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Short communication

Extensions of posets with an antitone involution to residuated structures ☆

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Abstract

We show that every poset **P** with an antitone involution can be extended to a commutative integral residuated poset $\mathbb{E}(\mathbf{P})$. If, moreover, **P** is a lattice then so is $\mathbb{E}(\mathbf{P})$.

Keywords: Poset; Bounded poset; Commutative integral residuated poset; Commutative integral residuated lattice; Antitone involution; Extension

1. Introduction

Residuated posets in general and residuated lattices in particular form an algebraic axiomatization of certain substructural logics (see e.g. [4], [8], [9] and [10] and references therein), especially of fuzzy logic, see [1] for details. Residuated lattices were studied for a long time starting with the pioneering paper by Ward and Dilworth [12], see also [3] and [7]. Posets and lattices with an antitone involution can serve as a suitable model of such a logic because this involution can be considered as a negation and hence these logics satisfy the double negation law, see [3]. Let us mention that a kind of residuated posets was studied also in [5]. Moreover, residuated structures derived from semirings were treated in [6] and [9].

Recall that a *poset with an antitone involution* is an ordered triple $(P, \leq, ')$ such that $(P, \leq)$ is a poset and $'$ is a unary operation on P satisfying

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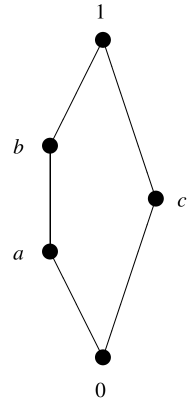

Fig. 1. The lattice $\mathbf{N}_5$.

- if $a \le b$ then $b' \le a'$,
- $a'' \approx a$

for all $a, b \in P$. Recall further that a *(bounded) commutative integral residuated poset* is an ordered six-tuple $\mathbf{P} = (P, \le, \odot, \rightarrow, 0, 1)$ such that

- $(P, \le, 0, 1)$ is a bounded poset,
- $(P, \odot, 1)$ is a commutative monoid with neutral element 1,
- $\rightarrow$ is a binary operation on P,
- $a \odot b \le c$ if and only if $a \le b \rightarrow c$

for all $a, b, c \in P$. The last property is called *adjointness*. If, moreover, $(P, \le)$ is a lattice then $\mathbf{P}$ is called a *commutative integral residuated lattice*. Every commutative integral residuated poset satisfies the identity $1 \rightarrow x \approx x$. By an *extension of a poset* $\mathbf{P} = (P, \le)$ is meant a poset $\mathbb{E}(\mathbf{P}) = (Q, \le)$ where $P \subseteq Q$ and for all elements $a, b \in P$ we have $a \le b$ in $\mathbb{E}(\mathbf{P})$ if and only if $a \le b$ in $\mathbf{P}$. If $\mathbf{P}$ is a lattice, then such an extension should preserve joins and meets of elements from P.

Unfortunately, not every bounded lattice $(L, \vee, \wedge, ', 0, 1)$ with an antitone involution $'$ can be converted into a commutative integral residuated lattice $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ satisfying the identity $x' \approx x \rightarrow 0$. For example, consider the non-modular lattice $\mathbf{N}_5 = (N_5, \vee, \wedge)$ depicted in Fig. 1.

It is easy to see that there exists exactly one antitone involution $'$ on $(N_5, \le)$, namely $0' = 1$, $a' = b$, $b' = a$, $c' = c$ and $1' = 0$. Suppose that $\mathbf{N}_5$ together with the involution $'$ could be converted into a commutative integral residuated poset $(N_5, \le, \odot, \rightarrow, 1)$ satisfying the identity $x' \approx x \rightarrow 0$. Since $x \odot y \le x$ and $x \odot y \le y$ for all $x, y \in N_5$ (see Theorem 2.17 in [1]) we have $c \odot a \le 0$ which implies $c \le a \rightarrow 0 = a' = b$, a contradiction. Thus $\mathbf{N}_5$ together with $'$ cannot be converted into a commutative integral residuated poset $(N_5, \le, \odot, \rightarrow, 1)$ satisfying the identity $x' \approx x \rightarrow 0$.

Hence, it is a question whether such a lattice (or poset in general) can be extended to a commutative integral residuated one by preserving the antitone involution. The aim of our paper is to show how such an extension can be constructed.

2. Results

At first, we show that the unary operation $x' := x \rightarrow 0$ in a commutative integral residuated lattice is antitone.

Lemma 1. *Let $(P, \le, \odot, \rightarrow, 0, 1)$ be a commutative integral residuated poset and put $x' := x \rightarrow 0$ for all $x \in P$ Then $x \le x''$ for all $x \in P$ and $'$ is antitone.*

Proof. Let $a, b \in P$. Then the every of the following assertions implies the next one:

$$\begin{aligned} a \to 0 &\leq a \to 0, \\ (a \to 0) \odot a &\leq 0, \\ a \odot (a \to 0) &\leq 0, \\ a &\leq (a \to 0) \to 0, \\ a &\leq a''. \end{aligned}$$

Moreover, every of the following assertions implies the next one:

$$\begin{aligned} a &\leq b, \\ a &\leq (b \to 0) \to 0, \\ a \odot (b \to 0) &\leq 0, \\ (b \to 0) \odot a &\leq 0, \\ b \to 0 &\leq a \to 0, \\ b' &\leq a'. \quad \square \end{aligned}$$

Our main result is as follows.

Theorem 2. *Let $\mathbf{P} = (P, \leq, ')$ be a poset with an antitone involution, assume $0 = c_1, \ldots, c_4 = 1 \notin P$, put $E(P) := P \cup \{c_1, \ldots, c_4\}$ and extend $\leq$ and $'$ from P to $E(P)$ by $0 < c_2 < x < c_3 < 1$ for all $x \in P$ and $c_i' := c_{5-i}$ for $i = 1, \ldots, 4$. Define binary operations $\odot$ and $\to$ on $E(P)$ as follows:*

$$0 \odot x = x \odot 0 := 0, 1 \odot x = x \odot 1 := x,$$
$$0 \to x = x \to 1 := 1, x \to 0 := x', 1 \to x := x$$

for $x \in E(P)$ and

$$x \odot y := \begin{cases} 0 & \text{if } x \leq y', \\ c_2 & \text{otherwise} \end{cases} \qquad x \to y := \begin{cases} 1 & \text{if } x \leq y, \\ c_3 & \text{otherwise} \end{cases}$$

for $x, y \in E(P) \setminus \{0, 1\}$. Then $\mathbb{E}(\mathbf{P}) := (E(P), \leq, \odot, \to, 0, 1)$ is a commutative integral residuated poset with the antitone involution $'$ satisfying the identity $x' \approx x \to 0$. (If $\mathbf{P}$ is already bounded then the least and greatest element of $\mathbf{P}$ may be identified with c_2 and c_3, respectively. If $\mathbf{P}$ has elements a, b, c, d satisfying $a < b \leq x \leq c < d$ for all $x \in P \setminus \{a, d\}$ then a, b, c, d may be identified with $c_1, \ldots, c_4$, respectively.)

Proof. Let $a, b, c \in E(P)$. Since $x'' = x$ for all $x \in P$ and $c_i'' = c_{5-i}' = c_{5-(5-i)} = c_i$ for all $i = 1, \ldots, 4$, $(E(P), \leq, ', 0, 1)$ is a bounded poset with an antitone involution.
If $\{a, b, c\} \cap \{0, 1\} \neq \emptyset$ then, obviously, $(a \odot b) \odot c = a \odot (b \odot c)$.
If $a, b, c \neq 0, 1$, $a \leq b'$ and $b \leq c'$ then $(a \odot b) \odot c = 0 \odot c = 0 = a \odot 0 = a \odot (b \odot c)$.
If $a, b, c \neq 0, 1$, $a \leq b'$ and $b \nleq c'$ then $(a \odot b) \odot c = 0 \odot c = 0 = a \odot c_2 = a \odot (b \odot c)$.
If $a, b, c \neq 0, 1$, $a \nleq b'$ and $b \leq c'$ then $(a \odot b) \odot c = c_2 \odot c = 0 = a \odot 0 = a \odot (b \odot c)$.
If $a, b, c \neq 0, 1$, $a \nleq b'$ and $b \nleq c'$ then $(a \odot b) \odot c = c_2 \odot c = 0 = a \odot c_2 = a \odot (b \odot c)$.
Therefore, $\odot$ is associative. Since $a \leq b'$ is equivalent to $b \leq a'$, $\odot$ is commutative.
If $a = 0$ then $a \odot b = 0$ and $a \leq b'$.
If $b = 0$ then $a \odot b = 0$ and $a \leq b'$.
If $a = 1$ then $a \odot b = 0$ and $a \leq b'$ are equivalent to $b = 0$.
If $b = 1$ then $a \odot b = 0$ and $a \leq b'$ are equivalent to $a = 0$.
If $a, b \neq 0, 1$ then $a \odot b = 0$ is equivalent to $a \leq b'$.
Hence $a \odot b = 0$ if and only if $a \leq b'$.
If $a = 0$ then $a \to b = 1$ and $a \leq b$.
If $b = 0$ then $a \to b = 1$ and $a \leq b$ are equivalent to $a = 0$.

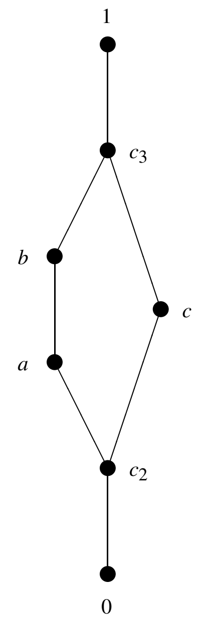

Fig. 2. Extended lattice $\mathbf{N}_5$.

If $a = 1$ then $a \to b = 1$ and $a \le b$ are equivalent to $b = 1$.
If $b = 1$ then $a \to b = 1$ and $a \le b$.
If $a, b \ne 0, 1$ then $a \to b = 1$ and $a \le b$ are equivalent.
Hence $a \to b = 1$ if and only if $a \le b$.
If $a = 0$ then $a \odot b \le c$ and $a \le b \to c$.
If $b = 0$ then $a \odot b \le c$ and $a \le b \to c$.
If $c = 0$ then $a \odot b \le c$ and $a \le b \to c$ are equivalent to $a \le b'$.
If $a = 1$ then $a \odot b \le c$ and $a \le b \to c$ are equivalent to $b \le c$.
If $b = 1$ then $a \odot b \le c$ and $a \le b \to c$ are equivalent to $a \le c$.
If $c = 1$ then $a \odot b \le c$ and $a \le b \to c$.
If $a, b, c \ne 0, 1$ then $a \odot b \le c_2$ and $c_3 \le b \to c$ and hence $a \odot b \le c$ and $a \le b \to c$.
Thus the adjointness property holds. □

As mentioned above, the non-modular lattice $\mathbf{N}_5$ with an antitone involution cannot be converted into a commutative integral residuated lattice $(N_5, \vee, \wedge, \odot, \to, 0, 1)$ with an antitone involution $'$ satisfying the identity $x' \approx x \to 0$. Using Theorem 2, we can extend $\mathbf{N}_5$ as follows.

Example 3. Theorem 2 applied to $\mathbf{N}_5$ yields the commutative integral residuated lattice depicted in Fig. 2.
Observe there is only one possibility for the antitone involution.

Let us note that if $\mathbf{P} = (P, \le, ')$ is a finite chain containing at least three elements (with unique antitone involution) then, using the construction from Theorem 2, $\mathbf{P}$ can be converted into a commutative integral residuated chain $\mathbb{E}(\mathbf{P}) = (P, \le, \odot, \to, 0, 1)$ satisfying the identity $x' \approx x \to 0$.

If the poset $\mathbf{P}$ in question is a lattice, we can also apply the construction of $'$, $\odot$ and $\to$ from Theorem 2 to obtain a commutative integral residuated lattice $\mathbb{E}(\mathbf{P})$. Hence, we can state the following.

Corollary 4. *Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a lattice with an antitone involution $'$. Then $\mathbf{L}$ can be extended to a commutative integral residuated lattice $\mathbb{E}(\mathbf{L})$ with an antitone involution where the operations $'$, $\odot$ and $\to$ are constructed as in Theorem 2 and $'$ coincides in L with the original one.*

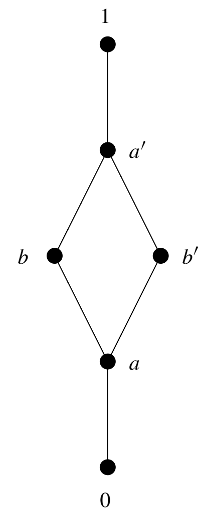

Fig. 3. A Kleene algebra.

Recall from [2] and [11] that a lattice $\mathbf{L} = (L, \vee, \wedge, ')$ with an antitone involution is called a *pseudo-Kleene algebra* if it satisfies the identities

$$x \wedge x' \leq y \vee y', \tag{1}$$

$$x \wedge (x' \vee y) \approx (x \wedge x') \vee (x \wedge y). \tag{2}$$

L is called a *Kleene algebra* if it is a distributive pseudo-Kleene algebra. In this case (2) can be omitted since it follows by distributivity. Kleene algebras and pseudo-Kleene algebras are considered as an algebraic axiomatization of a propositional logic satisfying De Morgan's laws and the double negation law, but not necessarily the law of excluded middle because the antitone involution $'$ need not be a complementation, see also [9]. An example of a Kleene algebra is depicted in Fig. 3.

A pseudo-Kleene algebra which is not a Kleene algebra is visualized in Fig. 4.

One can easily check that if **L** satisfies (1) and (2) then so does $\mathbb{E}(\mathbf{L})$ and if **L** is distributive then $\mathbb{E}(\mathbf{L})$ has this property, too. Hence, we can state

Corollary 5. *Every pseudo-Kleene algebra or every Kleene algebra can be extended to a commutative integral residuated pseudo-Kleene algebra or a commutative integral residuated Kleene-algebra, respectively.*

Hence, also the logic axiomatized by Kleene algebras or pseudo-Kleene algebras can be extended to a kind of fuzzy logics.

Let P be a set and $\mathbf{P}_1 = (P \times \{1\}, \leq)$ and $\mathbf{P}_2 = (P \times \{2\}, \leq)$ posets. We call $\mathbf{P}_2$ the *dual* of $\mathbf{P}_1$ if for all $x, y \in P$ we have $(x, 2) \leq (y, 2)$ if and only if $(y, 1) \leq (x, 1)$. Similarly as before, a poset together with its dual can be extended to a commutative integral residuated poset by means of a finite chain with at least four elements.

By Theorem 2 one can show that every poset with an antitone involution $'$ can be extended to a commutative integral residuated poset $(Q, \leq, \odot, \rightarrow, 1)$ of the form in Fig. 5 or of the form in Fig. 6 with $n > 1$ and $k \geq 0$ satisfying the identity $x' \approx x \rightarrow 0$ where P^d denotes the dual of P.

Conclusion. As we have seen, posets with an antitone involution can be extended to a kind of fuzzy logics. This has an important consequence. Namely, if there are operations $\odot$ and $\rightarrow$ satisfying adjointness then from the trivial identity

$$x \rightarrow y \approx x \rightarrow y$$

we infer

$$(x \rightarrow y) \odot x \leq y.$$

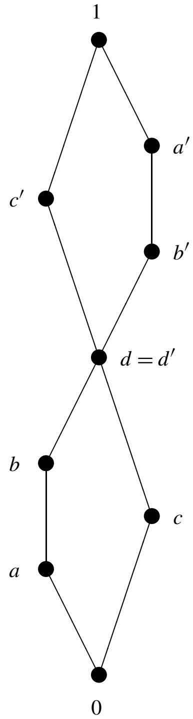

Fig. 4. A pseudo-Kleene algebra.

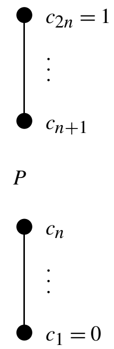

Fig. 5. Extended poset.

Hence, the truth value of proposition y cannot be smaller than the truth value of the conjunction of the proposition x and the implication $x \to y$. In other words, such a logic satisfies the derivation rule *Modus Ponens*. Hence, Corollary 5 shows that every De Morgan logic axiomatized by a Kleene or pseudo-Kleene algebra can be extended to a logic satisfying Modus Ponens.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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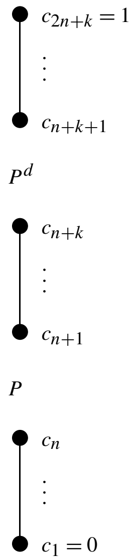

Fig. 6. Extended poset with its dual.

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