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Every Skew Effect Algebra can be Extended into a Total Algebra

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Skew effect algebras were already introduced as a non-associative modification of the so-called effect algebras which serve as an algebraic axiomatization of the propositional logic of quantum mechanics. Since skew effect algebras have a partial binary operation, we search for an algebra with a total binary operation which extends a given skew effect algebra and such that the underlying posets coincide. It turns out that the suitable candidate is a skew basic algebra introduced here. Algebraic properties of skew basic algebras are described and they are compared with the so-called pseudo basic algebras introduced by the authors recently.

Keywords: Skew basic algebra, pseudo basic algebra, commutative directoid, skew effect algebra, sectional switching involutions.

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Effect algebras were introduced by D.J. Foulis and M.K. Bennett [11] as a tool for axiomatization of propositional logic of quantum mechanics. Their non-associative modification was defined by the first author and H. Länger [7] under the name skew effect algebras. It turns out that this modification is successful and can be represented e.g. by means of the so-called conditionally skew residuated structures in the same manner as it was done for effect algebras by means of conditionally residuated structures by R. Halaš and the first author. It was shown in [7] that every skew effect algebra is in fact an ordered set with sectional switching involutions and an antitone global involution. This motivates us to describe this structure by a total algebra (with everywhere defined operations) similarly as it was done for effect algebras in [5]. However, a similar total algebra has been introduced recently by the authors and J. Krňávek [6] under the name pseudo basic algebra. In this paper we compare these algebras and yield several results describing ordered sets underlying these algebras.

1 PRELIMINARIES

Recall from [7] that by a **skew effect algebra** is meant a partial algebra $\mathcal{S} = (S; +, ', 0, 1)$ of type (2, 1, 0, 0) satisfying the following axioms:

(S1) if $x + y$ is defined then so is $y + x$ and $x + y = y + x$
(S2) $x + y = 1$ if and only if $y = x'$
(S3) if $x + 1$ is defined then $x = 0$
(S4) if $x + y = z$ then $x' = z' + y$
(S5) if $x' + (x + y)$ is defined then $y = 0$
(S6) if $(x + y) + z$ is defined then there exists an element $u \in S$ such that $(x + y) + z = x + u$.

The element x' is called a **supplement of** x.

It was shown in [7] that every effect algebra is a skew effect algebra and, moreover, a skew effect algebra is an effect algebra if and only if the partial operation $+$ is associative.

If $\mathcal{S} = (S; +, ', 0, 1)$ is a skew effect algebra and a binary relation $\leq$ is defined by

$$x \leq y \text{ if there exists } z \in S \text{ with } y = x + z$$

then $\leq$ is a partial order on S and 0 is the least and 1 the greatest element. It will be called an **induced order** of $\mathcal{S}$. A skew effect algebra $\mathcal{S}$ is called a **lattice skew effect algebra** if the induced ordered set $(S; \leq)$ is a lattice. Moreover, the mapping $x \mapsto x'$ is an antitone involution on S (i.e. $x'' = x$ and $x \leq y$ yields $y' \leq x'$). Defining for $a \in S$ a mapping $x \mapsto x^a$ on the interval (called **section**) $[a, 1]$ by $x^a = x' + a$, it is the so-called **sectional switching involution**, i.e. for every $x \in [a, 1]$ we have $x^{aa} = x$ and $a^a = 1$, $1^a = a$. Since this sectional involution is defined for every element $a \in S$, we define the so-called **induced poset with sectional switching involutions** $(S; \leq, (^a)_{a \in S}, 0, 1)$, where $x^0 = x'$ is the so-called **global antitone involution**.

It was proved in [7] that also conversely, if $(S; \leq, (^a)_{a \in S}, 0, 1)$ is such a poset and we define $x + y = (y^0)^x$ for $x \leq y^0$ then we get an induced skew effect algebra $\mathcal{A}(\mathcal{S})$. Moreover, these assignments of induced poset with sectional switching involutions and of a skew effect algebra $\mathcal{A}(\mathcal{S})$ are one-to-one correspondences.

2 SKEW BASIC ALGEBRAS

Our aim is to define an algebra (with everywhere defined operations) whose induced ordered set would have the same properties as that of a skew effect algebra. For this, we borrow the name from basic algebras which play a similar role for effect algebras, see [5].

Definition 1. *An algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$ *is called a* ***skew basic algebra*** *if it satisfies the following axioms:*

(A1) $x \oplus 0 = x$
(A2) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$
(A3) $x \oplus (\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus z) = 1$, *where* $1 = \neg 0$
(A4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
(A5) $(\neg(x \oplus y) \oplus y) \oplus x = 1$
(A6) $x \oplus y = y \oplus \neg(\neg(x \oplus y) \oplus y)$.

The following lemma gets several important properties of skew basic algebras.

Lemma 1. *Every skew basic algebra satisfies the following:*

(i) $\neg x \oplus x = 1$
(ii) $\neg\neg x = x$
(iii) $\neg x \oplus (y \oplus x) = 1$
(iv) $x \oplus \neg x = 1$
(v) $1 \oplus x = 1 = x \oplus 1$
(vi) $0 \oplus x = x$.

Proof.

(i) Putting $y = 0$ in (A5) and applying (A1) twice, we obtain $1 = (\neg(x \oplus 0) \oplus 0) \oplus x = \neg x \oplus x$.
(ii) Put $y = 0$ in (A2). We get $\neg\neg x = \neg(\neg(x \oplus 0) \oplus 0) \oplus 0 = x \oplus 0 = x$ using (A1) four times.

(iii) If we substitute $\neg x$ instead of x and $y \oplus x$ instead of y in (A2) and use (A4), (A2) and (i), we compute

$$\begin{aligned}\neg x \oplus (y \oplus x) &= \neg(\neg(\neg x \oplus (y \oplus x)) \oplus (y \oplus x)) \oplus (y \oplus x)\\ &= \neg(\neg(\neg(y \oplus x) \oplus x) \oplus x) \oplus (y \oplus x)\\ &= \neg(y \oplus x) \oplus (y \oplus x) = 1.\end{aligned}$$

(iv) follows directly from (i) by substitution $\neg x$ instead of x and applying (ii).

(v) By (iii) we have $\neg\neg x \oplus (x \oplus \neg x) = 1$. Applying (ii) and (iv) we get $x \oplus 1 = 1$. Further, (iii) yields $1 \oplus (x \oplus 0) = 1$ which gets $1 \oplus x = 1$ by (A1).

(vi) Putting $y = 0$ in (A4) and applying (A1) and (ii), we obtain $x = \neg(\neg 0 \oplus x) \oplus x = \neg(1 \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x$ using (v).

For the next, let us recall that an algebra $\mathcal{D} = (D; \sqcup)$ of type (2) is called a **commutative directoid** (see e.g. [8]) if it satisfies the axioms

(D1) $x \sqcup x = x$

(D2) $x \sqcup y = y \sqcup x$

(D3) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

If $\mathcal{D} = (D; \sqcup)$ is a commutative directoid and $\leq$ is a binary relation on D given as follows

$$x \leq y \quad \text{if and only if} \quad x \sqcup y = y$$

then $(D; \leq)$ is an ordered set which is **directed**, i.e. the set of upper bounds, so-called upper cone $U(a, b) = \{x \in D; a \leq x \text{ and } b \leq x\} \neq \emptyset$ for all $a, b \in D$. Also conversely, if $(D; \leq)$ is a directed ordered set and we define a binary operation $\sqcup$ as follows

- if $x \leq y$ then $x \sqcup y = y = y \sqcup x$
- if neither $x \leq y$ nor $y \leq x$ then $x \sqcup y = y \sqcup x \in U(x, y)$ is an arbitrary element from the upper cone $U(x, y)$

then $(D; \sqcup)$ is a commutative directoid. Let us note that every ordered set with a greatest element 1 is directed since $1 \in U(x, y)$ for all $x, y \in D$.

Lemma 2. *Let $(A; \oplus, \neg, 0)$ be a skew basic algebra. Define $x \leq y$ for $x, y \in A$ if $\neg x \oplus y = 1$. Then $(A; \leq, 0, 1)$ is an ordered set with the greatest element 1 and the least element 0.*

Proof. By (i) of Lemma 1, $\leq$ is reflexive. Assume $x \leq y$ and $y \leq x$. Then $\neg x \oplus y = 1$ and $\neg y \oplus x = 1$. These yield $x = 0 \oplus x = \neg 1 \oplus x = \neg(\neg y \oplus x) \oplus x = \neg(\neg x \oplus y) \oplus y = \neg 1 \oplus y = 0 \oplus y = y$ by (vi) of Lemma 1 and the axiom (A4). Suppose now $x \leq y$ and $y \leq z$. Then $\neg x \oplus y = 1$ and $\neg y \oplus z = 1$. We compute by (A3)

$$\begin{aligned}\neg x \oplus z &= \neg x \oplus (0 \oplus z) = \neg x \oplus (\neg(\neg y \oplus z) \oplus z)\\ &= \neg x \oplus (\neg(\neg(0 \oplus y) \oplus z) \oplus z)\\ &= \neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1.\end{aligned}$$

Hence, $x \leq z$ thus $\leq$ is reflexive, antisymmetric and transitive relation on A, i.e. $(A; \leq)$ is an ordered set. By (v) of Lemma 1 we conclude $\neg x \oplus 1 = 1$ thus $x \leq 1$ for each $x \in A$. Further, $\neg 0 \oplus x = 1 \oplus x = 1$ yields $0 \leq x$ for each $x \in A$.

The order $\leq$ defined on a skew basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ by

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1$$

will be called an **induced order** of $\mathcal{A}$.

Lemma 3. *Let $(A; \oplus, \neg, 0)$ be a skew basic algebra, $\leq$ its induced order. Then $\neg(\neg x \oplus y) \oplus y \in U(x, y)$ for all $x, y \in A$.*

Proof. By (iii) of Lemma 1 we have $y \leq \neg(\neg x \oplus y) \oplus y$. Due to (A4) we obtain $x \leq \neg(\neg y \oplus x) \oplus x = \neg(\neg x \oplus y) \oplus y$ thus $\neg(\neg x \oplus y) \oplus y$ is a common upper bound of x, y.

Theorem 1. *The axioms (A1)–(A6) are independent.*

Proof.

(a) Define $\neg 0 = 1, \neg 1 = 0$ and the constant operation $x \oplus y = 1$ for every $x, y \in \{0, 1\}$. Then $(\{0, 1\}; \oplus, \neg, 0)$ satisfies (A2)–(A6) but not (A1) because $0 \oplus 0 = 1 \neq 0$.

(b) Define $\neg 0 = \neg 1 = 1$ and $0 \oplus 0 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1 \oplus 1 = 1$. Then $(\{0, 1\}; \oplus, \neg, 0)$ satisfies (A1), (A3)–(A6) but not (A2) because $\neg(\neg(0 \oplus 0) \oplus 0) \oplus 0 = 1 \neq 0 = 0 \oplus 0$.

(c) Define $\oplus$ and $\neg$ by the following tables

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	b	1	b	1
b	b	a	1	1	1	1
c	c	1	1	a	d	1
d	d	b	1	c	c	1
1	1	1	1	1	1	1

x	0	a	b	c	d	1
$\neg x$	1	c	d	a	b	0

Then $(\{0, a, b, c, d, 1\}; \oplus, \neg, 0)$ satisfies (A1), (A2) and (A4)–(A6) but not (A3) because

$$a \oplus (\neg(\neg(\neg(a \oplus a) \oplus a) \oplus b) \oplus b) = b \neq \neg 0 = 1.$$

(d) Define $\neg 0 = 1$, $\neg 1 = 0$ and $0 \oplus 0 = 1 \oplus 1 = 0$, $1 \oplus 0 = 0 \oplus 1 = 1$. Then $\mathcal{A} = (\{0, 1\}; \oplus, \neg, 0)$ satisfies (A1)–(A3), (A5) and (A6) but not (A4) because $\neg(\neg 0 \oplus 1) \oplus 1 = 0 \neq 1 = \neg(\neg 1 \oplus 0) \oplus 0$.

(e) Define $\oplus$ and $\neg$ by the following tables

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	1	1	b	1
b	b	b	1	1	1
c	c	b	1	b	1
1	1	1	1	1	1

x	0	a	b	c	1
$\neg x$	1	a	c	b	0

Then $(\{0, a, b, c, 1\}; \oplus, \neg, 0)$ satisfies (A1)–(A4) and (A6) but not (A5) because

$$(\neg(a \oplus b) \oplus b) \oplus a = b \neq \neg 0 = 1.$$

(f) Finally, define $\oplus$ and $\neg$ by the following tables

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	1	a	1	1
b	b	c	c	1	1
c	c	1	1	1	1
1	1	1	1	1	1

x	0	a	b	c	1
$\neg x$	1	a	c	b	0

Then $(\{0, a, b, c, 1\}; \oplus, \neg, 0)$ satisfies (A1)–(A5) but not (A6) because

$$a \oplus b = a \neq c = b \oplus \neg(\neg(a \oplus b) \oplus b).$$

Theorem 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a skew basic algebra, $\leq$ its induced order. Define $1 = \neg 0$, $x \sqcup y = \neg(\neg x \oplus y) \oplus y$ and for $a \leq x$, $x^a = \neg x \oplus a$. Then $\mathcal{D}(\mathcal{A}) = (A; \sqcup, ({}^a)_{a \in A}, 0, 1)$ is a bounded commutative directoid with sectional switching involutions where*

(a) the global involution $x \mapsto x^0$ is antitone
(b) if $y \leq x$ then $x^y = (y^0)^{(x^0)}$.

Moreover, the order of the directoid $(A; \sqcup)$ coincides with that of $\mathcal{A}$.

Proof. By (A4) we have $x \sqcup y = y \sqcup x$. By (i) and (vi) of Lemma 1 we get $x \sqcup x = x$. Applying (A3) and (vi) of Lemma 1 we obtain

$$\begin{aligned} x \sqcup ((x \sqcup y) \sqcup z) &= \neg(\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z)) \\ &\quad \oplus(\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) \\ &= \neg 1 \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = (x \sqcup y) \sqcup z. \end{aligned}$$

Hence, $(A; \sqcup)$ is a commutative directoid. Further, $x \sqcup 0 = \neg(\neg x \oplus 0) \oplus 0 = \neg\neg x = x$ and $x \sqcup 1 = \neg(\neg x \oplus 1) \oplus 1 = 1$ thus it is a bounded commutative directoid. Of course, if $\leq$ is the order of the directoid $(A; \sqcup)$ and $x \leq y$ then $x \sqcup y = y$ and, by (A2) and (i) of Lemma 1,

$$\neg x \oplus y = \neg(\neg(\neg x \oplus y) \oplus y) \oplus y = \neg(x \sqcup y) \oplus y = \neg y \oplus y = 1.$$

Conversely, if $\neg x \oplus y = 1$ then $x \sqcup y = \neg(\neg x \oplus y) \oplus y = \neg 1 \oplus y = 0 \oplus y = y$ thus $x \leq y$ in the directoid $(A; \sqcup)$. Together,

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1, \qquad (*)$$

i.e. $\leq$ is the induced order of $\mathcal{A} = (A; \oplus, \neg, 0)$. Assume $a \leq x$. Then, by (iii) of Lemma 1, $\neg a \oplus (\neg x \oplus a) = 1$ thus $a \leq \neg x \oplus a = x^a$, i.e. $x \mapsto x^a$ is a mapping of $[a, 1]$ into itself. Further, $x^{aa} = \neg(\neg x \oplus a) \oplus a = x \sqcup a = x$. Next, $a^a = \neg a \oplus a = 1$ and $1^a = \neg 1 \oplus a = 0 \oplus a = a$ thus $x \mapsto x^a$ is a sectional switching involution for each $a \in A$. Finally, if $x \leq y$ then, by (A5),

$$y \oplus \neg x = (x \sqcup y) \oplus \neg x = (\neg(\neg x \oplus y) \oplus y) \oplus \neg x = 1$$

thus, by (∗), $\neg y \leq \neg x$, i.e. $\neg$ is a global antitone involution $x \mapsto x^0$ on A. Moreover, if $y \leq x$ then, by (A6) and (ii) of Lemma 1,

$$\begin{aligned} x^y &= \neg x \oplus y = y \oplus \neg(\neg(\neg x \oplus y) \oplus y) = y \oplus \neg(x \sqcup y) \\ &= y \oplus \neg x = \neg\neg y \oplus \neg x = (\neg y)^{(\neg x)} = (y^0)^{(x^0)}. \end{aligned}$$

In what follows, we can prove the converse.

Theorem 3. *Let $\mathcal{D} = (D; \sqcup, (^a)_{a\in D}, 0, 1)$ be a bounded commutative directoid with sectional switching involutions such that the global involution $x \mapsto x^0$ is antitone and $y \leq x \Rightarrow x^y = (y^0)^{(x^0)}$. Define $x \oplus y = (x^0 \sqcup y)^y$ and $\neg x = x^0$. Then $\mathcal{A}(\mathcal{D}) = (D; \oplus, \neg, 0)$ is a skew basic algebra.*

Proof.

(A1) $x \oplus 0 = (x^0 \sqcup 0)^0 = x^{00} = x$.

(A2) $\neg(\neg(x \oplus y) \oplus y) \oplus y = (((((x^0 \sqcup y)^y)^{00} \sqcup y)^y)^{00} \sqcup y)^y = (((x^0 \sqcup y)^y \sqcup y)^y \sqcup y)^y = (x^0 \sqcup y)^y = x \oplus y$.

(A3) $x \oplus (\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus z) = (x^0 \sqcup ((((x^0 \sqcup y)^y \sqcup y)^y \sqcup z)^z \sqcup z)^z)^{((((x^0\sqcup y)^y\sqcup y)^y\sqcup z)^z\sqcup z)^z} = (x^0 \sqcup ((x^0 \sqcup y) \sqcup z))^{((x^0\sqcup y)\sqcup z)} = ((x^0 \sqcup y) \sqcup z)^{((x^0\sqcup y)\sqcup z)} = 1$.

(A4) $\neg(\neg x \oplus y) \oplus y = ((x \sqcup y)^y \sqcup y)^y = (x \sqcup y)^{yy} = x \sqcup y$, analogously $\neg(\neg y \oplus x) \oplus x = y \sqcup x = x \sqcup y$.

(A5) $\neg(x \oplus y) \oplus y = \neg x \sqcup y$ and hence $(\neg(x \oplus y) \oplus y) \oplus x = (\neg(\neg x \sqcup y) \sqcup x)^x$. Since $\neg$ is an antitone involution, we obtain $\neg x \sqcup y \geq \neg x \Rightarrow \neg(\neg x \sqcup y) \leq \neg\neg x = x$, thus $(\neg(\neg x \sqcup y) \sqcup x)^x = x^x = 1$.

(A6) $y \oplus \neg(\neg(x \oplus y) \oplus y) = y \oplus \neg(\neg x \sqcup y) = (\neg y \sqcup \neg(\neg x \sqcup y))^{\neg(\neg x\sqcup y)} = (\neg y)^{\neg(\neg x\sqcup y)}$.

By the assumption, it is equal to $(\neg x \sqcup y)^y = x \oplus y$.

Let us note that if $\mathcal{A} = (A; \oplus, \neg, 0)$ is a skew basic algebra and $\mathcal{D}(\mathcal{A})$ the commutative directoid assigned by Theorem 2 then $\mathcal{A}(\mathcal{D}(\mathcal{A})) = \mathcal{A}$.

Example 1. *Let $A = \{0, a, b, c, d, 1\}$. Consider the skew basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are given by the tables*

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	c	1	1
b	b	d	c	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	1	1
1	1	1	1	1	1	1

x	0	a	b	c	d	1
$\neg x$	1	d	c	b	a	0

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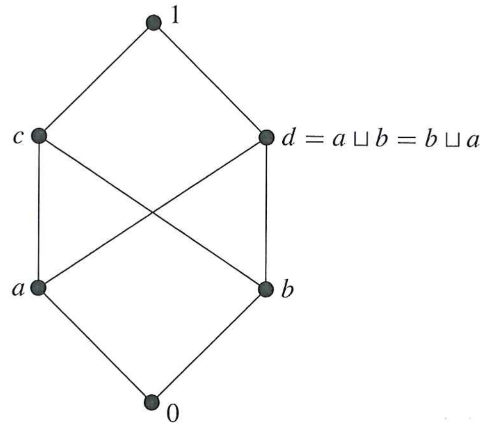

FIGURE 1

Its underlying poset is depicted in Figure 1. Note that the operation $\oplus$ is commutative, i.e. $x \oplus y = y \oplus x$ for each $x, y \in A$. By Theorem 2, $a \sqcup b = b \sqcup a = \neg(\neg a \oplus b) \oplus b = d$ and $a \sqcup c = c$, $a \sqcup d = d$, $b \sqcup c = c$, $b \sqcup d = d$, $c \sqcup d = 1$ determining the commutative directoid $\mathcal{D}(\mathcal{A})$.

The sectional switching involutions are as follows:

In $[0, 1]$, $x^0 = \neg x$.
In $[a, 1]$ it is: $a^a = 1$, $c^a = \neg c \oplus a = d$, $d^a = \neg d \oplus a = c$, $1^a = a$.
In $[b, 1]$ it is: $b^b = 1$, $c^b = \neg c \oplus b = c$, $d^b = \neg d \oplus b = d$, $1^b = b$.
In $[c, 1]$, $[d, 1]$ and $[1, 1]$ it is determined uniquely.

Note that here the involutions are antitone in each section.

Example 2. *Consider the skew basic algebra $\mathcal{A} = (\{0, a, b, c, d, e, 1\}; \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are given by the tables*

$\oplus$	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	e	c	d	d	1	1
b	b	c	d	e	1	e	1
c	c	d	e	1	1	1	1
d	d	a	1	1	1	1	1
e	e	1	b	1	1	1	1
1	1	1	1	1	1	1	1

x	0	a	b	c	d	e	1
$\neg x$	1	e	d	c	b	a	0

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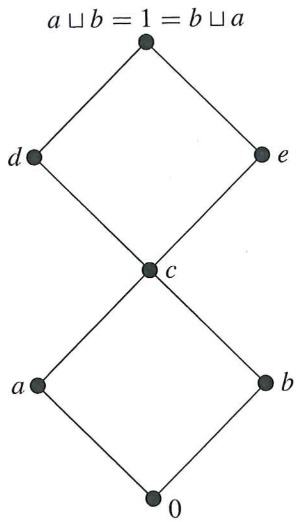

FIGURE 2

Note that the operation $\oplus$ *is not commutative since e.g.* $d \oplus a = a \neq d = a \oplus d$. *The induced bounded commutative directoid* $\mathcal{D}(\mathcal{A})$ *is visualized in Figure 2. Let us note that this poset is a lattice with respect to suprema and infima but it is not a semilattice with respect to the directoid operation* $\sqcup$ *because* $a \sqcup b = b \sqcup a = \neg(\neg a \oplus b) \oplus b = \neg b \oplus b = 1$ *but* $\sup(a, b) = c$.

The sectional switching involutions are as follows:

In $[0, 1]$, $x^0 = \neg x$.
In $[a, 1]$ *it is:* $a^a = 1, c^a = d, d^a = c, e^a = e, 1^a = a$.
In $[b, 1]$ *it is:* $b^b = 1, c^b = e, d^b = d, e^b = c, 1^b = b$.
In $[c, 1]$ *it is:* $c^c = 1, d^c = e, e^c = d$ *and* $1^c = c$.
In $[d, 1], [e, 1]$ *and* $[1, 1]$ *it is determined uniquely.*

Note that the involutions in sections $[a, 1]$ *and* $[b, 1]$ *are not antitone because*

$$c \leq e \quad \text{but} \quad c^a = d \parallel e = e^a$$

and

$$c \leq d \quad \text{but} \quad c^b = e \parallel d = d^b.$$

A skew basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ will be called a **lattice skew basic algebra** if the induced poset $(A; \leq)$ is a lattice and $x \vee y = \neg(\neg x \oplus y) \oplus y$.

Of course, since $x \mapsto x^0 = \neg x$ is an antitone involution, we get $x \wedge y = \neg(\neg x \vee \neg y)$ by DeMorgan laws.

Theorem 4. *A skew basic algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$ *is a lattice skew basic algebra if and only if it satisfies the axiom*

(L) $\neg(\neg x \oplus (\neg(\neg y \oplus z) \oplus z)) \oplus (\neg(\neg y \oplus z) \oplus z) = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z$.

Proof. Applying Theorem 2, the axiom (L) can be translated in the language of directoids as follows

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z. \qquad (**)$$

It is well-known (see e.g. [8]) that a directoid is a semilattice if and only if the operation $\sqcup$ is associative, i.e. if and only if it satisfies $(**)$. Then $\sqcup = \vee$ and due to Theorem 2 and DeMorgan laws, $(A; \leq)$ is a lattice where $x \wedge y = \neg(\neg x \vee \neg y)$.

In the previous two examples we have seen directoids, which are not lattices. Obviously the identity (L) does not hold for them, e.g. for $x = a, y = b$ and $z = c$ because

$$\neg(\neg a \oplus (\neg(\neg b \oplus c) \oplus c)) \oplus (\neg(\neg b \oplus c) \oplus c) = \neg 1 \oplus c = c \neq 1$$
$$= \neg c \oplus c = \neg(\neg(\neg(\neg a \oplus b) \oplus b) \oplus c) \oplus c.$$

We finish this section with an interesting connection between skew basic algebras and MV-algebras.

Theorem 5. *A skew basic algebra is an MV-algebra if and only if the operation* $\oplus$ *is associative.*

Proof. We need to prove that if the operation $\oplus$ is associative then the skew basic algebra satisfies the following identities:

(M1) $x \oplus 0 = x$
(M2) $\neg\neg x = x$
(M3) $x \oplus \neg 0 = \neg 0$
(M4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
(M5) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
(M6) $x \oplus y = y \oplus x$.

These identities define an MV-algebra $\mathcal{A} = (A; \oplus, \neg, 0)$. The second author proved in [12] that axiom (M6) is redundant, i.e. that it follows from the remaining axioms: (M1)–(M5). These five axioms follow directly from (A1) and (A4) of the axiomatic system of skew basic algebra, from the conditions (ii) and (v) of Lemma 1 and from the assumption that the operation $\oplus$ is associative.

Conversely, it is clear that every MV-algebra is a skew basic algebra whose operation $\oplus$ is associative.

3 THE COMPLETION OF SKEW EFFECT ALGEBRAS

Let $\mathcal{S} = (S; +, ', 0, 1)$ be a skew effect algebra and $\leq$ its induced order. As proved in [7], the operation $x + y$ is defined if and only if the elements x, y are **orthogonal**, i.e. if $x \leq y'$ which is equivalent to $y \leq x'$. Then $x + y = (y')^x = (x')^y$ in the induced poset with sectional switching involutions.

Theorem 6. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a skew basic algebra and $\leq$ its induced order. Define $1 = \neg 0$, $x' = \neg x$ and*

$$x + y = x \oplus y \quad \text{if and only if} \quad x \leq y'.$$

Then $\mathcal{S}(\mathcal{A}) = (A; +, ', 0, 1)$ is a skew effect algebra whose induced order coincides with $\leq$.

Proof.

(S1) If $x + y$ is defined then $x \leq y'$ which is equivalent to $y \leq x'$ due to antitony of the global involution $x \mapsto x'$ and hence also $y + x$ exists. Further, by (A6), we have $x + y = x \oplus y = y \oplus \neg(\neg(x \oplus y) \oplus y) = y \oplus \neg(\neg(\neg\neg x \oplus y) \oplus y) = y \oplus \neg(\neg x \sqcup y) = y \oplus x = y + x$, using $y \leq \neg x$.

(S2) Clearly $x = \neg\neg x = (x')'$ and hence $x + x'$ is defined and $x + x' = x \oplus \neg x = 1$. Conversely, if $x + y = 1$ then $x \leq y'$ if and only if $y \leq x'$ and $x \oplus y = 1$ if and only if $\neg\neg x \oplus y = 1$ whence $x' \leq y$. Thus $y = x'$.

(S3) If $x + 1$ is defined then $x \leq 1' = 0$ and hence $x = 0$.

(S4) If $x + y = z$ then $x \leq y'$ whence $y \leq x'$ and thus $x' = \neg x = \neg x \sqcup y = \neg(\neg\neg x \oplus y) \oplus y = \neg(x \oplus y) \oplus y = (x + y)' + y = z' + y$.

(S5) Assume $x' + (x + y)$ to be defined. Then $x' \leq (x + y)'$, i.e. $x + y \leq x$. However, $x \leq y'$ whence $x \leq x + y$ and thus $y + x = x + y = x$ according to (S1). Applying (S4) and (S2), we have $y' = x' + x = 1$, i.e. $y = 1' = 0$.

(S6) Suppose $(x + y) + z$ to be defined. Put $d := x + y$ and $e := d + z$. Then $x \leq y'$ and hence $x \leq x + y = d$ and $d \leq z'$, and hence $d \leq d + z = e$. Now $x \leq d \leq e$ yields $x \leq e = \neg\neg e$. Thus, $x + e'$ is defined and so is $e' + x$ according to (S1). Applying (S4) and (S1), from $e' + x = e' + x$ we conclude $(x + y) + z = d + z = e = \neg\neg e = (e' + x)' + x = x + (e' + x)'$.

Obviously, the induced order of the skew effect algebra $(A; +, ', 0, 1)$ coincides with $\leq$.

Example 3. *Let $A = \{0, a, b, c, d, 1\}$. Consider the skew basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are given by the tables*

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	c	1	1
b	b	c	d	1	d	1
c	c	a	1	1	1	1
d	d	1	b	1	1	1
1	1	1	1	1	1	1

x	0	a	b	c	d	1
$\neg x$	1	d	c	b	a	0

.

Its underlying poset is depicted in Figure 3, $x \sqcup y = \neg(\neg x \oplus y) \oplus y$. Now, by Theorem 6, we define $x' = \neg x$ and

$$x + y = x \oplus y \quad \text{if and only if} \quad x \leq y'.$$

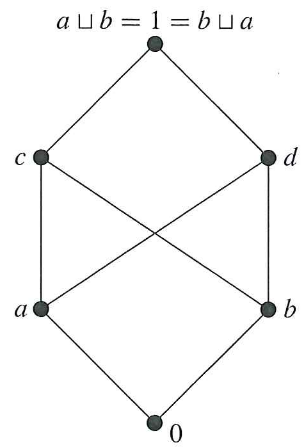

FIGURE 3

Then $\mathcal{S}(\mathcal{A}) = (A; +, ', 0, 1)$ is a skew effect algebra where the partial operation $+$ is given by the following table

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	–	1	–
b	b	c	d	1	–	–
c	c	–	1	–	–	–
d	d	1	–	–	–	–
1	1	–	–	–	–	–

Let us note that the partial operation $+$ in the skew effect algebra $\mathcal{S}(\mathcal{A})$ is commutative contrary to the fact that skew basic algebra $\mathcal{A}$ is not commutative because e.g. $a \oplus c \neq c \oplus a$.

Now, we are ready to describe a completion of a skew effect algebra into a total algebra which is a skew basic algebra.

Theorem 7. *Let $\mathcal{S} = (S; +, ', 0, 1)$ be a skew effect algebra and $\leq$ its induced order. Then there exists a skew basic algebra $\mathcal{A}(\mathcal{S}) = (S; \oplus, \neg, 0)$ such that $\leq$ coincides with the induced order of $\mathcal{S}$ and*

$$x + y = x \oplus y \quad \text{if} \quad x \leq y'.$$

Moreover, if $\mathcal{S}$ is a lattice skew effect algebra then $\mathcal{A}(\mathcal{S})$ is a lattice skew basic algebra and the underlying lattices coincide.

Proof. Assume $\mathcal{S} = (S; +, ', 0, 1)$ is a skew effect algebra and $\leq$ its induced order, i.e. $x \leq y$ if there is $z \in S$ such that $y = x + z$. Since $(S; \leq)$ is directed, we can convert it into a bounded commutative directoid $\mathcal{D}(\mathcal{S}) = (S; \sqcup, (^a)_{a \in S}, 0, 1)$ with sectional switching involutions and the global antitone involution $x^0 = x'$, $x^a = x' + a$ for $x \in [a, 1]$, where

$x \sqcup y = y \sqcup x = x \vee y$ provided the supremum $x \vee y$ exists and arbitrarily $x \sqcup y = y \sqcup x \in U(x, y)$ otherwise.

Hence, $\mathcal{D}(\mathcal{S})$ is assigned in a non-unique way if $(S; \leq)$ is not a lattice but it is assigned uniquely if $(S; \leq)$ is a lattice.

Now, we define $\neg x = x^0$ and $x \oplus y = (x^0 \sqcup y)^y$. By Theorem 3, $\mathcal{A}(\mathcal{D}(\mathcal{S})) = (S; \oplus, \neg, 0)$ is a skew basic algebra. Moreover, the induced order of commutative directoid $(S; \sqcup)$ coincides with $\leq$ (see e.g. [7]), i.e. $x \leq y$ if and only if $x \sqcup y = y$ which is equivalent to

$$\neg x \oplus y = (x \sqcup y)^y = y^y = 1.$$

Hence, the underlying ordered sets of $\mathcal{S}$ and of $\mathcal{A}(\mathcal{D}(\mathcal{S}))$ coincide.

Now, if $x \leq y'$ (i.e. $x \leq y^0$ if and only if $y \leq x^0$), then

$$x \oplus y = (x^0 \sqcup y)^y = (x^0)^y = x + y$$

as mentioned in Preliminaries. The last assertion follows directly from the construction of $\sqcup$.

As shown above, a skew basic algebra $\mathcal{A}(\mathcal{S})$ is assigned to a skew effect algebra in a unique way if and only if $\mathcal{S}$ is a lattice skew effect algebra. On the other hand, we have shown that the assignment $\mathcal{S} \mapsto \mathcal{A}(\mathcal{S})$ captures the whole information about $\mathcal{S}$ for any assigned $\mathcal{A}(\mathcal{S})$ because $x + y = x \oplus y$ whenever $x + y$ is defined.

4 PSEUDO BASIC ALGEBRAS, SECTION SKEW BASIC ALGEBRAS AND CONGRUENCE PROPERTIES

The concept of a pseudo basic algebra was introduced by the authors and J. Krňávek [6] in the sake to get a non-associative generalization of MV-algebras. Another useful modification of MV-algebras are basic algebras, see e.g. [3], which need not be commutative or associative. However, it was shown by M. Botur [1] that every commutative basic algebra is either an MV-algebra or it is infinite. Let us mention that the second author proved in [12] that every associative basic algebra is necessarily an MV-algebra. For pseudo basic algebras, the situation is different. Namely, there exist finite commutative pseudo basic algebras which are not MV-algebras, see [6]. Hence, pseudo basic algebras serve as a successful modification of both MV-algebras and basic algebras. On the other hand, they cannot be used as extensions of skew effect algebras because orthogonal elements in pseudo basic algebras need not commute. Moreover, the underlying poset of a pseudo basic algebra is a $\vee$-semilattice which need not be the case of a skew effect algebra, see the previous examples. In what follows, we are going to show that every strict pseudo basic algebra can become a lattice skew basic algebra when the orthogonal elements commute.

Recall from [6] that an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities

(P1) $\neg x \oplus x = 1$, where $1 = \neg 0$
(P2) $x \oplus 0 = x$
(P3) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$
(P4) $\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z = \neg(\neg(\neg(\neg y \oplus z) \oplus z) \oplus x) \oplus x$

is called a **pseudo basic algebra**. If, moreover, $\mathcal{A}$ satisfies also the identity

(A) $(\neg(x \oplus y) \oplus y) \oplus x = 1$,

it is called a **strict pseudo basic algebra**.

The following proposition was proved in [6].

Proposition 1. *Every pseudo basic algebra satisfies the following identities:*

(i) $0 \oplus x = x$
(ii) $\neg\neg x = x$
(iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
(iv) $\neg x \oplus (y \oplus x) = 1$, *where* $1 = \neg 0$
(v) $x \oplus \neg x = 1$
(vi) $1 \oplus x = 1 = x \oplus 1$.

In a pseudo basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, the induced order $\leq$ is introduced by $x \leq y$ if and only if $\neg x \oplus y = 1$ and, if $\mathcal{A}$ is strict, then $(A; \leq)$ is a lattice where $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$ and $x \leq y$ implies $\neg x \leq \neg y$.

In the following example, we get a strict pseudo basic algebra, which is not a skew basic algebra.

Example 4. *Define* $\oplus$ *and* $\neg$ *on the set* $\{0, a, b, c, 1\}$ *by the following tables*

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	c	c	1	1
b	b	b	1	1	1
c	c	1	1	1	1
1	1	1	1	1	1

x	0	a	b	c	1
$\neg x$	1	c	b	a	0

.

One can easily check that $\mathcal{A} = (\{0, a, b, c, 1\}; \oplus, \neg, 0)$ *is a strict pseudo basic algebra.*

On the other hand $\mathcal{A}$ *does not satisfy the axiom (A6) because*

$$b \oplus a = b \neq c = a \oplus b = a \oplus \neg(\neg(b \oplus a) \oplus a)$$

and hence it is not a skew basic algebra.

We can show that the axiom (A6) is just the missing condition characterizing lattice skew basic algebras among strict pseudo basic algebras.

Theorem 8. *A strict pseudo basic algebra* $\mathcal{A}$ *is a lattice skew basic algebra if and only if it satisfies the identity (A6).*

Proof. We need to prove that strict pseudo basic algebras satisfy (A1)–(A5). Axioms (A1), (A2) and (A5) are part of an axiomatic system of strict pseudo basic algebras. Further axiom (A4) is the same as the condition (iii) of Proposition 1. Finally, by (P4), we have

$$\neg(\neg(\neg(\neg y \oplus z) \oplus z) \oplus \neg x) \oplus \neg x = \neg(\neg(\neg(\neg\neg x \oplus y) \oplus y) \oplus z) \oplus z. \quad (1)$$

By (iv) and (ii) of Proposition 1, we obtain

$$x \oplus (y \oplus \neg x) = 1. \quad (2)$$

From (2), (1) and (ii) of Proposition 1, we infer (A3) immediately.

In what follows, we are checking if a section $[p, 1]$ of a given skew basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ can be organized into a skew basic algebra again. Of course, it is not possible for every $p \in A$ because the sectional involution x^p need not be antitone and, in the section $[p, 1]$, it should become a global involution. However, it is the only constraint as shown in the following.

Theorem 9. *Let* $\mathcal{A} = (A; \oplus, \neg, 0)$ *be a skew basic algebra and* $a \in A$. *On a section* $[a, 1]$ *we define* $x \oplus_a y = \neg(\neg x \oplus a) \oplus y$ *and* $\neg_a x = \neg x \oplus a$ *for all* $x, y \in [a, 1]$. *Then* $([a, 1]; \oplus_a, \neg_a, a)$ *is a skew basic algebra if and only if the sectional involution* x^a *in* $\mathcal{A}$ *is antitone.*

The proof can be easily done via assigned directoid with sectional switching involutions and hence it is omitted.

Example 5. *Consider the skew basic algebra of Example 1. Then the sectional switching involutions in the nontrivial sections* $[a, 1]$, $[b, 1]$ *are antitone and hence these sections can be converted into skew basic algebras. On the contrary, in the skew basic algebra of Example 2, the sectional switching involutions in the nontrivial sections* $[a, 1]$, $[b, 1]$ *are not antitone and hence these sections cannot be organized in skew basic algebras.*

Our next task is to check some important congruence properties of skew basic algebras.

Theorem 10. *The variety* $\mathcal{S}$ *of skew basic algebras is congruence regular and arithmetical.*

Proof. By a theorem of Csákány [9], see also Theorem 6.1.3 in [4], a variety is congruence regular if and only if there exist an integer n and ternary terms $t_1(x, y, z), \dots, t_n(x, y, z)$ such that

$$t_1(x, y, z) = \cdots = t_n(x, y, z) = z \quad \text{if and only if} \quad x = y.$$

We can take $n = 2$ and $t_1(x, y, z) = ((\neg x \oplus y) \sqcap (\neg y \oplus x)) \sqcap z$, $t_2(x, y, z) = \neg((\neg x \oplus y) \sqcap (\neg y \oplus x)) \oplus z$, where $a \sqcup b = \neg(\neg a \oplus b) \oplus b$ and the derived operation $\sqcap$ is defined in the variety $\mathcal{S}$ via the DeMorgan laws as follows: $a \sqcap b = \neg(\neg a \sqcup \neg b)$. Then

$$\begin{aligned} t_1(x, x, z) &= (\neg x \oplus x) \sqcap z = 1 \sqcap z = z \\ t_2(x, x, z) &= \neg 1 \oplus z = 0 \oplus z = z. \end{aligned}$$

If, conversely, $t_1(x, y, z) = t_2(x, y, z) = z$ then

$$z \leq (\neg x \oplus y) \sqcap (\neg y \oplus x). \tag{3}$$

As it follows from $t_2(x, y, z) = z$ and (3),

$$\begin{aligned} z &= \neg((\neg x \oplus y) \sqcap (\neg y \oplus x)) \oplus z = (((\neg x \oplus y) \sqcap (\neg y \oplus x)) \sqcup z)^z \\ &= ((\neg x \oplus y) \sqcap (\neg y \oplus x))^z. \end{aligned}$$

Since the involution in $[z, 1]$ is switching, it yields

$$(\neg x \oplus y) \sqcap (\neg y \oplus x) = 1$$

thus also $\neg x \oplus y = 1$ and $\neg y \oplus x = 1$. Hence $x \leq y$ and $y \leq x$ giving $x = y$. We have shown that the variety $\mathcal{S}$ is congruence regular.

To prove arithmeticity, it is enough to find a so-called Pixley term, i.e. a ternary term $m(x, y, z)$ satisfying the identities

$$m(x, x, z) = z, \quad m(x, z, z) = x \quad \text{and} \quad m(x, y, x) = x,$$

see e.g. Theorem 3.2.11 in [4]. In the variety $\mathcal{S}$, we can take

$$m(x, y, z) = ((\neg(\neg x \oplus y) \oplus z) \sqcap (\neg(\neg z \oplus y) \oplus x)) \sqcap (x \sqcup z).$$

Since the operation $\sqcap$ is derived from $\sqcup$ by using of the DeMorgan laws, absorption laws $a \sqcup (a \sqcap b) = a$ and $a \sqcap (a \sqcup b)$ are valid in the induced commutative directoid and hence we compute

$$\begin{aligned} m(x, x, z) &= ((\neg(\neg x \oplus x) \oplus z) \sqcap (\neg(\neg z \oplus x) \oplus x)) \sqcap (x \sqcup z) \\ &= (z \sqcap (z \sqcup x)) \sqcap (x \sqcup z) = z, \\ m(x, z, z) &= ((\neg(\neg x \oplus z) \oplus z) \sqcap (\neg(\neg z \oplus z) \oplus x)) \sqcap (x \sqcup z) \\ &= ((x \sqcup z) \sqcap x) \sqcap (x \sqcup z) = x, \end{aligned}$$

and, finally,

$$\begin{aligned} m(x, y, x) &= ((\neg(\neg x \oplus y) \oplus x) \sqcap (\neg(\neg x \oplus y) \oplus x)) \sqcap (x \sqcup x) \\ &= (\neg(\neg x \oplus y) \oplus x) \sqcap x. \end{aligned}$$

By (iii) of Lemma 1 we have $x \leq a \oplus x$ for all $a, x \in A$ thus $x \leq \neg(\neg x \oplus y) \oplus x$. This yields immediately $m(x, y, x) = x$ proving that $m(x, y, z)$ is really a Pixley term.

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Characterizations of Abel-Grassmann's Groupoids

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In this paper, we investigate some characterizations of regular and intra-regular Abel-Grassmann's groupoids in terms of $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy quasi-ideals.

Keywords: AG-groupoid, left invertive law, medial law, paramedial law and $(\in, \in \vee q_k)$-fuzzy (quasi-)ideal.

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1 INTRODUCTION

Usually the models of real world problems in almost all disciplines like engineering, medical sciences, mathematics, physics, computer science, management sciences, operations research and artificial intelligence are mostly full of complexities and consist of several types of uncertainties while dealing them in several occasion. To overcome these difficulties of uncertainties, many theories have been developed such as rough sets theory, probability theory, fuzzy sets theory, theory of vague sets, theory of soft ideals and the theory of intuitionistic fuzzy sets. Zadeh discovered the relationships of probability and fuzzy set theory which has appropriate approach to deal with uncertainties. Many authors have applied the fuzzy set theory to generalize the basic theories of Algebra. Mordeson *et al.* [16] has discovered the grand exploration