

INDEPENDENCE OF THE AXIOMATIC SYSTEM FOR MV-ALGEBRAS

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ABSTRACT. We prove that the well-known axiomatic system of MV-algebras is not independent. The axiom of commutativity can be deleted and the remaining axioms are shown to be independent.

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We describe an independent axiomatic system for MV-algebras. The concept of an MV-algebra was introduced by C. C. Chang [2] as an axiomatization of the Łukasiewicz many-valued logic. The definition used in nowadays is taken from the monograph [3] (with a different order of axioms):

DEFINITION 1. By an *MV-algebra* is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following axioms

$$(MV1) \quad x \oplus 0 = x$$

$$(MV2) \quad \neg \neg x = x$$

$$(MV3) \quad x \oplus y = y \oplus x$$

$$(MV4) \quad (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

$$(MV5) \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

$$(MV6) \quad x \oplus \neg 0 = \neg 0.$$

We show that the axiomatic system (MV1)–(MV6) is redundant.

THEOREM 1. *An algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ is an MV-algebra if and only if it satisfies the axioms (MV1), (MV2), (MV4), (MV5), (MV6).*

Proof. We need to show that (MV3) follows from the axioms (MV1), (MV2), (MV4), (MV5) and (MV6). For this, take $y = 0$ and substitute z by y in (MV4) to obtain

$$(x \oplus 0) \oplus y = x \oplus (0 \oplus y).$$

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Applying (MV1), we get

$$x \oplus y = x \oplus (0 \oplus y). \quad (1)$$

Further, take $x = 0$ and substitute y by x in (MV5) to compute

$$\neg(\neg 0 \oplus x) \oplus x = \neg(\neg x \oplus 0) \oplus 0$$

thus, applying (MV1) and (MV2), we obtain

$$\neg(\neg 0 \oplus x) \oplus x = x. \quad (2)$$

For the next step, we start with (2) where instead of all x is $0 \oplus x$. Thus we have

$$0 \oplus x = \neg(\neg 0 \oplus (0 \oplus x)) \oplus (0 \oplus x).$$

The right hand side of the last identity can be reduced using (1) twice and (2) as follows

$$\neg(\neg 0 \oplus (0 \oplus x)) \oplus (0 \oplus x) = \neg(\neg 0 \oplus (0 \oplus x)) \oplus x = \neg(\neg 0 \oplus x) \oplus x = x.$$

Therefore, we have

$$0 \oplus x = x. \quad (3)$$

Now, put $\neg(\neg x \oplus y)$ instead of x in (MV4) to obtain

$$\neg(\neg x \oplus y) \oplus (y \oplus z) = (\neg(\neg x \oplus y) \oplus y) \oplus z.$$

Since $(\neg(\neg x \oplus y) \oplus y) \oplus z = (\neg(\neg y \oplus x) \oplus x) \oplus z = \neg(\neg y \oplus x) \oplus (x \oplus z)$ by (MV5) and (MV4), we get

$$\neg(\neg x \oplus y) \oplus (y \oplus z) = \neg(\neg y \oplus x) \oplus (x \oplus z). \quad (4)$$

For $y = \neg 0$ in (MV5) we compute

$$\neg(\neg x \oplus \neg 0) \oplus \neg 0 = \neg(\neg \neg 0 \oplus x) \oplus x.$$

From this, applying (MV6), (MV2) and (3), we have

$$\neg 0 = \neg(0 \oplus x) \oplus x,$$

which, using (3) again, give us

$$\neg 0 = \neg x \oplus x. \quad (5)$$

Now, put $\neg y$ instead of y and y instead of z in (4) to obtain

$$\neg(\neg x \oplus \neg y) \oplus (\neg y \oplus y) = \neg(\neg \neg y \oplus x) \oplus (x \oplus y).$$

By (5) and (MV2) we reduce this to

$$\neg(\neg x \oplus \neg y) \oplus \neg 0 = \neg(y \oplus x) \oplus (x \oplus y).$$

Using (MV6), we have

$$\neg 0 = \neg(y \oplus x) \oplus (x \oplus y). \quad (6)$$

Finally, we are going to prove that $x \oplus y = y \oplus x$. Using (3) and (MV2) we compute

$$x \oplus y = 0 \oplus (x \oplus y) = \neg \neg 0 \oplus (x \oplus y).$$

Further, with (6) and (MV5),

$$\neg\neg 0 \oplus (x \oplus y) = \neg(\neg(y \oplus x) \oplus (x \oplus y)) \oplus (x \oplus y) = \neg(\neg(x \oplus y) \oplus (y \oplus x)) \oplus (y \oplus x)$$

which is equal to $\neg\neg 0 \oplus (y \oplus x)$ by (6), where x is substituted by y and vice versa. Since $\neg\neg 0 \oplus (y \oplus x) = 0 \oplus (y \oplus x) = y \oplus x$, by (MV2) and (3), we are done. $\square$

Next, we show that the remaining axioms (MV1), (MV2), (MV4), (MV5) and (MV6) are independent. Hence, none of them can be removed.

THEOREM 2. *The axioms (MV1), (MV2), (MV4), (MV5) and (MV6) are independent.*

Proof. Denote by B the two-element set $\{0, 1\}$.

(I) Consider an algebra $(B; \oplus, \neg, 0)$ where $\oplus$ is a constant operation: $x \oplus y = 0$ for all $x, y \in B$ and $\neg 0 = 0$, $\neg 1 = 1$. One can easily check that this algebra satisfies (MV2), (MV4), (MV5), (MV6) but not (MV1) since $1 \oplus 0 = 0 \neq 1$.

(II) Now, let $(B; \oplus)$ be a join-semilattice and $\neg x = 1$ for all $x \in B$. Then $(B; \oplus, \neg, 0)$ satisfies (MV1), (MV4), (MV5) and (MV6) but, trivially, not (MV2).

(III) Let $\mathcal{C} = (\{0, 1, 2\}; \oplus, \neg, 0)$ be an algebra of type $(2, 1, 0)$ where the operations $\oplus$ and $\neg$ are defined by the following tables:

$\oplus$	0	1	2
0	0	1	1
1	1	1	2
2	2	1	2

x	0	1	2
$\neg x$	1	0	2

Evidently $\mathcal{C}$ satisfies (MV1), (MV2), (MV6). We can show that (MV4) is not satisfied: take $x = 0$, $y = 1$, $z = 2$. Then

$$(0 \oplus 1) \oplus 2 = 1 \oplus 2 = 2 \neq 1 = 0 \oplus 2 = 0 \oplus (1 \oplus 2).$$

It remains to prove that $\mathcal{C}$ satisfies (MV5).

(a) If $y = 0$ then (MV5) is reduced to $x = \neg(1 \oplus x) \oplus x$ which is plain to check.

(b) If $y = 1$ then (MV5) is $1 = \neg(0 \oplus x) \oplus x$ which one can easily check.

(c) For $y = 2$ is (MV5) as follows: $\neg(\neg x \oplus 2) \oplus 2 = \neg(\neg 2 \oplus x) \oplus x$ which also holds for each $x \in \{0, 1, 2\}$.

(IV) Let $\mathcal{D} = (\{0, 1, 2\}; \oplus, \neg, 0)$ be an algebra of type $(2, 1, 0)$ where the operations $\oplus$ and $\neg$ are defined by the following tables:

$\oplus$	0	1	2
0	0	1	0
1	1	1	1
2	2	1	2

x	0	1	2
$\neg x$	1	0	2

Evidently $\mathcal{D}$ satisfies (MV1), (MV2) and (MV6). We can show that (MV5) is not satisfied: take $x = 0$, $y = 2$. Then

$$\neg(\neg 0 \oplus 2) \oplus 2 = \neg(1 \oplus 2) \oplus 2 = \neg 1 \oplus 2 = 0 \neq 2 = \neg\neg 2 = \neg(\neg 2 \oplus 0) \oplus 0.$$

It remains to prove that $\mathcal{D}$ satisfies (MV4).

(a) If $z = 0$ or $z = 2$ then (MV4) is reduced to $x \oplus y = x \oplus y$ which is always true.

(b) If $z = 1$ then (MV4) is $(x \oplus y) \oplus 1 = x \oplus (y \oplus 1)$ which is evidently true.

(V) Finally, let $(B; \oplus)$ be a join-semilattice and $\neg$ be the identity mapping on B . Then clearly (MV1), (MV2) and (MV4) are satisfied. To prove (MV5) we mention that for $x = y$ it is trivial as well as for the general case $\{x, y\} = \{0, 1\}$ since

$$\neg(\neg 1 \oplus 0) \oplus 0 = 1 \oplus 0 = 1 = 1 \oplus 1 = \neg(\neg 0 \oplus 1) \oplus 1.$$

It remains to show that (MV6) is violated. For this, take $x = 1$. Then clearly the left-hand side of (MV6) equals to 1 but the right-hand side is 0. $\square$

It is an open problem whether another axiom, different from (MV3), can be removed from (MV1)–(MV6) to obtain an axiomatic system of MV-algebras. After a preliminary inspection, the author conjectures that this is not possible.

Remark 1. In [1: Theorem 2.2, Theorem 3.2.] it was shown another independent axiomatic system for MV-algebras $(A; \oplus, \neg, 0)$:

$$(P1) \quad (x \oplus y) \oplus z = y \oplus (z \oplus x)$$

$$(P2) \quad x \oplus 0 = x$$

$$(P3) \quad x \oplus \neg 0 = \neg 0$$

$$(P4) \quad \neg\neg 0 = 0$$

$$(P5) \quad \neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x.$$

For more details see [1].

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