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ORIGINAL PAPER

# Independence of axiom system of basic algebras

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**Abstract** We prove that the axiom system of basic algebras as given in Chajda and Emanovský (Discuss Math Gen Algebra Appl 24:31–42, 2004) is not independent. The axiom (BA3) can be deleted and the remaining axioms are shown to be independent. The case when the axiom of double negation is deleted is also treated.

**Keywords** Basic algebra · Axiom system · Independence · Double negation law

The concept of a basic algebra (as a certain generalization of an MV-algebra) was introduced in Chajda and Emanovský (2004). For reader's convenience, we recommend Chajda et al. (2007) as a source of contemporary terminology and the definition. Connections of basic algebras with MV-algebras are treated in Chajda and Halaš (2008) and Botur and Halaš (2008) and basic algebras are used in Chajda et al. (2008) for axiomatization of the so-called many-valued quantum logic.

We are going to describe an independent axiom system for basic algebras, following the work done for MV-algebras by Cattaneo and Lombardo (1998).

We repeat the definition of Chajda et al. (2007).

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**Definition** By a basic algebra is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following axioms

(BA1) $x \oplus 0 = x$;
(BA2) $\neg\neg x = x$ (double negation);
(BA3) $x \oplus \neg 0 = \neg 0 = \neg 0 \oplus x$;
(BA4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Łukasiewicz axiom);
(BA5) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

It is familiar to denote $\neg 0$ by 1 (as it is usual for MV-algebras). It is plain to show that every basic algebra satisfies also the identities $0 \oplus x = x$ and $\neg x \oplus x = \neg 0$ (see Chajda et al. 2007).

Our aim is to prove that the axiom system (BA1)–(BA5) is redundant since (BA3) follows from the remaining ones.

**Theorem 1** *An algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$ *of type* $(2, 1, 0)$ *is a basic algebra if and only if it satisfies the axioms* (BA1), (BA2), (BA4) *and* (BA5).

*Proof* All we need to show is that (BA3) is a conclusion of the axioms (BA1), (BA2), (BA4) and (BA5). For this, take $z = 0$, $x = \neg 0$ and substitute $y$ by $x$ in (BA5) to obtain

$$\neg(\neg(\neg(\neg 0 \oplus x) \oplus x) \oplus 0) \oplus (\neg 0 \oplus 0) = \neg 0.$$

Applying (BA1) and (BA2), we get

$$(\neg(\neg 0 \oplus x) \oplus x) \oplus \neg 0 = \neg 0.$$

We use (BA4) to compute

$$(\neg(\neg x \oplus 0) \oplus 0) \oplus \neg 0 = \neg 0$$

thus, applying (BA1) and (BA2), we obtain

$$x \oplus \neg 0 = \neg\neg x \oplus \neg 0 = \neg 0. \quad (1)$$

Now, put $y = \neg 0$ and $z = 0$ in (BA5) to obtain

$\neg(\neg(\neg(x \oplus \neg 0) \oplus \neg 0) \oplus 0) \oplus (x \oplus 0) = \neg 0.$

By (BA1) we reduce this to

$\neg(\neg(\neg(x \oplus \neg 0) \oplus \neg 0)) \oplus x = \neg 0.$

Using (BA2) and (1), we have

$\neg 0 \oplus x = (0 \oplus \neg 0) \oplus x = \neg 0.$

Together with (1), this yields (BA3). □

We are going to show that the remaining axioms (BA1), (BA2), (BA4) and (BA5) are independent and hence none of them can be reduced.

**Theorem 2** *The axioms* (BA1), (BA2), (BA4) *and* (BA5) *are independent.*

*Proof* Denote by $B$ the two-element set $\{0, 1\}$.

(I) Consider an algebra $(B; \oplus, \neg, 0)$ where $\oplus$ is a constant operation: $x \oplus y = 1$ for all $x, y \in B$ and $\neg 0 = 1$, $\neg 1 = 0$. One can easily check that this algebra satisfies (BA2), (BA4), (BA5) but not (BA1) since $0 \oplus 0 = 1 \neq 0$.

(II) Now, let $(B; \oplus)$ be a join-semilattice and $\neg x = 1$ for all $x \in B$. Then $(B; \oplus, \neg, 0)$ satisfies (BA1), (BA4) and (BA5) but, trivially, not (BA2).

(III) Let $(B; \oplus)$ be a ring $\mathbb{Z}_2$ and $\neg 0 = 1$, $\neg 1 = 0$. Then $(B; \oplus, \neg, 0)$ satisfies evidently (BA1), (BA2). We can show that (BA4) is not satisfied: take $x = 1$, $y = 0$. Then

$$\begin{aligned}\neg(\neg 1 \oplus 0) \oplus 0 &= \neg(0 \oplus 0) \oplus 0 = 1 \oplus 0 = 1 \neq 0\\ &= 1 \oplus 1 = \neg(1 \oplus 1) \oplus 1\\ &= \neg(\neg 0 \oplus 1) \oplus 1.\end{aligned}$$

It remains to prove that $(B; \oplus, \neg, 0)$ satisfies (BA5).

(a) If $y = 0$ then (BA5) is reduced to
$$\neg(x \oplus z) \oplus (x \oplus z) = \neg 0 \tag{2}$$
which is plain to check.

(b) If $y = 1$ and $x = 0$ then (BA5) is
$\neg(\neg(\neg(0 \oplus 1) \oplus 1) \oplus z) \oplus (0 \oplus z) = \neg 0$
which is $\neg(0 \oplus z) \oplus (0 \oplus z) = \neg 0$, i.e (2) with $x = 0$.

(c) If $y = 1$ and $x = 1$ then (BA5) is reduced by means of (2) again.

(IV) Finally, let $(B; \oplus)$ be a join-semilattice and $\neg$ be the identity mapping on $B$. Then clearly (BA1) and (BA2) are satisfied. To prove (BA4) we mention that for $x = y$ it is trivial as well as for the general case $\{x, y\} = \{0, 1\}$ since

$$\begin{aligned}\neg(\neg 1 \oplus 0) \oplus 0 &= 1 \oplus 0 = 1 = 1 \oplus 1\\ &= \neg(\neg 0 \oplus 1) \oplus 1.\end{aligned}$$

It remains to show that (BA5) is violated. For this, take $x = y = z = 1$. Then clearly the left-hand side of (BA5) equals to 1 but the right-hand side is 0.

□

We wonder if any other of the axioms (BA1)–(BA5) can be deleted to obtain a system still determining basic algebras. Unfortunately, we could not find any. We can only prove the following:

**Theorem 3** *Let* $\mathcal{A} = (A; \oplus, \neg, 0)$ *be an algebra of type* $(2, 1, 0)$ *satisfying the axioms* (BA1), (BA3) *and* (BA4). *The following conditions are equivalent*:

(a) $\mathcal{A}$ *satisfies* (BA2);
(b) $\mathcal{A}$ *satisfies the identities* $0 \oplus x = x$ *and* $\neg\neg 0 = 0$.

*Proof* If we apply (BA3), (BA4) and (BA1), we infer

$$\neg\neg 0 \oplus x = \neg(\neg 0 \oplus x) \oplus x = \neg(\neg x \oplus 0) \oplus 0 = \neg\neg x. \tag{3}$$

Assume (BA2). Then $\neg\neg 0 = 0$ and hence (3) yields $0 \oplus x = \neg\neg 0 \oplus x = \neg\neg x = x$ proving (a) ⇒ (b).

Conversely, assume (b). Then (3) yields $x = 0 \oplus x = \neg\neg 0 \oplus x = \neg\neg x$ proving (b) ⇒ (a). □

**Corollary** *An algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$ *of type* $(2, 1, 0)$ *is a basic algebra if and only if it satisfies the axioms* (BA4), (BA5) *and the identities*

(BA1+) $x \oplus 0 = x = 0 \oplus x$;
(BA2-0) $\neg\neg 0 = 0$;
(BA3−) $\neg 0 \oplus x = \neg 0$.

*Proof* It follows from Theorem 3 and the proof of Theorem 1. □

*Remark* The axiom (BA2-0) is essentially weaker than the corresponding axiom of double negation (BA2) since the axioms (BA1+), (BA2-0) and (BA3–) are satisfied also in pseudocomplemented $\vee$−semilattices (where $\oplus$ stands for $\vee$ and $\neg$ is pseudocomplementation).

Let us note that the axiom system presented in the Corollary for basic algebras corresponds to that for MV-algebras from Theorem 2.1 in Cattaneo and Lombardo (1998).

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