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FOUNDATIONS

Evolution of objects and concepts

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Abstract

The method for producing concepts within a given context was developed by R. Wille, and it is known under the name formal concept analysis. Every concept is fully determined by its extent and intent where extent is the set of all objects and intent the set of all attributes of this concept. We show in examples that in situations of real world this method need not be satisfactory because time dimension plays a crucial role in human thinking. Hence, it is necessary to consider tense operators on time depending objects or on the whole concepts. A formal method how to evaluate these operators is investigated in this paper.

Keywords Formal concept analysis · Formal context · Concept · Tense operators

1 Introduction

The bases of every reasoning are concepts. An effective method to determine all possible concepts within a given context was introduced by R. Wille; see, e.g., Ganter and Wille (1999) and numerous references there. We now briefly recap this method. Let a set O of *objects* be given as well as a set M of *attributes*. Besides of these, we have given a relation $R \subseteq O \times M$ such that for $g \in O$ and $m \in M$ we have $(g, m) \in R$ if and only if the object g shares the attribute m. The triple $\mathcal{K} = (O, M, R)$ is called *a (formal) context*. We usually assume that the context is given; however, it is possible to enlarge the set O or the set M whenever it is necessary. We consider the pairs (C, D) such that $C \subseteq O$ and $D \subseteq M$. Denoting

$$C' = \{m \in M;\ (g, m) \in R \text{ for each } g \in C\},$$

it is the set of all attributes shared by all the objects of C, and

$$D' = \{g \in O;\ (g, m) \in R \text{ for each } m \in D\},$$

analogously as above, it is the set of all objects which share all the attributes from D. The pair (C, D) is called *a concept* if $D = C'$ and $C = D'$. Then, C is called an *extent* and D an *intent* of the concept (C, D). One can easily check that $C \subseteq C''$, $C' = C'''$, $D \subseteq D''$ and $D' = D'''$. It was proved by R. Wille (see, e.g., Ganter and Wille 1999) that the set of all concepts $C(\mathcal{K})$ of a given context $\mathcal{K}$ forms a complete lattice with respect to the order $(C_1, D_1) \leqslant (C_2, D_2)$ if and only if $C_1 \subseteq C_2$ or, equivalently, $D_2 \subseteq D_1$. The least element of $C(\mathcal{K})$ is the concept $(\emptyset'', M)$ generated by the empty set of objects, and the greatest element is the concept (O, O'). If (C, D) is a concept, $g \in O$ an object and $g \in C$, we will write alternatively $g \in (C, D)$ to express that g belongs to this concept. Then, of course $\{g\}' \supseteq D$.

One should also mention that in Tříska and Vychodil (2017) the authors have introduced attribute implications

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(that represent data dependencies) annotated by time points and investigated their semantic entailment and its axiomatization.

For a given context $\mathcal{K} = (O, M, R)$, a procedure determining all concepts of $\mathcal{K}$ was developed. One of the first approaches was the computer program TOSCANA produced by P. Burmeister in 1980s. Nowadays, other fast computer programs and algorithms were settled by Andrews (2015) and Outrata and Vychodil (2012). This method is very appropriate because it enables to produce concepts automatically by computer, and hence, it can serve efficiently in artificial intelligence (AI). However, this method has its limits which will be discussed here and a modification will be proposed.

Consider a context which has among its attributes also "to be small," "to be a bowl" and "to have a handle." Within it, the concept of a cup can be determined as a thing having these three attributes. Up to now, all looks positively. Imagine a situation when you boil water for coffee and ask your friend to go into the kitchen and bring a cup from the table. When he comes back, he says: "There was only one cup but without a handle." For a man, such a proposition is possible. However, for AI in our sense it is impossible because there cannot be "a cup without handle," since a cup is defined as a small bowl *with* handle. How it is possible that a man can create such a concept? The answer is easy. He recognizes that this bowl had a handle in the past, but now it is broken. In other words, there *has been* a time when it was a cup, but now the handle is broken.

Another example concerns animals. You can go for a walk with your little son, and in the lake, you can see some small animals. Your son can ask what an animal it is. You can answer: "It is a young frog." However, in an appropriate context which contains attributes like "to be green," "to have legs," "to live in water," "to have a tail," a frog (e.g., green toad) can be determined as an animal having the first three attributes but does not have the last one. On the contrary, our animal called a tadpole has no legs, it is not green, but it has a tail. How you can call it "a young frog"? The answer is as follows. You know that in the future this animal will be green, will have legs and will not have a tail, i.e., it will be a frog. Thus, now it can be called "a young frog."

2 Tense operators

The foregoing two examples show that when creating concepts, time dimension plays an important role. The aim of our paper is to introduce this time dimension in the formal concept analysis.

Moreover, we develop also a general theory of closure operators on ordered sets with a couple of order-preserving mappings which can form Galois connection and are connected with that closure operator via a certain relation. This is useful for a more general setting which completes the study of tense operators on formal concepts which can be considered either in the fuzzification or in concept formalization in psychology (see, e.g., Bělohlávek (2002); Bělohlávek and Klir (2011) for details).

From now on, consider a couple $(T, \leqslant)$ where a non-empty set T is considered as *a timescale* and the relation $\leqslant$ (not necessary linear order) will express time preference, i.e., $t_1 \leqslant t_2$ for $t_1, t_2 \in T$ means that t_1 is *before* t_2 or, equivalently, t_2 is *after* t_1. The couple $(T, \leqslant)$ is called a *time frame*. In *temporary logic*, called also *tense logic*, see the paper (Burgess 1984) or, e.g., the monograph (Chajda and Paseka 2015), four of the so-called *tense operators* are introduced. They are denoted by the symbols P, F, H and G, and they are in fact modal operators whose intended meaning is as follows:

P "It has at some time been the case that …"
F "It will at some time be the case that …"
H "It has always been the case that …"
G "It will always be the case that …."

Now, we are going to show how the description of these operators can be formalized. Let a context $\mathcal{K} = (O, M, R)$ be given. The sets O, M are usually considered to be finite. Let (C, D) be a concept in this context $\mathcal{K}$ and let $g \in C$ be a given object. Denote by $\{a_1, \ldots, a_n\}$ the set of all attributes from M and put $g_i = 1$ if g shares the attribute a_i and $g_i = 0$ otherwise. Hence, to every $g \in C$ can be assigned a vector $g = (g_1, g_2, \ldots, g_n)$ whose entries are 1 or 0 depending on the relation R. We will identify the object g with this vector, and hence, it will be denoted by the same symbol. We can assume that $O \subseteq 2^M$.

As pointed in our examples, the nature of g can be changed in time, and thus, g will be considered to be *time depending* and denoted by $\widetilde{g} \in \left(2^M\right)^T \cong 2^{M\times T}$, i.e., $\widetilde{g}(t) = (g_1(t), \ldots, g_n(t))$ where $g_i(t) = 1$ if g shares the attribute a_i at time $t \in T$ and $g_i(t) = 0$ otherwise. It is the *vector assigned* to $\widetilde{g}(t)$. For the concept (C, D), denote by $v(D)$ the vector $(v_1, \ldots, v_n)$ such that $v_i = 1$ if $a_i \in D$ and $v_i = 0$ otherwise. This is the *vector assigned* to the concept (C, D). In the set $\{0, 1\}^n$, we can introduce a partial order as follows:

$$(v_1, \ldots, v_n) \leqslant (w_1, \ldots, w_n) \quad \text{if and only if} \quad v_i \leqslant w_i \quad \text{for all } i = 1, \ldots, n.$$

Let $\widetilde{O}$ be a system of time depending objects and $\widetilde{g}$ be from $\widetilde{O}$. Let (C, D) be a concept within $\mathcal{K} = (O, M, R)$ and let $t_0 \in T$. Then, we can formalize tense operators as follows

- $P(\widetilde{g})(t_0) \in (C, D)$ if there exists $t_1 \leqslant t_0$ such that $\widetilde{g}(t_1) = (g_1(t_1), \ldots, g_n(t_1)) \geqslant v(D)$; (1)

- $F(\widetilde{g})(t_0) \in (C, D)$ if there exists $t_2 \geqslant t_0$ such that $\widetilde{g}(t_2) = (g_1(t_2), \ldots, g_n(t_2)) \geqslant v(D)$; (2)
- $H(\widetilde{g})(t_0) \in (C, D)$ if $\widetilde{g}(t_1) = (g_1(t_1), \ldots, g_n(t_1)) \geqslant v(D)$ for all $t_1 \leqslant t_0$; (3)
- $G(\widetilde{g})(t_0) \in (C, D)$ if $\widetilde{g}(t_2) = (g_1(t_2), \ldots, g_n(t_2)) \geqslant v(D)$ for all $t_2 \geqslant t_0$. (4)

It is clear that if we have some evidence on behavior of a given object $\widetilde{g}$ during the time interval either in the past or in future, this evaluation can be provided by a formal procedure, and hence, it can be done by using a device of AI.

The next step is to consider an evolution of the whole concept (C, D), not only of a given object $\widetilde{g}$ from it. Using our previous examples, we can consider how the concept of "a cup" can be changed in the concept "a cup without handle" and a concept "young frog (alias tadpole)" can be changed into the concept "frog." For this, we can use our previous investigation of the tense operators on objects.

Let (C, D) be a concept from $\mathcal{K} = (O, M, R)$ and let $S \subseteq \widetilde{O}$ be a system of time depending objects. Assume that $t_0 \in T$ (meaning "now"). We define tense operators P, F, H and G as follows

- $P(S)(t_0) \sqsubseteq C$ if for each $\widetilde{g} \in S$ there exists $t_1 \leqslant t_0$ such that $\widetilde{g}(t_1) \in C$. This means "Every time depending object from S is or has been before in the concept (C, D)";
- $F(S)(t_0) \sqsubseteq C$ if for each $\widetilde{g} \in S$ there exists $t_2 \geqslant t_0$ such that $\widetilde{g}(t_2) \in C$. This means "Every time depending object from S is or will be in the concept (C, D)";
- $H(S)(t_0) \sqsubseteq C$ if for each $\widetilde{g} \in S$ and each $t_1 \leqslant t_0$ we have $\widetilde{g}(t_1) \in C$. This means "Every time depending object from S is or has been always in the concept (C, D)";
- $G(S)(t_0) \sqsubseteq C$ if for each $\widetilde{g} \in S$ and each $t_2 \geqslant t_0$ we have $\widetilde{g}(t_2) \in C$. This means "Every time depending object from S is or will be always in the concept (C, D)."

Reading these formulas carefully, one can mention that for every $\widetilde{g} \in S$ and all $t_0 \in T$ there can be different time t_1 or t_2 from T for which we have $P(g)(t_0) \in C$ or $F(g)(t_0) \in C$, respectively. However, it is in accordance with our everyday experience because not all handles of all cups will be broken at the same time and not all tadpoles become frogs during the same day.

To show how it can be checked whether $P(\widetilde{g})$ or $F(\widetilde{g})$ belong in a given concept (C, D), we start with the following example.

Example 1 We put O = {glass, pot, coffee cup, tee cup, white cup, decorated cup}, M = {bowl, small, with a handle, white, decoration}. Let $R \subseteq O \times M$ be given by the following table.

	glass	pot	coffee cup	tee cup	white cup	decorated cup
bowl	1	1	1	1	1	1
small	1	0	1	1	1	1
with a handle	0	1	1	1	1	1
white	0	0	0	1	1	0
decoration	0	0	1	0	0	1

Then, we have a context $\mathcal{K} = (O, M, R)$. Let D = {bowl, small, with a handle} be the intent of the context (C, D), where C = {coffee cup, tee cup, white cup, decorated cup}.

Assume that $T = \{t_0, t_1, t_2, t_3\}$ such that $t_3 \leqslant t_2 \leqslant t_1 \leqslant t_0$ is our time scale. Let further $\widetilde{g}_1, \widetilde{g}_2$ be two time depending objects given by the following tables

	t_3	t_2	t_1	t_0		t_3	t_2	t_1	t_0
$\widetilde{g}_1(t)$	$\begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\\1\end{pmatrix}$	$\widetilde{g}_2(t)$	$\begin{pmatrix}1\\1\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\1\\0\end{pmatrix}$

Neither $\widetilde{g}_1$ nor $\widetilde{g}_2$ are now, i.e., in the time t_0 contained in the context (C, D). But we immediately see that $\widetilde{g}_1$ was in the time t_1 or t_2 in (C, D) and $\widetilde{g}_2$ was in the time t_1 or t_2 or t_3 in (C, D). If we use a restriction of vectors g_i onto attributes from D, then this can be equivalently expressed as follows:

$$\max(\widetilde{g}_1(t)|D; t \leqslant t_0, t \in T) = \begin{pmatrix}1\\1\\1\end{pmatrix} = v(D)|D$$

and

$$\max(\widetilde{g}_2(t)|D; t \leqslant t_0, t \in T) = \begin{pmatrix}1\\1\\1\end{pmatrix} = v(D)|D.$$

The above reasoning can be extended from a single object $\widetilde{g}$ to a set S of time depending objects as follows.

Let $S = \{\widetilde{g}_1, \widetilde{g}_2\}$. The fact that both $\widetilde{g}_1$ and $\widetilde{g}_2$ were in the past in the context (C, D) can be expressed as follows:

$$\min(\max(\widetilde{g}_1(t)|D; t \leqslant t_0, t \in T), \max(\widetilde{g}_2(t)|D; t \leqslant t_0, t \in T)) = v(D)|D.$$

The above condition should read as follows:

$P(S)(t_0) \sqsubseteq C.$

The just-described procedure can be formulated in a general setting as follows.

Theorem 1 *Let a context $\mathcal{K} = (O, M, R)$ be given and (C, D) be a concept of $\mathcal{K}$. Let $(T, \leqslant)$ be a time frame, $t_0 \in T$ and $S \subseteq \left(2^M\right)^T$ be a subset of time depending objects. Then, $P(S)(t_0) \sqsubseteq C$ if and only if*

$$\min(\max(\widetilde{g}(t)|D;\, t \leqslant t_0, t \in T);\, \widetilde{g} \in S) = v(D)|D$$

and $F(S)(t_0) \sqsubseteq C$ if and only if

$$\min(\max(\widetilde{g}(t)|D;\, t \geqslant t_0, t \in T);\, \widetilde{g} \in S) = v(D)|D.$$

Proof Let $\widetilde{g} \in \left(2^M\right)^T$ be a time depending object. Then, we have the following sequence of equivalences:

$$\begin{aligned} P(\widetilde{g})(t_0) \in (C, D) &\iff \text{there exists } t_1 \leqslant t_0 \text{ such that } \widetilde{g}(t_1) \geqslant v(D) \\ &\iff \text{there exists } t_1 \leqslant t_0 \text{ such that } \widetilde{g}(t_1)|D = v(D)|D \\ &\iff \max(\widetilde{g}(t)|D;\, t \leqslant t_0, t \in T) = v(D)|D. \end{aligned}$$

It follows that

$$\begin{aligned} P(S)(t_0) \sqsubseteq C &\iff \text{for each } \widetilde{g} \in S \text{ we have } P(\widetilde{g})(t_0) \in (C, D) \\ &\iff \text{for each } \widetilde{g} \in S \ \max(\widetilde{g}(t)|D;\, t \leqslant t_0, t \in T) = v(D)|D \\ &\iff \min(\max(\widetilde{g}(t)|D;\, t \leqslant t_0, t \in T);\, \widetilde{g} \in S) = v(D)|D. \end{aligned}$$

By the same considerations, we obtain that

$$\begin{aligned} F(\widetilde{g})(t_0) \in (C, D) &\iff \max(\widetilde{g}(t)|D;\, t \geqslant t_0, t \in T) = v(D)|D \quad \text{and} \\ F(S)(t_0) \sqsubseteq C &\iff \min(\max(\widetilde{g}(t)|D;\, t \geqslant t_0, t \in T);\, \widetilde{g} \in S) = v(D)|D. \end{aligned}$$

□

Remark 1 Reformulating the above condition in the language of subsets, we have that

$$P(\widetilde{g})(t_0) \in (C, D) \quad \text{if and only if} \quad D \in \bigcup_{t \leqslant t_0} \downarrow(\widetilde{g}(t_0))$$

and

$$P(S)(t_0) \sqsubseteq C \quad \text{if and only if} \quad D \in \bigcap_{\widetilde{g} \in S} \bigcup_{t \leqslant t_0} \downarrow(\widetilde{g}(t_0)).$$

Similarly,

$$F(\widetilde{g})(t_0) \in (C, D) \quad \text{if and only if} \quad D \in \bigcup_{t \geqslant t_0} \downarrow(\widetilde{g}(t_0))$$

and

$$F(S)(t_0) \sqsubseteq C \quad \text{if and only if} \quad D \in \bigcap_{\widetilde{g} \in S} \bigcup_{t \geqslant t_0} \downarrow(\widetilde{g}(t_0)).$$

So we may prescribe, to any $S \in \mathbf{2}^{\mathbf{2}^{M \times T}} \cong \mathbf{2}^{(\mathbf{2}^M)^T}$, a sequence of subsets $\left(\bigcap_{\widetilde{g} \in S} \bigcup_{t \leqslant t_0} \downarrow(\widetilde{g}(t_0))\right)_{t_0 \in T}$ or $\left(\bigcap_{\widetilde{g} \in S} \bigcup_{t \geqslant t_0} \downarrow(\widetilde{g}(t_0))\right)_{t_0 \in T}$ from $(\mathbf{2}^{\mathbf{2}^M})^T \cong \mathbf{2}^{\mathbf{2}^M \times T}$, respectively.

Because the evaluation of P and F for objects is a formal procedure as mentioned above, then also the evaluation of these operators for concepts is a formal procedure, and hence, it can be provided by means of a device of AI.

To develop our study of tense operators on concepts, we can mention that our approach is not the only way how to define and evaluate these operators. An alternative way is to consider the relation R of a given context $\mathcal{K} = (O, M, R)$ as time depending (i.e., $R(t)$) and study the operators P, F, H and G on the matrix of $R(t)$ (see, e.g., Wolff 2001). However, this was not our intent in this paper.

3 Galois connections and closure operators

For our construction of tense operators on a given formal context, we need several advanced concepts and methods from algebra and the theory of ordered sets. These are collected in this section.

Let $\mathbf{A} = (A, \leqslant)$ and $\mathbf{B} = (B, \leqslant)$ be ordered sets. A mapping $f : A \to B$ is called *residuated* if there exists a mapping $g : B \to A$ such that

$$f(a) \leqslant b \quad \text{if and only if} \quad a \leqslant g(b)$$

for all $a \in A$ and $b \in B$.

In this situation, we say that f and g form a *residuated pair* or that the pair (f, g) is a (monotone) *Galois connection* (shortly *adjunction*). The mapping f is called a *lower adjoint of* g or a *left adjoint of* g, and the mapping g is called an *upper adjoint of* f or a *right adjoint of* f. We also say that f and g are *tense operators* if $\mathbf{A} = \mathbf{B}$.

The following description of Galois connections is a folklore in the theory of ordered sets.

Lemma 1 *Let $(A, \leqslant)$ and $(B, \leqslant)$ be ordered sets. Let $f : A \to B$ and $g : B \to A$ be mappings. The following conditions are equivalent:*

(1) (f, g) is a Galois connection.
(2) f and g are monotone, $id_A \leqslant g \circ f$ and $f \circ g \leqslant id_B$.
(3) $g(b) = \bigvee(\{x \in A \mid f(x) \leqslant b\})$ and $f(a) = \bigwedge(\{y \in B \mid a \leqslant g(y)\})$ for all $a \in A$ and $b \in B$.

In the above case, g is determined uniquely by f and, similarly, f is determined uniquely by g. Moreover, f preserves all existing joins in $(A, \leqslant)$ and g preserves all existing meets in $(B, \leqslant)$. If, in addition, both $(A, \leqslant)$ and $(B, \leqslant)$ are complete lattices, we have the converse, i.e., if f preserves all joins in $(A, \leqslant)$, then f has an upper adjoint g given by the condition $g(b) = \bigvee(\{x \in A \mid f(x) \leqslant b\})$, for all $b \in B$. Similarly, if g preserves all meets in $(A, \leqslant)$, then g has a lower adjoint f given by the condition $f(a) = \bigwedge(\{y \in B \mid a \leqslant g(y)\})$, for all $a \in A$.

Definition 1 Let $\mathbf{A} = (A, \leqslant)$ be an ordered set. A *closure operator on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(c1) $a \leqslant j(a)$ (j is *extensive*)
(c2) $a \leqslant b$ implies $j(a) \leqslant j(b)$ (j is *order preserving*)
(c3) $(j \circ j)(a) = j(a)$ (j is *idempotent*).

A *coclosure operator on* $\mathbf{A}$ is a map $j\colon A \to A$ such that for any $a, b \in A$:

(co1) $a \geqslant j(a)$ (j is *coextensive*)
(c2) $a \leqslant b$ implies $j(a) \leqslant j(b)$
(c3) $(j \circ j)(a) = j(a)$

We put $A_j = \{a \in A \mid j(a) = a\}$. Clearly, A_j with the induced order is a sub-poset of $\mathbf{A}$. Recall that a coclosure operator on an ordered set is a closure operator on the ordered set with a dual order.

Note that a self-map $j\colon A \to A$ is a closure operator on $\mathbf{A}$ if and only if

$$a \leqslant j(b) \iff j(a) \leqslant j(b).$$

Moreover, for any Galois connection (f, g), the composition $j = g \circ f\colon A \to A$ is a closure operator on $\mathbf{A}$ and we say that *j is a closure operator induced by f*.

In what follows, we will explain the connection between ordered sets $\mathbf{A}$ with a closure operator $j\colon A \to A$ and formal contexts.

We can define a formal context $\mathcal{K}(\mathbf{A}, j) = (A, A_j, \leqslant)$ where $\leqslant$ is the order on $\mathbf{A}$ restricted to $A \times A_j$. We say that $\mathcal{K}(\mathbf{A}, j)$ is the *context induced by* $(\mathbf{A}, j)$. We denote by $\mathcal{E}(\mathbf{A}, j)$ the set of all extents of the context $\mathcal{K}(\mathbf{A}, j)$.

Having a formal context $\mathcal{K} = (O, M, R)$, we know that the mapping $X \mapsto X''$ for $X \subseteq O$ or the mapping $Y \mapsto Y''$ for $Y \subseteq M$ is the closure operators on $\mathbf{2}^O$ or $\mathbf{2}^M$, respectively. However, now we show that also conversely, every ordered set with a closure operator can be represented by means of a certain context. In fact, such an ordered set $\mathbf{A} = (A, \leqslant)$ with a closure operator j can be embedded into the context $\mathcal{K}(\mathbf{A}, j)$ defined above.

Theorem 2 Representation theorem of closure operators. *Let $\mathbf{A} = (A, \leqslant)$ be an ordered set and let $j\colon A \to A$ be a closure operator on $\mathbf{A}$. Let $\mathcal{K}(\mathbf{A}, j)$ be the context induced by $(\mathbf{A}, j)$. Then, the mapping $i_{\mathbf{A}}\colon A \to \mathbf{2}^A$ given by $i_{\mathbf{A}}(a) = \downarrow(a)$ is an order-reflecting morphism of ordered sets such that $i_{\mathbf{A}}(A)$ is a sub-poset of $\mathbf{2}^A$, $i_{\mathbf{A}}(A_j) = \mathcal{E}(\mathbf{A}, j) \cap i_{\mathbf{A}}(A)$ and $\{a\}'' = \downarrow(j(a))$ for all $a \in A$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\;j\;} & A_j \\ {\scriptstyle i_{\mathbf{A}}}\downarrow & & \downarrow{\scriptstyle i_{\mathbf{A}}|A_j} \\ \mathbf{2}^{\mathbf{A}} & \xrightarrow[(-)'']{} & \mathcal{E}(\mathbf{A}, j) \end{array}$$

Proof Since $a \leqslant b$ if and only if $\downarrow(a) \subseteq \downarrow(b)$, for all $a, b \in A$, we have that $i_{\mathbf{A}}$ is an order-reflecting morphism of ordered sets, and hence, $i_{\mathbf{A}}(A)$ is a sub-poset of $\mathbf{2}^A$.

Let $a \in A$. We compute:

$$(i_{\mathbf{A}}|A_j \circ j)(a) = i_{\mathbf{A}}|A_j(j(a)) = \downarrow(j(a)) \quad \text{and}$$
$$((-)'' \circ i_{\mathbf{A}})(a) = (\downarrow(a))'' = \{a\}'' = \downarrow(j(a)).$$

□

Hence, any ordered set $\mathbf{A}$ with a closure operator $j\colon A \to A$ can be realized as a restriction of the ordered set $\mathbf{2}^A$ of all subsets of A with the closure operator $(-)''$ given by the induced context $\mathcal{K}(\mathbf{A}, j)$.

We denote by **Cord** the category of ordered sets with a closure operator with order-preserving mappings as morphisms, and by **cCord** the category of ordered sets with a coclosure operator with order-preserving mappings as morphisms.

In what follows, we want to provide a meaningful construction giving tense operators which will be in accordance with the intuitive idea of time dependency.

For this reason, we introduce tense operators which can be easily evaluated on a given complete lattice accompanied with a time frame.

Consider a complete lattice $\mathbf{L} = (L; \leqslant, 0, 1)$. Let $(T, \leqslant)$ be a time frame. Define the following mappings $\widehat{G}$, $\widehat{P}$, $\widehat{H}$ and $\widehat{F}$ on $\mathbf{L}^T$ as follows:
for all $p \in L^T$, $s \in T$

$$\widehat{G}(p)(s) = \bigwedge_L \{p(t) \mid s \leqslant t\}$$
$$\widehat{P}(p)(s) = \bigvee_L \{p(t) \mid t \leqslant s\};$$
$$\widehat{H}(p)(s) = \bigwedge_L \{p(t) \mid t \leqslant s\};$$
$$\widehat{F}(p)(s) = \bigvee_L \{p(t) \mid s \leqslant t\}.$$

The just constructed operator $\widehat{G}$ or $\widehat{P}$ or $\widehat{H}$ or $\widehat{F}$ will be called a *tense operator on $\mathbf{L}^T$ constructed by means of the time frame* $(T, \leqslant)$. Moreover, $(\widehat{P}, \widehat{G})$ and $(\widehat{F}, \widehat{H})$ are Galois connections [see Chajda and Paseka (2015, Theorem 6.1)]. If we assume that the time frame $(T, \leqslant)$ is a preordered set, i.e., $\leqslant$ is reflexive and transitive, then we have again from Chajda and Paseka (2015, Theorem 6.1) that $\widehat{P}$ and $\widehat{F}$ are closure operators and $\widehat{G}$ and $\widehat{H}$ are coclosure operators such that $\widehat{G} \leqslant \widehat{F}$ and $\widehat{H} \leqslant \widehat{P}$.

Let $\mathbf{A} = (A, \leqslant)$ be an ordered set and let $P : A \to A$ and $G : A \to A$ be mappings. Let $\mathbf{C} = (C, \leqslant)$ be an ordered set and let T be a set of order-preserving mappings from $\mathbf{A}$ to $\mathbf{C}$.

We put

$$\rho^P = \{(s,t) \in T \times T \mid (\forall a \in A)(s(a) \leqslant t(P(a)))\} \quad \text{and}$$
$$\rho_G = \{(s,t) \in T \times T \mid (\forall b \in A)(s(G(b)) \leqslant t(b))\}.$$

We say that ρ_G is the *G-induced relation* and ρ^P is the *P-induced relation*.

The situation can be simplified as shown in the following lemma.

Lemma 2 *Let $\mathbf{A} = (A, \leqslant)$ be an ordered set. Let $P : A \to A$ and $G : A \to A$ be mappings such that (P, G) is a Galois connection. Let $\mathbf{C} = (C, \leqslant)$ be an ordered set and let T be a set of order-preserving mappings from $\mathbf{A}$ to $\mathbf{C}$. Then,*

$$\rho^P = \rho_G.$$

Proof Assume first that $(s,t) \in \rho_G$, $a \in A$. Then, $s(a) \leqslant s(G(P(a))) \leqslant t(P(a))$. Conversely, let $s(a) \leqslant t(P(a))$ for all $a \in A$. It follows, for any $b \in A$, that $s(G(b)) \leqslant t(P(G(b))) \leqslant t(b)$, i.e., $(s,t) \in \rho_G$. Hence,

$$\rho_G = \{(s,t) \in T \times T \mid (\forall a \in A)(t(P(a)) \geqslant s(a))\} = \rho^P.$$

□

Proposition 1 *Let $\mathbf{A} = (A, \leqslant)$ be an ordered set and let $P : A \to A$ and $G : A \to A$ be order-preserving mappings. Let $\mathbf{C} = (C, \leqslant)$ be an ordered set and let T be a set of order-preserving mappings from $\mathbf{A}$ to $\mathbf{C}$. The following statements hold:*

(i) Let $t \in T$, $t \circ P \in T$. Then, $(t \circ P, t) \in \rho^P$ and, for all $a \in A$, the set $\{s(a) \mid s \ \rho^P \ t\}$ has a greatest element $t(P(a))$, i.e.,

$$t(P(a)) = \max_C \{s(a) \mid s \ \rho^P \ t\}.$$

(ii) Let $s \in T$, $s \circ G \in T$. Then, $(s, s \circ G) \in \rho_G$ and, for all $b \in A$, the set $\{t(b) \mid s \ \rho_G \ t\}$ has a smallest element $s(G(b))$, i.e.,

$$s(G(b)) = \min_C \{t(b) \mid s \ \rho_G \ t\}.$$

Proof (i): Since, for all $a \in A$, $t(P(a)) = (t \circ P)(a)$ we have that $(t \circ P, t) \in \rho^P$ and clearly for all $a \in A$, $(t \circ P)(a) = \min_C \{s(a) \mid s \ \rho^P \ t\}$. (ii): It is enough to note that the claim follows from (i) applied to the ordered sets $\mathbf{A}^{op}$ and $\mathbf{C}^{op}$. □

One can now ask, for a given ordered set $\mathbf{A} = (A, \leqslant)$ and mappings $P : A \to A$, $G : A \to A$ such that (P, G) is a Galois connection, whether there exist a frame (T, ρ) and a non-trivial ordered set $\mathbf{M} = (M; \leqslant, 0, 1)$ such that the mappings G, P can be derived by this construction where $\mathbf{A}$ is embedded into the power algebra $\mathbf{M}^T$. Hence, we ask whether every element p of A is of the form $(p(t))_{t \in T}$ in M^T and, moreover, $G(p)(s) = \bigwedge_M \{p(t) \mid s \ \rho \ t\}$ and $P(p)(s) = \bigvee_M \{p(t) \mid t \ \rho \ s\}$. If such a representation exists, then one can recognize the time variability of elements of A expressed as time-dependent functions $(p(t))_{t \in T}$.

We have

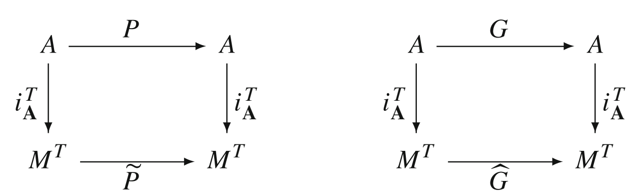
Proposition 2 [Chajda and Paseka (2015, Proposition 6.4)] *Let $\mathbf{A} = (A; \leqslant)$ be an ordered set equipped with a full set T of order-preserving mappings into a non-trivial complete lattice $\mathbf{M}$. Then, the map $i^T_{\mathbf{A}} : A \to M^T$ given by $i^T_{\mathbf{A}}(x)(s) = s(x)$ for all $x \in A$ and all $s \in T$ is an order-reflecting order-preserving mapping such that $i^T_{\mathbf{A}}(A)$ is embedded into $\mathbf{M}^T$.*

Let $\mathbf{A} = (A; \leqslant)$ be an ordered set with a full set T of order-preserving mappings into a complete lattice $\mathbf{M}$. Let $P : A \to A$ and $G : A \to A$ be order-preserving mappings, ρ_G the G-induced relation by $\mathbf{M}$ and ρ^P the P-induced relation by $\mathbf{M}$. Let us denote by $(\star)$ the following condition:

(1) for all $b \in B$ and for all $s \in S$, $s(G(b)) = \bigwedge_M \{t(b) \mid s \ \rho_G \ t\}$,$(\star)$
(2) for all $a \in A$ and for all $t \in T$, $t(P(a)) = \bigvee_M \{s(a) \mid s \ \rho^P \ t\}$.

By the same arguments as in Chajda and Paseka (2015), Theorem 6.3, we can prove the following.

Theorem 3 *Let $\mathbf{A}$ be an ordered set with a full set T of order-preserving mappings into a complete lattice $\mathbf{M}$. Let $P : A \to A$ and $G : A \to A$ be order-preserving mappings, ρ_G the G-induced relation by $\mathbf{M}$ and ρ^P the P-induced relation by $\mathbf{M}$, respectively, such that the condition $(\star)$ is satisfied. Then, the map $i^T_{\mathbf{A}}$ is an order-reflecting order-preserving mapping from $\mathbf{A}$ into the complete lattice $\mathbf{M}^T$ such that the following diagrams commute:*



Here $\widehat{G}$ is constructed by means of (T, ρ_G) and $\widetilde{P}$ is constructed by means of (T, ρ^P).

In particular, if $\rho_G = \rho^P$ or (P, G) is a Galois connection, then $\widetilde{P} = \widehat{P}$.

4 Logical complete lattices and ordered sets with a closure operator

In Sect. 2, we studied behavior and formalization of tense operators on concepts derived from the classical context $\mathcal{K}$. Here the main role is played by the tense operators F and P, and the operators G and H are not so essential for evolution of objects and their sets. On the other hand, there exist also other tense operators, e.g., the operators "since" and "until" [for other possible tense operators, the reader is referred to the monograph (see e.g., Rump (2013); Chajda and Paseka (2015)] Further, not only classical context is of some interest. It was shown in the monograph Bělohlávek and Klir (2011) that also the so-called fuzzy concepts play an important role in applications, in particular when the concepts are created by human beings. All of these ideas lead to a study of more general setting where our results from Sect. 2 are included as particular cases. In what follows, we will not study the possible applications, but we will develop a general theory. Because concepts in classical context are just the closed subsets with respect to the mentioned closure operator (denoted by $''$ in Sect. 1) and because P and F are order-preserving self-mappings on the complete lattice of formal concepts, it is natural to investigate a general closure operator on a given ordered set endowed with two order-preserving mappings which are connected with that closure operator via certain binary relations. It seems that the order-categorical approach can be used here with advantage.

In this section, we exhibit a duality between a class of ordered sets with a closure operator to a class of complete lattices with a coclosure operator. Since every dual of a complete lattice with a coclosure operator is an ordered set with a closure operator, this leads to a self-embedding of the category of complete lattices with a coclosure operator.

An element c of a complete lattice L is said to be *supercompact* if for any non-empty subset $X \subseteq L$, the inequality $c \leqslant \bigvee X$ implies that $c \leqslant x$ for some $x \in X$. The set of supercompact elements of L will be denoted by L^{sc}.

For every ordered set $\mathbf{A}$, the *upper sets* $X \subseteq A$ (i.e., the subsets X of A with $a \geqslant b \in X$ implies $a \in X$) can be made into a complete lattice $(\mathcal{U}(\mathbf{A}), \subseteq)$ with the smallest element $\emptyset$ and the greatest element A (since $\mathcal{U}(\mathbf{A})$ is closed under arbitrary unions and intersections) such that supercompact elements of $\mathcal{U}(\mathbf{A})$ are exactly the principal upper sets $\uparrow(a)$ (Banaschewski and Niefield 1991). The lattices of the form $\mathcal{U}(\mathbf{A})$ are (up to isomorphism) exactly the *superalgebraic lattices*, i.e., those which are join generated by their supercompact elements. If $\mathbf{A} = (A, \leqslant)$, $\mathbf{B} = (B, \leqslant)$ are ordered sets and $h\colon A \to B$ an order-preserving mapping, it is well known that the induced mapping $\mathcal{U}(h)\colon \mathcal{U}(\mathbf{B}) \to \mathcal{U}(\mathbf{A})$, $Y \mapsto h^{-1}[Y]$ is a complete lattice homomorphism (preserving arbitrary joins and meets).

Conversely, for superalgebraic lattices $\mathbf{L}$, $\mathbf{K}$ and a complete lattice homomorphism $g\colon L \to K$, we put $\mathcal{V}(\mathbf{L}) = (L^{sc}; \geqslant)$ and $\mathcal{V}(g)\colon K^{sc} \to L^{sc}$ will be a restriction of the left adjoint $f\colon K \to L$ of g to the ordered set K^{sc} [see Erné et al. (2007, Proposition 2.5)].

The functors $\mathcal{U}$ and $\mathcal{V}$ provide a duality between the category of ordered sets with order-preserving mappings and the category of superalgebraic lattices with complete homomorphisms (Erné et al. 2007, Proposition 2.2).

Note also that a complete lattice is superalgebraic if and only if it is completely embeddable in a discrete cube $\{0, 1\}^X$.

Definition 2 Let $\mathbf{L}$ be a complete lattice with a coclosure operator c. We say that $\mathbf{L}$ is a *logical lattice* if the following conditions are satisfied.

(i) Every $a \in L$ can be represented as a join $a = \bigvee C$ with $C \subseteq L^{sc}$.
(ii) c is a complete lattice endomorphism.
(iii) $c(L^{sc}) \subseteq L^{sc}$.

A *morphism of logical lattices* $\mathbf{L}$, $\mathbf{K}$ is a mapping $h\colon L \to K$ which satisfies $h(\bigvee X) = \bigvee h(X)$ and $h(\bigwedge X) = \bigwedge h(X)$ for any subset $X \subseteq L$. The category of logical lattices will be denoted by **LogL**.

Now we define two functors $\overline{\mathcal{U}}\colon$ **Cord** $\to$ **LogL** and $\overline{\mathcal{V}}\colon$ **LogL** $\to$ **Cord** such that $\overline{\mathcal{U}}$ is a restriction of $\mathcal{U}$ to **Cord**, $\overline{\mathcal{V}}$ is a restriction of $\mathcal{V}$ to **LogL**, and, for a closure operator j on an ordered set $\mathbf{A}$ and a coclosure operator c on a logical lattice $\mathbf{L}$, we put $c_j(X) = \bigcup\{\uparrow(j(x)) \mid x \in X\}$, $X \in \mathcal{U}(\mathbf{A})$, and $j_c(a) = c(a)$, $a \in L^{sc}$. Clearly, $c_j(X) \in \mathcal{U}(\mathbf{A})$ and $j_c(a) \in L^{sc}$.

Now let us prove our first main result.

Theorem 4 *Let* $\mathbf{A} \in$ **Cord** *and* $\mathbf{L} \in$ **LogL**. *Then,* $\overline{\mathcal{U}}(\mathbf{A}) \in$ **LogL** *and* $\overline{\mathcal{V}}(\mathbf{L}) \in$ **Cord**. *The functors* $\overline{\mathcal{U}}$ *and* $\overline{\mathcal{V}}$ *are well defined and provide a duality between the category of ordered sets with a closure operator and the category of logical lattices.*

Proof From Erné et al. (2007, Proposition 2.2), we know that $\overline{\mathcal{U}}(\mathbf{A})$ is an superalgebraic lattice and $\overline{\mathcal{V}}(\mathbf{L})$ is an ordered set. It is enough to show that, for a closure operator j on $\mathbf{A}$ and a coclosure operator c on $\mathbf{L}$, $c_j\colon \overline{\mathcal{U}}(\mathbf{A}) \to \overline{\mathcal{U}}(\mathbf{A})$ is a coclosure operator on $\overline{\mathcal{U}}(\mathbf{A})$, and $j_c\colon \overline{\mathcal{V}}(\mathbf{L}) \to \overline{\mathcal{V}}(\mathbf{L})$ is a closure operator on $\overline{\mathcal{V}}(\mathbf{L})$ such that $\mathbf{A} \cong \overline{\mathcal{V}}(\overline{\mathcal{U}}(\mathbf{A}))$ in **Cord** and $\mathbf{L} \cong \overline{\mathcal{U}}(\overline{\mathcal{V}}(\mathbf{L}))$ in **LogL**.

Let $X, Y \in \overline{\mathcal{U}}(\mathbf{A})$, $X \subseteq Y$. Assume that $a \in c_j(X)$. Then, there is $x \in X$ such that $x \leqslant j(x)$ and $a \in\uparrow(j(x))$. Hence, $a \in\uparrow(x)$, i.e., $a \in X$. Moreover, since $x \in X$ we obtain that $x \in Y$, i.e., $a \in c_j(Y)$. It follows that c_j is coextensive and order preserving. Hence, also $c_j(c_j(X)) \subseteq c_j(X)$. Now, assume that $b \in c_j(X)$. Then, there is an element $a \in X$ such that $b \in \uparrow(j(a)) = \uparrow(j(j(a)))$ and $j(a) \in c_j(X)$. It follows that $b \in c_j(c_j(X))$ and we have $c_j(X) \subseteq c_j(c_j(X))$. To show that c_j is a complete lattice homomorphism, note that $c_j(\uparrow(a)) = \uparrow(j(a)) = j^{-1}(\uparrow(a))$ for all $a \in A$ (this also yields that $c(\overline{\mathcal{U}}(\mathbf{A})^{sc}) \subseteq \overline{\mathcal{U}}(\mathbf{A})^{sc}$), and

$$\begin{aligned} c_j(X) &= \bigcup\{\uparrow(j(x)) \mid x \in X\} = \bigcup\{j^{-1}(\uparrow(x)) \mid x \in X\} \\ &= j^{-1}(\bigcup\{\uparrow(x) \mid x \in X\}) = j^{-1}(X). \end{aligned}$$

Since j^{-1} preserves all unions and intersections so does c_j.

The fact that $\overline{\mathcal{V}}(\mathbf{L}) \in \mathbf{LogL}$ follows from the fact that L^{sc} is ordered with the reverse partial order to the order on $\mathbf{L}$. In particular, the restriction of the coclosure operator c to L^{sc} will be a closure operator.

Next, consider the order-reversing injection

$$\uparrow(-)\colon A \to \overline{\mathcal{U}}(\mathbf{A}).$$

Evidently, $\uparrow(-)$ maps A bijectively onto $\overline{\mathcal{U}}(\mathbf{A})^{sc}$. For $a, x \in A$, we have $\uparrow(x) \geqslant j_{c_j}(\uparrow(a))$ in $\overline{\mathcal{U}}(\mathbf{A})^{sc}$ if and only if $\uparrow(x) \subseteq j_{c_j}(\uparrow(a))$ if and only if $\uparrow x \subseteq c_j(\uparrow(a))$ if and only if $x \geqslant j(a)$. This yields that $\uparrow(j(a)) = j_{c_j}(\uparrow(a))$, i.e., the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\quad j \quad} & A \\ {\scriptstyle \uparrow(-)}\downarrow & & \downarrow{\scriptstyle \uparrow(-)} \\ \overline{\mathcal{U}}(\mathbf{A})^{sc} & \xrightarrow[\quad j_{c_j} \quad]{} & \overline{\mathcal{U}}(\mathbf{A})^{sc} \end{array}$$

and $\uparrow(-)$ induces an isomorphism $\mathbf{A} \cong \overline{\mathcal{V}}(\overline{\mathcal{U}}(\mathbf{A}))$ in $\mathbf{Cord}$.

Let us verify that every logical lattice $\mathbf{L}$ is isomorphic to $\overline{\mathcal{U}}(\overline{\mathcal{V}}(\mathbf{L})) = \overline{\mathcal{U}}(L^{sc})$. Let us consider the mapping $\kappa\colon L \to \overline{\mathcal{U}}(L^{sc})$ with

$$\kappa(p) := \{x \in L^{sc} \mid x \leqslant p\}.$$

Similarly as in Rump and Yang (2014) we have that $\kappa(p)$ is an upper set in the ordered set $\overline{\mathcal{V}}(\mathbf{L})$, i.e., with respect to the reverse order and that κ induces an isomorphism $\mathbf{L} \cong \overline{\mathcal{U}}(\overline{\mathcal{V}}(\mathbf{L}))$ in ordered sets. In particular, κ preserves supercompact elements. Let us check that κ preserves the coclosure operator. For $p \in L, x \in L^{sc}$, we have $\kappa(x) = \uparrow_{L^{sc}}(x) \subseteq c_{j_c}(\kappa(p))$ in $\overline{\mathcal{U}}(\overline{\mathcal{V}}(\mathbf{L}))$ if and only if $x \in c_{j_c}(\kappa(p))$ if and only if there is $y \in L^{sc}$, $y \leqslant p$ such that $x \in \uparrow_{L^{sc}}(j_c(y))$ if and only if there is $y \in L^{sc}$, $y \leqslant p$ such that $x \leqslant c(y)$ if and only if $x \leqslant c(p)$ (since p is a join of supercompact elements x, $x \in L^{sc}$ and c preserves arbitrary joins). Thus, κ is an isomorphism of logical lattices.

Since $\overline{\mathcal{U}}$ coincides with $\mathcal{U}$ on morphisms, i.e., on order-preserving mappings, and $\overline{\mathcal{V}}$ coincides with $\mathcal{V}$ on morphisms, i.e., on complete lattice homomorphisms, the proof is completed. □

5 Being before and being after

Now, let us express the property "Every time depending object from S is or has been before in the concept (C, D)." in the language of upper sets on $\mathbf{2}^O$. More precisely, let a context $\mathcal{K} = (O, M, R)$ be given and (C, D) be a concept of $\mathcal{K}$. Let $(T, \leqslant)$ be a time frame, $t_0 \in T$, $g \in O^T$, and $S \subseteq O^T$ be a subset of time depending objects (now as functions from T to O). Then, $P(g)(t_0) \in (C, D)$ if and only if there exists $t_1 \leqslant t_0$ such that $g(t_1) \in C$ if and only if there exists $t_1 \leqslant t_0$ such that $C \in \uparrow_{\mathbf{2}^O}(\{g(t_1)\})$ if and only if there exists $t_1 \leqslant t_0$ such that $\uparrow_{\mathbf{2}^O} C \subseteq \uparrow_{\mathbf{2}^O}(\{g(t_1)\})$ if and only if $\uparrow_{\mathbf{2}^O} C \subseteq \bigcup_{t_1 \leqslant t_0} \uparrow_{\mathbf{2}^O}(\{g(t_1)\})$.

Hence, $P(S)(t_0) \sqsubseteq C$ if and only if $\uparrow_{\mathbf{2}^O} C \subseteq \bigcap_{g \in S} \bigcup_{t_1 \leqslant t_0} \uparrow_{\mathbf{2}^O}(\{g(t_1)\})$.

Since $g(t_1)$ uniquely corresponds to $\uparrow_{\mathbf{2}^O}(\{g(t_1)\})$, we obtain a tense operator $P\colon \left(\overline{\mathcal{U}}(\mathbf{2}^O)\right)^T \to \left(\overline{\mathcal{U}}(\mathbf{2}^O)\right)^T$ defined by $P((X_t)_{t \in T})(t_0) = \bigcup_{t_1 \leqslant t_0} X_t$ which is in accordance with the standard definition of tense operators on complete lattices (see Chajda and Paseka 2015). Moreover, we can extend P to $\mathbf{2}^{\left(\overline{\mathcal{U}}(\mathbf{2}^O)\right)^T}$ as follows

$$P(S)(t_0) = \bigcap\{P((X_t)_{t \in T})(t_0) \mid (X_t)_{t \in T} \in S\}.$$

So we obtain a mapping $P\colon \mathbf{2}^{\left(\overline{\mathcal{U}}(\mathbf{2}^O)\right)^T} \to \left(\overline{\mathcal{U}}(\mathbf{2}^O)\right)^T$.

Let $\mathbf{A} = (A, \leqslant)$ be an ordered set and let $j\colon A \to A$ be a closure operator on $\mathbf{A}$. Note that the set of all order-preserving mappings $\mathbf{O}(\mathbf{A}, j)$ is a monoid with respect to the composition of mappings containing always the identity mapping id_A and the closure operator j. In what follows, we shall assume that $T_j \subseteq \mathbf{O}(\mathbf{A}, j)$ will be a submonoid of the ordered set $\mathbf{O}(\mathbf{A}, j)$ of all order-preserving mappings on $\mathbf{A}$ containing j and all order-preserving mappings which will be used in the assumptions of our statements.

We introduce a mapping $\bar{i}_{\mathbf{A}}^{T_j}\colon A \to \left(\overline{\mathcal{U}}(\mathbf{A})\right)^{T_j}$ as follows:

$$\bar{i}_{\mathbf{A}}^{T_j}(a) = (\uparrow(h(a)))_{h \in T_j}$$

for all $a \in A$.

Clearly, $\bar{i}_{\mathbf{A}}^{T_j}$ is an order-preserving mapping from $(A, \geqslant)$ into $\left(\overline{\mathcal{U}}(\mathbf{A})\right)^{T_j}$ since $a \geqslant b, a, b \in A$ implies $h(a) \geqslant h(b)$ (since any $h \in T_j$ is order preserving) which in turn yields that $\uparrow(h(a)) \subseteq \uparrow(h(b))$, i.e., $(\uparrow(h(a)))_{h \in T_j} \leqslant (\uparrow(h(b)))_{h \in T_j}$. Conversely, if $(\uparrow(h(a)))_{h \in T_j} \leqslant (\uparrow(h(b)))_{h \in T_j}$ we have (since $\mathrm{id}_A \in T_j$) that $\uparrow(a) \subseteq \uparrow(b)$, i.e., $a \geqslant b$ and $\bar{i}_{\mathbf{A}}^{T_j}$ is order reflecting.

Proposition 3 *Let $\mathbf{A} = (A, \leqslant)$ be an ordered set, j a closure operator on $\mathbf{A}$. Let $P\colon A \to A$ and $G\colon A \to A$ be order-preserving mappings, T_j a submonoid of the ordered* *set $\mathbf{O}(\mathbf{A}, j)$ containing j, G and P such that ρ^P is the P-induced relation on T_j and ρ_G is the G-induced relation on T_j with respect to the dual order.*

Then, the mapping $\bar{i}^{T_j}_{\mathbf{A}}$ is an order-reflecting morphism of ordered sets such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{P} & A \\ \big\downarrow{\scriptstyle \bar{i}^{T_j}_{\mathbf{A}}} & & \big\downarrow{\scriptstyle \bar{i}^{T_j}_{\mathbf{A}}} \\ (\overline{\mathcal{U}}(\mathbf{A}))^{T_j} & \xrightarrow[\widetilde{P}]{} & (\overline{\mathcal{U}}(\mathbf{A}))^{T_j} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{G} & A \\ \big\downarrow{\scriptstyle \bar{i}^{T_j}_{\mathbf{A}}} & & \big\downarrow{\scriptstyle \bar{i}^{T_j}_{\mathbf{A}}} \\ (\overline{\mathcal{U}}(\mathbf{A}))^{T_j} & \xrightarrow[\widehat{G}]{} & (\overline{\mathcal{U}}(\mathbf{A}))^{T_j} \end{array}$$

Here $\widetilde{P}$ is constructed by means of the time frame (T_j, ρ^P) and $\widehat{G}$ is constructed by means of the time frame (T_j, ρ_G) on the logical lattice $\overline{\mathcal{U}}(\mathbf{A})$.

Proof We have only to check that the part (2) of the condition $(\star)$ is satisfied. This in turn translates to the following two conditions:

(a) for all $a \in A$ and for all $t \in T_j$,

$$\bar{i}^{T_j}_{\mathbf{A}}(P(a))(t) = \bigcup\{\bar{i}^{T_j}_{\mathbf{A}}(a)(s) \mid s\ \rho^P\ t\} = \widetilde{P}(\bar{i}^{T_j}_{\mathbf{A}}(a))(t),$$

(b) for all $b \in A$ and for all $s \in T_j$,

$$\bar{i}^{T_j}_{\mathbf{A}}(G(b))(s) = \bigcap\{\bar{i}^{T_j}_{\mathbf{A}}(b)(t) \mid s\ \rho_G\ t\} = \widehat{G}(\bar{i}^{T_j}_{\mathbf{A}}(b))(s).$$

Let $a \in A$ and $t \in T_j$ be arbitrary, fixed. Let $x \in \bar{i}^{T_j}_{\mathbf{A}}(P(a))(t) = \uparrow(t(P(a)))$. Put $h = t \circ P$. Then, $h \in T_j$, $h\ \rho^P\ t$ and we obtain that $x \in \uparrow(h(a)) \subseteq \bigcup\{\bar{i}^{T_j}_{\mathbf{A}}(a)(s) \mid s\ \rho^P\ t\}$. Conversely, let $s\ \rho^P\ t$. Then, $s(a) \geqslant t(P(a))$, i.e., $\uparrow(s(a)) \subseteq \uparrow(t(P(a)))$. This in turn implies that $\bigcup\{\bar{i}^{T_j}_{\mathbf{A}}(a)(s) \mid s\ \rho^P\ t\} \subseteq \bar{i}^{T_j}_{\mathbf{A}}(P(a))(t)$. Hence, the condition (a) is satisfied.

Similarly, let $b \in A$ and $s \in T_j$ be arbitrary, fixed. If $s\ \rho_G\ t$, we have that $s(G(a)) \geqslant t(a)$ for all $a \in A$. Hence, $\uparrow(s(G(b))) \subseteq \uparrow(t(b))$, i.e., $\bar{i}^{T_j}_{\mathbf{A}}(G(b))(s) \subseteq \bigcap\{\bar{i}^{T_j}_{\mathbf{A}}(b)(t) \mid s\ \rho_G\ t\}$. Since $s\ \rho_G\ s \circ G$, we obtain that $\bar{i}^{T_j}_{\mathbf{A}}(G(b))(s) = \bigcap\{\bar{i}^{T_j}_{\mathbf{A}}(b)(t) \mid s\ \rho_G\ t\}$ and the condition (b) is satisfied. □

The above theorem allows us to lift any order-preserving mapping P on $\mathbf{A}$ to a left adjoint $\widetilde{P}$ on $(\overline{\mathcal{U}}(\mathbf{A}))^{T_j}$ using the time frame (T_j, ρ^P). In particular, we are able to lift our closure operator $j\colon A \to A$ to a coclosure operator and a right adjoint $\widehat{j}$ on $(\overline{\mathcal{U}}(\mathbf{A}))^{T_j}$ using the time frame (T_j, ρ_j).

Theorem 5 *Let $\mathbf{A} = (A, \leqslant)$ be an ordered set, j a closure operator on $\mathbf{A}$ and $X \subseteq A$. Let $P\colon A \to A$ be an order-preserving mapping and T_j a submonoid of the ordered set $\mathbf{O}(\mathbf{A}, j)$ containing j and P such that ρ^P is the P-induced relation and ρ_j is the j-induced relation on T_j.*

Then, the mapping $\bar{i}^{T_j}_{\mathbf{A}}$ is an order-reflecting morphism of ordered sets such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{P} & A \\ \big\downarrow{\scriptstyle \bar{i}^{T_j}_{\mathbf{A}}} & & \big\downarrow{\scriptstyle \bar{i}^{T_j}_{\mathbf{A}}} \\ (\overline{\mathcal{U}}(\mathbf{A}))^{T_j} & \xrightarrow[\widetilde{P}]{} & (\overline{\mathcal{U}}(\mathbf{A}))^{T_j} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{j} & A \\ \big\downarrow{\scriptstyle \bar{i}^{T_j}_{\mathbf{A}}} & & \big\downarrow{\scriptstyle \bar{i}^{T_j}_{\mathbf{A}}} \\ (\overline{\mathcal{U}}(\mathbf{A}))^{T_j} & \xrightarrow[\widehat{j}]{} & (\overline{\mathcal{U}}(\mathbf{A}))^{T_j} \end{array}$$

Here $\widetilde{P}$ and $\widehat{j}$ are constructed by means of time frames (T_j, ρ^P) and (T_j, ρ_j), respectively, on the complete lattice $(\overline{\mathcal{U}}(\mathbf{A}))^{T_j}$. Moreover, we obtain an operator $\widetilde{F}$ constructed by means of the time frame $(T_j, (\rho^P)^{op})$, i.e., for any $g \in (\overline{\mathcal{U}}(\mathbf{A}))^{T_j}$,

$$\widetilde{F}(g)(s) = \bigcup\{g(t) \mid s\ \rho^P\ t\}.$$

In particular, if we identify the identity operator on A with the present tense "now," then the time depending object $g \in (\overline{\mathcal{U}}(\mathbf{A}))^{T_j}$ has been in the closure $j(a)$ (which plays the role of our concept (C, D)) if and only if $\uparrow(j(a)) = \bar{i}^{T_j}_{\mathbf{A}}(j(a))(id_A) = \bar{i}^{T_j}_{\mathbf{A}}(a)(j) \subseteq \widetilde{P}(g)(id_A)$.

Similarly, $g \in (\overline{\mathcal{U}}(\mathbf{A}))^{T_j}$ will be in the closure $j(a)$ if and only if $\bar{i}^{T_j}_{\mathbf{A}}(a)(j) \subseteq \widetilde{F}(g)(id_A)$.

Therefore, a subset $S \subseteq (\overline{\mathcal{U}}(\mathbf{A}))^{T_j}$ has been in the closure $j(a)$ if and only if $\bar{i}^{T_j}_{\mathbf{A}}(a)(j) \subseteq \bigcap_{g \in S} \widetilde{P}(g)(id_A)$, and S will be in the closure $j(a)$ if and only if $\bar{i}^{T_j}_{\mathbf{A}}(a)(j) \subseteq \bigcap_{g \in S} \widetilde{F}(g)(id_A)$.

It is transparent that, for a context $\mathcal{K} = (O, M, R)$ and a concept (C, D) of $\mathcal{K}$, our considerations from Sect. 2 and from the beginning of this section are covered by Theorem 5 and the above observations.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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