in T$.
"$\Leftarrow$": If $(a,b,c),(a,d,b) \in T$ then $(d,b,c) = (t(a,d,t(a,b,c)),t(a,b,c),c) \in T$.
(vii) t satisfies $t(x,t(x,u,t(x,y,z)),z) = t(x,u,t(x,y,z))$ if and only if $(x,t(x,u,t(x,y,z)),z) \in T$ for all $x,y,z,u \in A$.
"$\Rightarrow$": $(a,t(a,b,c),c),(a,t(a,d,t(a,b,c)),t(a,b,c)) \in T$ and hence $(a,t(a,d,t(a,b,c)),c) \in T$.
"$\Leftarrow$": If $(a,b,c),(a,d,b) \in T$ then $(a,d,c) = (a,t(a,d,t(a,b,c)),c) \in T$.
(viii) t satisfies $t(t(x,y,z),x,z) = x$ if and only if $(t(x,y,z),x,z) \in T$ for all $x,y,z \in A$.
"$\Rightarrow$": $(a,t(a,b,c),c) \in T$ and hence $(t(a,b,c),a,c) \in T$.
"$\Leftarrow$": If $(a,b,c) \in T$ then $(b,a,c) = (t(a,b,c),a,c) \in T$.
(ix) If $(a,b,c),(b,a,c) \in T$ then $a = t(b,a,c) = t(t(a,b,c),a,c) = t(a,b,c) = b$.
(x) This is clear.
(xi) t satisfies $t(x,t(x,y,t(x,z,u)),u) = t(x,y,t(x,z,u))$ if and only if $(x,t(x,y,t(x,z,u)),u) \in T$ for all $x,y,z,u \in A$.
"$\Rightarrow$": $(a,t(a,b,t(a,c,d)),t(a,c,d)),(a,t(a,c,d),d) \in T$ and hence $(a,t(a,b,t(a,c,d)),d) \in T$.
"$\Leftarrow$": If $(a,b,c),(a,c,d) \in T$ then $t(a,b,d) = t(a,t(a,b,t(a,c,d)),d) = t(a,b,t(a,c,d)) = b$.

□

Lemma 2. *Let T be a ternary relation on A. Then*

$$|T| = \sum_{(a,b)\in A^2} |Z_T(a,b)|.$$

Proof.

$$T = \dot{\bigcup_{(a,b)\in A^2}} (\{a\}\times Z_T(a,b)\times\{b\}).$$

□

Corollary 1. *Let A be finite, $|A| = n$. If T is a centred ternary relation on A then $|T| \geq n^2$. Moreover, if T is centred then $|T| = n^2$ if and only if $|Z_T(x,y)| = 1$ for each $x, y \in A$.*

2. Congruences, homomorphisms and subsystems of ternary relational systems

By a **ternary relational system** is meant a couple $\mathcal{T} = (A;T)$ where T is a ternary relation on A. $\mathcal{T}$ is called **centred** if T is centred. As shown in the previous section, to every centred ternary relational system $\mathcal{T} = (A;T)$ there can be assigned an algebra $\mathcal{A}(T) = (A;t)$ with one ternary operation $t : A^3 \to A$ such that t is assigned to T. Now, we can introduce an inverse construction. It means that to every algebra $\mathcal{A} = (A;t)$ of type (3) there can be assigned a ternary relational system $\mathcal{T}(A) = (A;T_t)$ where T_t is defined by

$$T_t := \{(x,y,z) \in A^3 \mid t(x,y,z) = y\}. \tag{2.1}$$

Of course, an assigned ternary relational system $\mathcal{T}(A) = (A;T_t)$ need not be centred. However, if $\mathcal{T} = (A;T)$ is a centred ternary relational system and $\mathcal{A}(T) = (A;t)$ an assigned algebra then T_t is centred despite the fact that t is not determined uniquely. In fact, we have $(a,b,c) \in T_t$ if and only if $t(a,b,c) = b$ if and only if $(a,b,c) \in T$. Hence, we have proved the following

Lemma 3. *Let $\mathcal{T} = (A;T)$ be a centred ternary relational system, $\mathcal{A}(T) = (A;t)$ an assigned algebra and $\mathcal{T}(\mathcal{A}(T)) = (A;T_t)$ the ternary relational system assigned to $\mathcal{A}(T)$. Then $\mathcal{T}(\mathcal{A}(T)) = \mathcal{T}$.*

The best known correspondence between centred ternary relational systems and corresponding algebras of type (3) is the case of "betweenness"-relations and median algebras which was initiated by J. R. Isbell [5] and essentially developed by H.-J. Bandelt and J. Hedlíková [1]. However, there are also some essential differences between relational systems and the corresponding algebras. For binary relational systems it was described by the first and the third author in [2]. In what follows, we are going to handle it for the ternary case.

If $\mathcal{T} = (A;T)$ is a ternary relational system and E an equivalence relation on A then the **quotient relational system** $\mathcal{T}/E$ is defined as the relational system $(A/E, T/E)$ where $T/E := \{([x]_E, [y]_E, [z]_E) \mid (x,y,z) \in T\}$. It is evident that E need not be a congruence on the assigned algebra $\mathcal{A}(T) = (A;t)$ and hence congruences on $\mathcal{T} = (A;T)$, respectively on $\mathcal{A}(T)$ are different concepts.

Similarly, by a **subsystem** of $\mathcal{T} = (A;T)$ is meant a couple of the form $(B, T|B)$ with a non-empty subset B of A and $T|B := T \cap B^3$. One can easily see that this need not be a subalgebra of $\mathcal{A}(T) = (A;t)$.

Finally, by a **homomorphism** of a ternary relational system $\mathcal{T} = (A;T)$ into a ternary relational system $\mathcal{S} = (B;S)$ is meant a mapping $h : A \to B$ satisfying

$$(a,b,c) \in T \quad \Longrightarrow \quad (h(a), h(b), h(c)) \in S.$$

A homomorphism h is called **strong** if for each triple $(p,q,r) \in S$ there exists $(a,b,c) \in T$ such that $(h(a), h(b), h(c)) = (p,q,r)$.

Now, we define the following concept.

Definition 5. A t**-homomorphism** from a centred ternary relational system $\mathcal{T} = (A;T)$ to a ternary relational system $\mathcal{S} = (B;S)$ is a homomorphism from $\mathcal{T}$ to $\mathcal{S}$ such that there exists an algebra $(A;t)$ assigned to $\mathcal{T}$ such that $a,b,c,a',b',c' \in A$ and $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ together imply $h(t(a,b,c)) = h(t(a',b',c'))$.

Theorem 3. *Let $\mathcal{T} = (A;T)$ and $\mathcal{S} = (B;S)$ be centred ternary relational systems and $\mathcal{A}(T) = (A;t)$ and $\mathcal{B}(S) = (B;s)$ assigned algebras. Then every homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$ is a t-homomorphism from $\mathcal{T}$ to $\mathcal{S}$.*

Proof. Let $a,b,c,a',b',c' \in A$. If $(a,b,c) \in T$ then $t(a,b,c) = b$ and hence $s(h(a), h(b), h(c)) = h(t(a,b,c)) = h(b)$ showing $(h(a), h(b), h(c)) \in S$. Thus h is a homomorphism from $\mathcal{T}$ to $\mathcal{S}$.
Moreover, if $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ then

$$h(t(a,b,c)) = s(h(a), h(b), h(c)) = s(h(a'), h(b'), h(c')) = h(t(a',b',c')).$$

Hence h is a t-homomorphism from $\mathcal{T}$ to $\mathcal{S}$. $\square$

The theorem just proved says that every homomorphism of assigned algebras is a t-homomorphism of the original relational systems. Now we can show under which conditions the converse assertion becomes true.

Theorem 4. *Let $\mathcal{T} = (A;T)$ and $\mathcal{S} = (B;S)$ be centred ternary relational systems. Then for every strong t-homomorphism h from $\mathcal{T}$ to $\mathcal{S}$ with assigned algebra $\mathcal{A}(T) = (A;t)$ there exists an algebra $\mathcal{B}(S) = (B;s)$ assigned to $\mathcal{S}$ such that h is a homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$.*

Proof. Let h be a strong t-homomorphism from $\mathcal{T}$ to $\mathcal{S}$. By definition there exists an algebra $\mathcal{A}(T) = (A;t)$ assigned to $\mathcal{T}$ such that for all $a,b,c,a',b',c' \in A$ with

$(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ it holds $h(t(a,b,c)) = h(t(a',b',c'))$. Define a ternary operation s on B as follows:

$$s(h(x), h(y), h(z)) := h(t(x, y, z))$$

for all $x, y, z \in A$. Since h is strong and a t-homomorphism, s is correctly defined. For $a, b, c \in A$, if $(h(a), h(b), h(c)) \in S$ then there exists $(d, e, f) \in T$ such that $(h(d), h(e), h(f)) = (h(a), h(b), h(c))$. Now

$$s(h(a), h(b), h(c)) = h(t(a, b, c)) = h(t(d, e, f)) = h(e) = h(b).$$

If $(h(a), h(b), h(c)) \notin S$ then $(a, b, c) \notin T$ since h is a homomorphism from $\mathcal{T}$ to $\mathcal{S}$ and hence $t(a, b, c) \in Z_T(a, c)$, i. e. $(a, t(a, b, c), c) \in T$. Thus $(h(a), h(t(a, b, c)), h(c)) \in S$, i. e.

$$(h(a), s(h(a), h(b), h(c)), h(c)) \in S$$

whence $s(h(a), h(b), h(c)) \in Z_S(h(a), h(c))$. This shows that $\mathcal{B}(S)$ is an algebra assigned to $\mathcal{B}$. It is easy to see that h is a homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$. $\square$

We are going to get connections between t-homomorphisms of relational systems and congruences on the assigned algebras.

Theorem 5. *Let $\mathcal{T} = (A; T)$, $\mathcal{S} = (B; S)$ be centred ternary relational systems. Then the following hold:*

(i) *If h is a strong t-homomorphism from $\mathcal{T}$ to $\mathcal{S}$ then there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that* $\ker h \in \mathrm{Con}\mathcal{A}(T)$.

(ii) *If $\mathcal{A}(T) = (A; t)$ is an algebra assigned to $\mathcal{T}$ and $\theta \in$* $\mathrm{Con}\mathcal{A}(T)$ *then the canonical mapping $h : A \to A/\theta$ is a strong t-homomorphism from $\mathcal{T}$ onto $\mathcal{T}/\theta$.*

Proof. (i) Let h be a strong t-homomorphism from $\mathcal{T}$ to $\mathcal{S}$. By definition and Theorem 4, there exist assigned algebras $\mathcal{A}(T) = (A; t)$, respectively $\mathcal{B}(S) = (B; s)$ such that h is a homomorphism of $\mathcal{A}(T)$ to $\mathcal{B}(S)$ and hence $\ker h \in \mathrm{Con}\mathcal{A}(T)$.

(ii) Let $\mathcal{A}(T) = (A; t)$ be an algebra assigned to $\mathcal{T}$, $\theta \in \mathrm{Con}\mathcal{A}(T)$ and $h : A \to A/\theta$ denote the canonical mapping. By definition of T/θ, if $(a, b, c) \in T$ then $(h(a), h(b), h(c)) \in T/\theta$ and hence h is a homomorphism from $\mathcal{T}$ to $\mathcal{T}/\theta$. If, moreover, $a, b, c, a', b', c' \in A$ and $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ then

$$h(t(a, b, c)) = t(h(a), h(b), h(c)) = t(h(a'), h(b'), h(c')) = h(t(a', b', c')).$$

Therefore h is a t-homomorphism from $\mathcal{T}$ onto $\mathcal{T}/\theta$. Obviously, h is strong. $\square$

Definition 6. Let $\mathcal{T} = (A; T)$ be a centred ternary relational system. An equivalence relation θ on A is called a t**-congruence** on $\mathcal{T}$ if there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that $\theta \in \mathrm{Con}\mathcal{A}(T)$. A subset B of A is called a t**-subsystem** of $\mathcal{T}$ if there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that $(B; t)$ is a subalgebra of $\mathcal{A}(T)$.

Example 2. Consider $A = \{a,b,c,d\}$ and the ternary relation T on A defined as follows: $T := A \times \{d\} \times A$. Then $d \in Z_T(x,y)$ for each $x,y \in A$ and hence T is centred and its median is non-empty, in fact $M_T(x,y,z) = \{d\}$ for all $x,y,z \in A$. For $B = \{a,b,c\}$, $\mathcal{B} = (B;T|B)$ is a subsystem of $\mathcal{A} = (A;T)$ but it is not a t-subsystem. Namely, for every $x,y,z \in A$ t can be defined in the unique way as follows: $t(x,y,z) := d$. Hence, $(\{a,b,c\};t)$ is not a subalgebra of $(A;t)$. On the contrary, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$ are t-subsystems of $\mathcal{A}$.

Remark 2. Let $\mathcal{A} = (A;t)$, $\mathcal{B} = (B;s)$ be algebras of type (3) and $h : A \to B$ a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Put $\mathcal{T}(A) := (A;T_t)$ and $\mathcal{S}(B) := (B;S_s)$ where T_t, S_s are defined by (2.1). Then h need not be a t-homomorphism of $\mathcal{T}(A)$ to $\mathcal{S}(B)$, see the following example.

Example 3. Let $A = \{-1,0,1\}$, $B = \{1,0\}$ and $t(x,y,z) = x \cdot y$, $s(x,y,z) = x \cdot y$, where "$\cdot$" is the multiplication of integers. Let $h : A \to B$ be defined by $h(x) = |x|$. Then h is clearly a homomorphism from $\mathcal{A} = (A;t)$ to $\mathcal{B} = (B;s)$ and

$$T_t = (A \times \{0\} \times A) \cup (\{1\} \times A^2).$$

There exists exactly one algebra $(A;t^*)$ assigned to $\mathcal{T}(A)$, namely where

$$t^*(x,y,z) := \begin{cases} y & \text{if } y = 0 \text{ or } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now $h(-1) = h(1)$ but $h(t^*(-1,-1,1)) = h(0) = 0 \neq 1 = h(1) = h(t^*(1,1,1))$. Thus h is not a t-homomorphism.

We can prove the following:

Theorem 6. *If $\mathcal{A} = (A;t)$ and $\mathcal{B} = (B;s)$ are algebras of type (3), $\mathcal{A}$ satisfies the identity*

$$t(x,t(x,y,z),z) = t(x,y,z)$$

and $\mathcal{T}(A) = (A;T_t)$ and $\mathcal{S}(B) = (B;S_s)$ denote the relational systems corresponding to $\mathcal{A}$ and $\mathcal{B}$, respectively, as defined by (2.1) then every homomorphism h from $\mathcal{A}$ to $\mathcal{B}$ is a t-homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$.

Proof. Let $a,b,c,d,e,f \in A$. If $(a,b,c) \in \mathcal{T}(A)$ then $t(a,b,c) = b$ and hence

$$s(h(a),h(b),h(c)) = h(t(a,b,c)) = h(b)$$

whence $(h(a),h(b),h(b)) \in \mathcal{S}(B)$. This shows that h is a homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$. Obviously, t is assigned to T_t. Finally, $(h(a),h(b),h(c)) = (h(d),h(e),h(f))$ implies

$$h(t(a,b,c)) = s(h(a),h(b),h(c)) = s(h(d),h(e),h(f)) = h(t(d,e,f))$$

which shows that h is a t-homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$. □

3. Derived binary systems

Let T be a ternary relation on A and p an arbitrary, but fixed element of A. Then

$$R_T := \{(x,y) \in A^2 \mid (x,y,p) \in T\}$$

is called the binary relation p**-derived from T**. Moreover, put $x \circ y := t(x,y,p)$ for all $x,y \in A$ if T is centred and t is an assigned operation.

If T is reflexive then R_T is reflexive, too. If, moreover, T is centred then Theorem 2 implies $x \circ x = x$, the idempotency of the operation $\circ$ which is in accordance with (i) of Theorem 8 in [3].

Similarly, if T is R-symmetric then R_T is symmetric. If, moreover, T is centred then Theorem 2 implies $(x \circ y) \circ x = x$ which is identity (ii) of Theorem 8 in [3] characterizing symmetric binary relations (for directed relational systems).

If T is R-antisymmetric then R_T is antisymmetric. If, moreover, T is centred then Theorem 2 yields that $(x \circ y) \circ x = x \circ y$ which, if satisfied for all $p \in A$, is a sufficient condition for the antisymmetry of R_T. This condition is also a sufficient condition for the antisymmetry of binary relations (see (v) of Theorem 8 in [3]).

If T is R-transitive then R_T is transitive. If, moreover, T is centred then Theorem 2 implies $x \circ ((x \circ y) \circ u) = (x \circ y) \circ u$ which is just identity (iii) of Theorem 8 in [3] characterizing transitivity of binary relations.

Let us recall from [3] that a binary relation R on A is **(upward) directed** if

$$U_R(a,b) := \{x \in A \mid (a,x),(b,x) \in R\} \neq \varnothing \text{ for all } a,b \in A.$$

Although reflexivity, R-symmetry, R-antisymmetry and R-transitivity of a ternary relation T on A yields the corresponding property of R_T, we are not able to show that if T is centred then R_T is directed. However, our characterization of the corresponding properties for binary relations by means of the induced binary operations in [3] are possible for directed relations only.

Example 4. Put $A := \{x,y,z\}$ and

$$T := \{(x,z,y)\} \cup \{(a,y,b) \mid (a,b) \in A^2 \setminus \{(x,y)\}\}.$$

Then T is centred because $Z_T(x,y) = \{z\}$ and $Z_T(a,b) = \{y\}$ for $(a,b) \in A^2 \setminus \{(x,y)\}$. Put $p := y$ and consider the p-derived binary relation R_T on A. Then

$$x \circ (x \circ y) = t(x,t(x,y,y),y) = t(x,z,y) = z = t(x,y,y) = x \circ y,$$

but

$$y \circ (x \circ y) = t(y,t(x,y,y),y) = t(y,z,y) = y \neq z = t(x,y,y) = x \circ y.$$

Thus $y \circ (x \circ y) \neq (x \circ y)$. According to (ii) of Theorem 6 in [3], R_T is not directed.

Remark 3. Theorem 6 in [3] says that for a groupoid $(G;\circ)$ the following are equivalent:

(i) There exists a directed relational system $(G;R)$ with a reflexive relation R such that $(G;\circ)$ corresponds to $(G;R)$.
(ii) $(G;\circ)$ satisfies the identities $x\circ x = x$ and $x\circ(x\circ y) = y\circ(x\circ y) = x\circ y$.

We are going to show a sufficient condition for R_T to be directed.

Theorem 7. *Let T be a reflexive ternary relation on A such that $Z_T(a,c)\cap Z_T(b,c)\neq\varnothing$ for all $a,b,c\in A$. Let $p\in A$ and R_T denote the binary relation p-derived from T. Then R_T is directed.*

Proof. Due to the assumption, T is centred and hence we can consider a ternary operation t on A assigned to T such that $t(a,b,c)\in Z_T(a,c)\cap Z_T(b,c)$ if $(a,b,c)\in A^3\setminus T$. Since T is reflexive, we have $x\circ x = t(x,x,p) = x$.
First assume $(x,y)\in R_T$. Then $(x,y,p)\in T$. Thus $t(x,y,p) = y$ and hence

$$x\circ(x\circ y) = t(x,t(x,y,p),p) = t(x,y,p) = x\circ y.$$

Since T is reflexive, we obtain

$$y\circ(x\circ y) = t(y,t(x,y,p),p) = t(y,y,p) = y = t(x,y,p) = x\circ y.$$

Now suppose $(x,y)\notin R_T$. Then

$$x\circ(x\circ y) = t(x,t(x,y,p),p) = t(x,y,p) = x\circ y.$$

Since $t(x,y,p)\in Z_T(x,p)\cap Z_T(y,p)$ we have also $t(x,y,p)\in Z_T(y,p)$ and hence

$$y\circ(x\circ y) = t(y,t(x,y,p),p) = t(x,y,p) = x\circ y.$$

We have shown that $\circ$ satisfies (ii) of Theorem 6 in [3]. Thus R_T is directed. □

The converse assertion is also true. For a binary relation R on A and a fixed element $p\in A$ we define

$$T_p(R) := \{(x,y,p)\mid(x,y)\in R\}\cup\{(x,x,y)\mid x,y\in A\}. \tag{3.1}$$

Then we can prove

Proposition 1. *Let R be a reflexive binary relation on A, $p\in A$ and $T_p(R)$ defined by (3.1). Then $T_p(R)$ is a centred ternary relation on A and its p-derived binary relation is just R.*

Proof. It is evident that $T_p(R)$ is a ternary relation on A, its p-derived binary relation is just R and $Z_{T_p(R)}(x,y)\supseteq\{y\}\neq\varnothing$ for all $x,y\in A$, i. e. $T_p(R)$ is centred. □

In what follows, we focus on the relation between ternary relations preserving a given function and properties of assigned operations.

Definition 7. Let T be a ternary relation and f an m-ary operation on A. We say that f **preserves** T if

$$(a_1,b_1,c_1),\dots,(a_m,b_m,c_m)\in T \text{ implies } (f(a_1,\dots,a_m),f(b_1,\dots,b_m),f(c_1,\dots,c_m))\in T.$$

It is worth noticing that the set of all operations on A preserving a given relation T forms a so-called clone. This topic is intensively investigated in contemporary algebra.

Definition 8. Let f be an m-ary and g an n-ary operation on A. We say that f and g **commute with each other** if

$$f(g(x_{11},\dots,x_{1n}),\dots,g(x_{m1},\dots,x_{mn})) = g(f(x_{11},\dots,x_{m1}),\dots,f(x_{1n},\dots,x_{mn}))$$

for all $x_{11},\dots,x_{1n},\dots,x_{m1},\dots,x_{mn}\in A$.

We remark that also the set of all operations on A commuting with a given operation f forms a clone. Our next task is to compare both of these concepts.

Lemma 4. *If T is a centred ternary relation, f an m-ary operation on A commuting with a ternary operation t assigned to T then f preserves T.*

Proof. Let t be a ternary operation assigned to T. Assume $(a_1,b_1,c_1),\dots,(a_m,b_m,c_m)\in T$. Let f commute with t. Then $t(a_i,b_i,c_i)=b_i$ for $i=1,\dots,m$ and hence

$$\begin{aligned}&t(f(a_1,\dots,a_m),f(b_1,\dots,b_m),f(c_1,\dots,c_m))\\&=f(t(a_1,b_1,c_1),\dots,t(a_m,b_m,c_m))\\&=f(b_1,\dots,b_m)\end{aligned}$$

showing $(f(a_1,\dots,a_m),f(b_1,\dots,b_m),f(c_1,\dots,c_m))\in T$. □

Clearly the sufficient condition used in the previous Lemma is not necessary. Such a condition is as follows.

Theorem 8. *If T is a centred ternary relation, f an m-ary operation on A and t a ternary operation assigned to T then f preserves T if and only if it satisfies the following identity:*

$$\begin{aligned}&t(f(x_1,\dots,x_m),f(t(x_1,y_1,z_1),\dots,t(x_m,y_m,z_m)),f(z_1,\dots,z_m))\\&= f(t(x_1,y_1,z_1),\dots,t(x_m,y_m,z_m)).\end{aligned}\tag{3.2}$$

Proof. Assume that f preserves T. Since t is assigned to T we have $(x_i,t(x_i,y_i,z_i),z_i)\in T$ for all $i=1,\dots,m$. Hence

$$(f(x_1,\dots,x_m),f(t(x_1,y_1,z_1),\dots,t(x_m,y_m,z_m)),f(z_1,\dots,z_m))\in T.$$

Thus (3.2) holds.

Conversely, assume that f satisfies (3.2) and $(a_1,b_1,c_1),\dots,(a_m,b_m,c_m) \in T$. Then

$$t(a_i,b_i,c_i) = b_i$$

for $i = 1,\dots,m$, and hence

$$\begin{aligned}
&t(f(a_1,\dots,a_m), f(b_1,\dots,b_m), f(c_1,\dots,c_m)) \\
&= t(f(a_1\dots,a_m), f(t(a_1,b_1,c_1),\dots,t(a_m,b_m,c_m)), f(c_1,\dots,c_m)) \\
&= f(t(a_1,b_1,c_1),\dots,t(a_m,b_m,c_m)) \\
&= f(b_1,\dots,b_m)
\end{aligned}$$

proving $(f(a_1,\dots,a_m), f(b_1,\dots,b_m), f(c_1,\dots,c_m)) \in T$. Hence, f preserves T. $\square$

4. Median-like algebras

The concept of a median algebra was introduced by J. R. Isbell (see [5]) as follows: An algebra $\mathcal{A} = (A;t)$ of type (3) is called a **median algebra** if it satisfies the following identities:

(M1) $t(x,x,y) = x$;
(M2) $t(x,y,z) = t(y,x,z) = t(y,z,x)$;
(M3) $t(t(x,y,z),v,w) = t(x,t(y,v,w),t(z,v,w))$.

It is well-known (see e.g. [1], [5]) that the ternary relation T_t on A assigned to t via (2.1) is centred and, moreover, $|M_{T_t}(a,b,c)| = 1$ for all $a,b,c \in A$. In fact, $t(a,b,c) \in M_{T_t}(a,b,c)$. In particular, having a distributive lattice $\mathcal{L} = (L;\vee,\wedge)$ then $m(x,y,z) = M(x,y,z)$ and putting $t(x,y,z) := m(x,y,z)$, one obtains a median algebra. Conversely, every median algebra can be embedded into a distributive lattice. Moreover, the assigned ternary relation T_t is the so-called "betweenness", see [10] and [11].

In what follows, we focus on the case when $M_T(a,b,c) \neq \varnothing$ for all $a,b,c \in A$ and $t(a,b,c) \in M_T(a,b,c)$ also in case $|M_T(a,b,c)| \geq 1$.

Definition 9. A **median-like algebra** is an algebra $(A;t)$ of type (3) where t satisfies (M1) and (M2) and where there exists a centred ternary relation T on A such that $t(x,y,z) \in M_T(x,y,z)$ for all $x,y,z \in A$.

Theorem 9. *An algebra $\mathcal{A} = (A;t)$ of type (3) is median-like if t satisfies (M1), (M2) and*

$$t(x,t(x,y,z),y) = t(y,t(x,y,z),z) = t(z,t(x,y,z),x) = t(x,y,z). \tag{4.1}$$

Proof. If $T := \{(x,y,z) \in A^3 \mid t(x,y,z) = y\}$ then $t(x,y,z) \in M_T(x,y,z)$ for all $x,y,z \in A$. $\square$

Lemma 5. *Every median algebra is a median-like algebra.*

Proof. As shown in [5], identities (M1), (M2), (M3) are equivalent to the identity

$$t(x,t(x,z,w),t(y,z,w)) = t(x,z,w).$$

Putting $w = y$ and using (M1) and (M2), we derive

$$t(x,t(x,z,y),y) = t(x,t(x,z,y),t(y,z,y)) = t(t(x,x,y),z,y) = t(x,z,y)$$

whence (4.1) follows since according to (M2) we have $t(u,v,w) = t(x,y,z)$ for any permutation (u,v,w) of (x,y,z). □

The following examples show that a median-like algebra need not be a median algebra.

Example 5. Put $A := \{1,2,3,4,5\}$, let t denote the ternary operation on A defined by $t(x,x,y) = t(x,y,x) = t(y,x,x) := x$ for all $x,y \in A$ and $t(x,y,z) := \min(x,y,z)$ for all $x,y,z \in A$ with $x \neq y \neq z \neq x$ and put $T := \{(x,x,y) \mid x,y \in A\} \cup \{(y,x,x) \mid x,y \in A\} \cup \{(x,y,z) \in A^3 \mid y < x < z\} \cup \{(x,y,z) \in A^3 \mid y < z < x\}$. Then t satisfies (M1) and (M2) and $t(x,y,z) \in M_T(x,y,z)$ for all $x,y,z \in A$. This shows that $(A;t)$ is median-like. However, this algebra is not a median algebra since

$$t(t(1,3,4),2,5) = t(1,2,5) = 1 \neq 2 = t(1,2,2) = t(1,t(3,2,5),t(4,2,5))$$

and hence (M3) is not satisfied.

Example 6. Consider the lattice M_3 given in FIGURE 1 below.

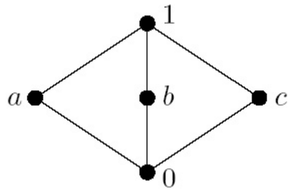

FIGURE 1.

Then M_3 is not distributive, $m(a,b,c) = 0$ and $M(a,b,c) = 1$. Define $(x,y,z) \in T$ if and only if $y \in [x \wedge z, x \vee z]$. Let t be an assigned operation defined as follows

$$t(x,y,z) := m(x,y,z).$$

Then $t(x,y,z) \in M_T(x,y,z)$ for all triples of elements x,y,z and hence $(M_3;t)$ is a median-like algebra. However, it is not a median algebra because identity (M3) is violated:

$$t(t(a,b,c),a,1) = t(0,a,1) = a \neq 1 = t(a,1,1) = t(a,t(b,a,1),t(c,a,1)).$$

The previous example motivated us to state a general construction for lattices which need not be neither distributive nor modular.

Theorem 10. *Let $\mathcal{L} = (L;\vee,\wedge)$ be a lattice. Define $t_1(x,y,z) := m(x,y,z)$, $t_2(x,y,z) := M(x,y,z)$. Then $\mathcal{A}_1 := (L;t_1)$ and $\mathcal{A}_2 := (L;t_2)$ are median-like algebras. Moreover, the following conditions are equivalent*

(a) $\mathcal{A}_1 = \mathcal{A}_2$;
(b) $\mathcal{A}_1$ *is a median algebra;*
(c) $\mathcal{L}$ *is distributive.*

Proof. Since both $m(x,y,z)$ and $M(x,y,z)$ satisfy (M1) and (M2) and $m(x,y,z), M(x,y,z) \in [m(x,y,z), M(x,y,z)] = M_T(x,y,z)$ for $(x,y,z) \in L^3$ and $T := \{(x,y,z) \in L^3 \mid x \wedge z \leq y \leq x \vee z\}$, $\mathcal{A}_1, \mathcal{A}_2$ are median-like algebras. It is well-known that $m(x,y,z) = M(x,y,z)$ if and only if $\mathcal{L}$ is distributive which proves $(a) \Leftrightarrow (c)$. The implication $(c) \Rightarrow (b)$ is well-known (see e.g. [1], [5]). Finally, we prove $(b) \Rightarrow (c)$. Assume that (b) holds but (c) does not. Then $\mathcal{L}$ contains either $\mathcal{M}_3 = (\{0,a,b,c,1\};\vee,\wedge)$ or $\mathcal{N}_5 = (\{0,a,b,c,1\};\vee,\wedge)$ (with $a < c$) as a sublattice. In the first case we have

$$t(t(a,b,c),a,1) = t(0,a,1) = a \neq 1 = t(a,1,1) = t(a,t(b,a,1),t(c,a,1))$$

whereas in the second case

$$t(t(c,b,a),a,1) = t(a,a,1) = a \neq c = t(c,1,a) = t(c,t(b,a,1),t(a,a,1))$$

which shows that (M3) does not hold. This is a contradiction to (b). Hence (c) holds. □

Comparing our definition with Theorem 2, we conclude:

Corollary 2. *An algebra $(A;t)$ of type (3) is median-like if t satisfies (M2) and if it is assigned to a centred antisymmetric or non-sharp ternary relation on A.*

Let us mention that median-like algebras form a variety because they are defined by identities. Moreover, this variety is congruence distributive, i. e. $\mathrm{Con}\mathcal{A}$ is distributive for every median-like algebra $\mathcal{A}$, because the operation t is a majority term, i. e. it satisfies by (M1) and (M2)

$$t(x,x,y) = t(x,y,x) = t(y,x,x) = x.$$

Theorem 11. *let $\mathcal{L} = (L;\vee,\wedge)$ be a lattice and t a ternary operation on L satisfying (M1) and (M2) and $t(x,y,z) \in [m(x,y,z), M(x,y,z)]$ for all $x,y,z \in A$. Then $\mathcal{A} := (L;t)$ is a median-like algebra.*

Proof. Put $T := \{(x,y,z) \in L^3 \mid x \wedge z \leq y \leq x \vee z\}$. Then $M_T(a,b,c) = [m(a,b,c), M(a,b,c)]$ for all $a,b,c \in L$. Hence $t(a,b,c) \in M_T(a,b,c)$ for all $a,b,c \in L$ showing that $\mathcal{A}$ is a median-like algebra. □

5. CYCLIC ALGEBRAS

Apart from the "betweenness" relation, another ternary relation plays an important role in mathematics. It is the so-called **cyclic order**, see e.g. [4], [9] and references there.

Definition 10. A ternary relation T on A is called **asymmetric** if

$$(a,b,c) \in T \text{ for } a \neq b \neq c \quad \text{implies} \quad (c,b,a) \notin T. \tag{5.1}$$

A ternary relation C on A is called a **cyclic order** if it is cyclic, asymmetric, cyclically transitive and satisfies $(a,a,a) \in C$ for each $a \in A$.

Remark 4. Let C be a cyclic order on a set A. Then $(a,b,a) \notin C$ for all $a,b \in A$ with $a \neq b$. Namely, if $(a,b,a) \in C$ then, by (5.1), $(a,b,a) \notin C$, a contradiction. Since C is cyclic, we have also $(a,a,b),(b,a,a) \notin C$.

Applying (5.1), we derive immediately

Lemma 6. *A centred ternary relation T on A is asymmetric if and only if any assigned ternary operation t satisfies the implication:*

$$(t(x,y,z) = y \text{ and } x \neq y \neq z) \quad \Longrightarrow \quad t(z,y,x) \neq y. \tag{5.2}$$

Similarly as for "betweenness" relations, we can derive an algebra of type (3) for a centred cyclic order by means of its assigned operation.

Definition 11. A **cyclic algebra** is an algebra assigned to a cyclic relation.

Cyclic algebras can be characterized by certain identities and the implication (5.2) as follows.

Theorem 12. *An algebra $\mathcal{A} = (A;t)$ of type (3) is a cyclic algebra if and only if it satisfies (5.2) and*

$$\begin{aligned}
&t(x,t(x,y,z),z) = t(x,y,z),\\
&t(t(x,y,z),z,x) = z,\\
&t(x,t(x,y,t(x,z,u)),u) = t(x,y,t(x,z,u)),\\
&t(x,x,x) = x.
\end{aligned}$$

Proof. Assume that $\mathcal{A} = (A;t)$ satisfies the above identities and (5.2). By Theorem 1 and the first identity, t is an assigned operation of a certain centred ternary relation C on A. By Theorem 2 and the second and third identity, C is cyclic and cyclically transitive. The fourth identity gets $(x,x,x) \in C$ for each $x \in A$. Finally, Lemma 6 yields that C is asymmetric and hence a cyclic order on A. Of course, t is an assigned operation of C and hence $\mathcal{A} = (A;t)$ is a cyclic algebra.

The converse follows directly by Definition 11. $\square$

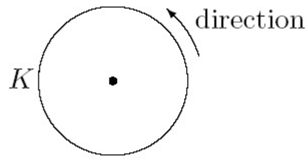

FIGURE 2.

Example 7. Let K be a circle in a plane with a given direction, see FIGURE 2. Define a ternary relation C on K as follows:

$(a,a,a) \in C$ for each $a \in K$ and
$(a,b,c) \in C$ if $a \to b$ and $b \to c$ for $a \neq b \neq c$.

It is an easy exercise to check that C is a cyclic order on K. If $a,b \in K$ then either $a = b$ and hence $Z_C(a,a) = \{a\}$ or $a \neq b$ thus $Z_C(a,b)$ equals the arc of K between a and b, i. e. it contains a continuum of points. Hence C is centred. For any assigned operation t, the algebra $\mathcal{A}(C) = (K;t)$ is a cyclic algebra.

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