

Lexicographic product vs \mathbb{Q} -perfect and \mathbb{H} -perfect pseudo effect algebras

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Abstract We study the Riesz decomposition property types of the lexicographic product of two po-groups. Then we apply them to the study of pseudo effect algebras which can be decomposed into a comparable system of non-void slices indexed by some subgroup of real numbers. Finally, we present their representation by the lexicographic product.

Keywords Pseudo effect algebra · po-group · Unital po-group · Strong unit · Riesz decomposition property · Lexicographic product · n -Perfect pseudo effect algebra · \mathbb{Q} -perfect pseudo effect algebra

1 Introduction

Quantum structures or quantum logics appeared in the early Thirties when the foundations of a new physics, called now quantum physics, have been founded. It was recognized that

the classical rules of statistics of quantum mechanical measurements do not satisfy the axioms of the Kolmogorov probability theory [Kolmogorov \(1933\)](#), and therefore, it was necessary to find a more general structures than Boolean algebras. The first quantum structures were Boolean algebras, orthomodular lattices and posets, orthoalgebras, D-posets and effect algebras. Effect algebras were introduced in [Foulis and Bennett \(1994\)](#) as a counter part to D-posets introduced in [Kôpka and Chovanec \(1994\)](#). D-posets have a primary notion difference of two comparable events while effect algebras have a primary notion addition of two mutually excluding events. A prototypical example of effect algebras is the system $\mathcal{E}(H)$ of all Hermitian operators of a Hilbert space H which are between the zero operator and the identity. In many important examples, an effect algebra is a unit interval in some Abelian partially ordered group. This is true also for $\mathcal{E}(H)$ because it is an interval in the po-group $\mathcal{B}(H)$ of all Hermitian operators of H . In [Ravindran \(1996\)](#) it was shown that every effect algebra with the Riesz Decomposition Property (RDP in short) is an interval in an Abelian po-group with strong unit and with interpolation. A special subfamily of interval effect algebras are MV-algebras, [Cignoli et al. \(2000\)](#).

In [Dvurečenskij and Vetterlein \(2001a,b\)](#), the addition $+$ was not more longer assumed to be commutative and, therefore, pseudo effect algebras were introduced as a noncommutative generalization of effect algebras. Also for these structures it can happen that they are an interval in a unital po-group not necessarily Abelian. This is allowed by a stronger type of RDP, called RDP_1 , which in the commutative case is equivalent to RDP.

Perfect effect algebras were introduced in [Dvurečenskij \(2007\)](#). For them it is assumed that every element is either an infinitesimal or it is a co-infinitesimal, i.e. a negation of an infinitesimal. In [Dvurečenskij \(2008\)](#), n -perfect GMV-

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algebras, which are pseudo effect algebras with the so-called RDP_2 have been introduced. In addition, n -perfect pseudo effect algebras were investigated in Dvurečenskij et al. (2013) as pseudo effect algebras which can be decomposed into $n+1$ comparable slices. For strong n -perfect algebras it was shown that they can be represented by a lexicographic product of a cyclic po-group $\frac{1}{n}\mathbb{Z}$ with some directed torsion-free po-group with RDP_1 .

Therefore, the question when the lexicographic product $\mathbb{Z} \overrightarrow{\times} G$ has RDP_1 was studied in Dvurečenskij and Krňávek (2013) in more detail, and here we present new results in this direction.

In this paper, we will also study pseudo effect algebras, E , which have a decomposition $(E_t : t \in [0, 1] \cap \mathbb{H})$ of E to comparable slices E_t 's, where \mathbb{H} is a subgroup of \mathbb{R} such that $1 \in \mathbb{H}$. We show that in a particular case $\mathbb{H} = \mathbb{Q}$, the group of rational numbers, we are able to present a so-called strong \mathbb{Q} -perfect pseudo effect algebra as an interval in the lexicographic product $\mathbb{Q} \overrightarrow{\times} G$ with some directed torsion-free po-group G . The same will be done for a general subgroup \mathbb{H} . The paper will generalize the results from Dvurečenskij et al. (2013), where a special case $\mathbb{H} = \frac{1}{n}\mathbb{Z}$ was studied as n -perfect pseudo effect algebras.

The paper is organized as follows. Section 2 is gathering the basic notions and results on pseudo effect algebras and po-groups. In Sect. 3, we extend the Riesz decomposition properties of the lexicographic product of two po-groups studied originally in Dvurečenskij and Krňávek (2013). Section 4 is concentrated to the description of \mathbb{H} -decompositions of pseudo effect algebras. Section 5 is devoted to the study of \mathbb{H} -perfect pseudo effect algebras. Section 6 describes some representations of strong \mathbb{Q} -perfect pseudo effect algebras by the lexicographic product of type $\mathbb{Q} \overrightarrow{\times} G$. Section 7 deals with the most general case - to strong \mathbb{H} -perfect pseudo effect algebras. In addition, some open problems are formulated.

2 Basic facts on pseudo effect algebras and po-groups

According to Dvurečenskij and Vetterlein (2001a,b), we say that a partial algebraic structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is said to be a *pseudo effect algebra*, if for all $a, b, c \in E$, the following hold.

- (PE1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$.
- (PE2) There are exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.
- (PE3) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.
- (PE4) If $a + 1$ or $1 + a$ exists, then $a = 0$.

In every pseudo effect algebra we can derive a partial order, \leq , defined as follows: $a \leq b$ if and only if there exists an element $c \in E$ such that $a + c = b$. Then 0 and 1 is the least and greatest element of E . In addition, $a \leq b$ if and only if $b = a + c = d + a$ for some $c, d \in E$. We write $c = a \setminus b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a \setminus b) = b,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a/1$ for the *left* and *right negation*, respectively, of any $a \in E$.

We recall that if $+$ is commutative, E is said to be an *effect algebra*; for a comprehensive survey on effect algebras we recommend e.g. Dvurečenskij and Pulmannová (2000).

A pseudo effect algebra E is said to be *symmetric* if $a^- = a^\sim$ for each $a \in E$. We note that it can happen that E is symmetric but $+$ is not commutative, see e.g. the proof of Proposition 4.7 below.

A non-empty subset I of a pseudo effect algebra E is said to be an *ideal* if (1) $a, b \in I, a + b \in E$, then $a + b \in I$, and (2) if $a \leq b \in I$, then $a \in I$. An ideal $I \neq E$ is *maximal* if it is not a proper subset of any ideal $J \neq E$.

A mapping h from a pseudo effect algebra E into another one F is said to be a *homomorphism* if (1) $h(1) = 1$, and (2) if $a + b$ is defined in E , so is defined $h(a) + h(b)$ and $h(a + b) = h(a) + h(b)$.

Let a be an element of a pseudo effect algebra E and $n \geq 0$ be an integer. We define

$$0a := 0, \quad 1a := a, \quad na := (n - 1)a + a, \quad n \geq 2,$$

supposing $(n - 1)a$ and $(n - 1)a + a$ are defined in E . An element a is *infinitesimal* if na exists in E for each $n \geq 1$. We denote by $\text{Infinit}(E)$ the set of all infinitesimal elements of E .

A state on E is any mapping $s : E \rightarrow [0, 1]$ such that (1) $s(1) = 1$, and (2) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in E . Let $\mathcal{S}(E)$ be the set of states on E . It can happen that $\mathcal{S}(E)$ is empty. A state s is *extremal* if from $s = \lambda s_1 + (1 - \lambda)s_2$ for some $s_1, s_2 \in \mathcal{S}(E)$ and $0 < \lambda < 1$ one follows $s_1 = s_2 = s$.

An ideal I is *normal* if $x + I := \{x + i : i \in I\} = I + x := \{j + x : j \in I\}$ for any $x \in E$. For example, if s is a state on E , then

$$\text{Ker}(s) := \{x \in E : s(x) = 0\}$$

is a normal ideal on E .

If A, B are two nonempty subsets of E , we set $A + B := \{a + b : a \in A, b \in B, a + b \text{ is defined in } E\}$. We say that $A + B$ is *defined* in E if $a + b$ is defined in E for each $a \in A$ and each $b \in B$. Similarly, we write $A \leq B$ if $a \leq b$ for each $a \in A$ and each $b \in B$.

We recall that a *po-group* (=partially ordered group) is a group $(G; +, 0)$ endowed with a partial order \leq such that if $a \leq b, a, b \in G$, then $x + a + y \leq x + b + y$ for all $x, y \in G$.

We denote by $G^+ = \{g \in G : g \geq 0\}$ the *positive cone* of G . If, in addition, G is a lattice under \leq , we call it an ℓ -group (=lattice ordered group). An element $u \in G^+$ is said to be a *strong unit* (=order unit) if given $g \in G$, there is an integer $n \geq 1$ such that $g \leq nu$, and the couple (G, u) with a fixed strong unit u is said to be a *unital po-group*.

The set $C(G) := \{c \in G : c + g = g + c, \forall g \in G\}$ is said to be the commutative center of G .

We say a po-group G is directed if, given $g_1, g_2 \in G$, there is $g \in G$ such that $g_1, g_2 \leq g$. This is equivalent to the statement: given $h_1, h_2 \in G$, there is $h \in G$ such that $h \leq h_1, h_2$. A subgroup H of a po-group G is *convex* if $a, b \in H$ and $c \in G$ with $a \leq c \leq b$ imply $c \in H$.

An important class of pseudo effect algebras can be obtained as follows. If (G, u) is a unital (not necessary Abelian) po-group with strong unit u , and

$$\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}, \tag{2.1}$$

then $(\Gamma(G, u); +, 0, u)$ is a pseudo effect algebra if we restrict the group addition $+$ to $\Gamma(G, u)$. Every pseudo effect algebra E that is isomorphic to some $\Gamma(G, u)$ is said to be an *interval pseudo effect algebra*.

For basic properties of pseudo effect algebras see [Dvurečenskij and Vetterlein \(2001a,b\)](#).

The following kinds of the Riesz decomposition properties were introduced in [Dvurečenskij and Vetterlein \(2001a,b\)](#) and are crucial for the study of pseudo effect algebras.

We say that a po-group G satisfies

- (1) the Riesz interpolation property (RIP for short) if, for $a_1, a_2, b_1, b_2 \in G$, $a_1, a_2 \leq b_1, b_2$ implies there exists an element $c \in G$ such that $a_1, a_2 \leq c \leq b_1, b_2$;
- (2) RDP_0 if, for $a, b, c \in G^+$, $a \leq b + c$, there exist $b_1, c_1 \in G^+$, such that $b_1 \leq b$, $c_1 \leq c$ and $a = b_1 + c_1$;
- (3) RDP if, for all $a_1, a_2, b_1, b_2 \in G^+$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$; this property will be formally denoted by the following table:

a_1	c_{11}	c_{12}
a_2	c_{21}	c_{22}
	b_1	b_2

- (4) RDP_1 if, for all $a_1, a_2, b_1, b_2 \in G^+$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$, and $0 \leq x \leq c_{12}$ and $0 \leq y \leq c_{21}$ imply $x + y = y + x$;
- (5) RDP_2 if, for all $a_1, a_2, b_1, b_2 \in G^+$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$, and $c_{12} \wedge c_{21} = 0$;

- (6) the *Strong Riesz Interpolation Property* (SRIP for short) if, for $a_1, a_2, b_1, b_2 \in G^+$ with $a_1, a_2 < b_1, b_2$, there is $c \in G$ such that $a_1, a_2 < c < b_1, b_2$.

If, for $a, b \in G^+$, we have for all $0 \leq x \leq a$ and $0 \leq y \leq b$, $x + y = y + x$, we denote this property by *a com b*.

For Abelian po-groups, RDP , RDP_1 , RDP_0 and RIP are equivalent.

We recall that the po-groups of reals numbers \mathbb{R} and rational numbers \mathbb{Q} satisfy SRIP, but the group of integers \mathbb{Z} does not. In addition, let \mathbb{H} be any subgroup of \mathbb{R} . Due to ([Good-earl 1986](#), Lem 4.21), \mathbb{H} is either cyclic, i.e. it is isomorphic to some $\frac{1}{n}\mathbb{Z}$ or is dense in \mathbb{R} . Also in the latter case, SRIP holds for \mathbb{H} .

A po-group G satisfies RDP_2 iff G is an ℓ -group, ([Dvurečenskij and Vetterlein 2001a](#), Prop 4.2(2)).

We say that a pseudo effect algebra E satisfies the above types of the Riesz decomposition properties, if in the definition of RDP 's, we change G^+ to E .

The basic result of pseudo effect algebras is the following representation theorem ([Dvurečenskij and Vetterlein 2001b](#), Thm 7.2).

Theorem 2.1 *For every pseudo effect algebra with RDP_1 , there is a unique (up to isomorphism of unital po-groups) unital po-group (G, u) with RDP_1 such that $E \cong \Gamma(G, u)$.*

We note that a pseudo effect algebra E satisfies RDP_2 iff E is a lattice and if it satisfies RDP_1 iff E is in fact a pseudo MV-algebra ([Dvurečenskij and Vetterlein 2001b](#), Thm 8.8). For them we have that the variety of pseudo MV-algebras is categorically equivalent to the category of unital ℓ -groups, see [Dvurečenskij \(2002\)](#).

A *state* on a unital po-group (G, u) is any mapping $s : G \rightarrow \mathbb{R}$ such that (i) $s(g) \geq 0$ for $g \in G^+$, (ii) $s(g_1 + g_2) = s(g_1) + s(g_2)$, $g_1, g_2 \in G$, and (iii) $s(1) = 1$. If a pseudo effect algebra $E = \Gamma(G, u)$ satisfies RDP_1 , then every state on E can be extended to a unique state on (G, u) , and conversely, the restriction of every state on (G, u) to $[0, u]$ gives a state on E , see [Dvurečenskij and Vetterlein \(2001b\)](#). In addition, an extremal state is extendible only to an extremal state on (G, u) and vice-versa.

3 Lexicographic product and RDP 's

In this section, we continue with the study of RDP 's properties of the lexicographic product which started in [Dvurečenskij and Krňávek \(2013\)](#) and we concentrate to the lexicographic product of two po-groups.

Let G_1 and G_2 be two po-groups and we define the direct product group $G_1 \times G_2$ with the group addition defined by coordinates. We define the *lexicographic order* \leq on $G_1 \times G_2$

by $(g_1, g_2) \leq (h_1, h_2)$ iff either $g_1 < g_2$ or $g_1 = g_2$ and $h_1 \leq h_2$, for $(g_1, g_2), (h_1, h_2) \in G_1 \times G_2$, and $G_1 \overrightarrow{\times} G_2$ will denote the *lexicographic product* of G_1 and G_2 endowed with the above defined lexicographic order.

Theorem 3.1 *Let G_1 be a linearly ordered po-group and let G_2 be a directed po-group with RDP. Then the po-group $G = G_1 \overrightarrow{\times} G_2$ satisfies RDP.*

Proof Let G_2 be a directed po-group with RDP. The positive cone $(G_1 \overrightarrow{\times} G_2)^+$ is of the form $\{(0, a) : a \in G_2^+\} \cup \{(n, a) : n \in G_1^+ \setminus \{0\}, a \in G_2\}$. Assume that

$$(m_1, a_1) + (m_2, a_2) = (n_1, b_1) + (n_2, b_2)$$

holds in $(G_1 \overrightarrow{\times} G_2)^+$.

- (1) Let $(0, a_1) + (0, a_2) = (0, b_1) + (0, b_2)$. Then $a_1, a_2, b_1, b_2 \in G_2^+$ and RDP for this case follows from RDP for G_2 .
- (2) Let $m_1 + m_2 > 0$ and $m_2 = 0$. For the subcase $n_2 > 0$, we use the same ideas as case (1) of (Dvurečenskij and Krňávek 2013, Prop 3.3, Thm 3.10), and we have the following decomposition table

$$\begin{array}{c|cc} (m_1, a_1) & (n_1, b_1) & (n_2, -b_1 + a_1) \\ (0, a_2) & (0, 0) & (0, a_2) \\ \hline & (n_1, b_1) & (n_2, b_2) \end{array} \text{ for } n_2 > 0.$$

For the subcase $n_2 = 0$, we have $a_2, b_2 \geq 0$. As G_2 is directed, there exists $d \in G_2$ with $d \leq a_1, b_1$. Then there exist $e_{11}, e_{12}, e_{21}, e_{22} \in G_2^+$ such that

$$\begin{array}{c|cc} -d + a_1 & e_{11} & e_{12} \\ a_2 & e_{21} & e_{22} \\ \hline & -d + b_1 & b_2 \end{array}$$

which yields

$$\begin{array}{c|cc} a_1 & d + e_{11} & e_{12} \\ a_2 & e_{21} & e_{22} \\ \hline & b_1 & b_2 \end{array},$$

and finally

$$\begin{array}{c|cc} (m_1, a_1) & (n_1, d + e_{11}) & (0, e_{12}) \\ (0, a_2) & (0, e_{21}) & (0, e_{22}) \\ \hline & (n_1, b_1) & (0, b_2) \end{array} \text{ for } n_2 = 0.$$

The case $(m_1, a_1) + (0, a_2) = (0, b_1) + (n_2, b_2)$ follows from the first case of (2).

In a similar way we deal with the case $(0, a_1) + (m_2, a_2) = (n_1, b_1) + (n_2, b_2)$, where $n_1 \geq 1$. Then $m_2 \geq 1, a_1 \geq 0$, and we use the decomposition

$$\begin{array}{c|cc} (0, a_1) & (0, a_1) & (0, 0) \\ (m_2, a_2) & (n_1, -a_1 + b_1) & (n_2, b_2) \\ \hline & (n_1, b_1) & (n_2, b_2) \end{array}.$$

- (3) Let $m_1, m_2, n_1, n_2 > 0$. We use the tables

$$\begin{array}{c|cc} -d + a_1 & e_{11} & e_{12} \\ a_2 - d & e_{21} & e_{22} \\ \hline & -d + b_1 & b_2 - d \end{array},$$

where $d \leq a_1, a_2, b_1, b_2$, and then

$$\begin{array}{c|cc} (m_1, a_1) & (n_1, d + e_{11}) & (-n_1 + m_1, e_{12}) \\ (m_2, a_2) & (0, e_{21}) & (m_2, e_{22} + d) \\ \hline & (n_1, b_1) & (n_2, b_2) \end{array} \text{ for } m_1 \geq n_1,$$

where $d \leq a_1, a_2, b_1, b_2$, and finally

$$\begin{array}{c|cc} (m_1, a_1) & (m_1, d + e_{11}) & (0, e_{12}) \\ (m_2, a_2) & (-m_1 + n_1, e_{21}) & (n_2, e_{22} + d) \\ \hline & (n_1, b_1) & (n_2, b_2) \end{array} \text{ for } n_1 \geq m_1,$$

where $d \leq a_1, a_2, b_1, b_2$. The second case follows also from the first one, if we take into account that $(n_1, b_1) + (n_2, b_2) = (m_1, a_1) + (m_2, a_2)$. □

Theorem 3.2 *Let G_1 be a linearly ordered po-group and let G_2 be an Abelian directed po-group with RDP. Then the po-group $G_1 \overrightarrow{\times} G_2$ satisfies RDP₁.*

Proof We use the same decompositions as in the proof of Theorem 3.1. It is enough to verify RDP₁ for (3) of Theorem 3.1. Assume that $(0, 0) \leq (x_1, x_2) \leq (0, e_{21})$ and $(0, 0) \leq (y_1, y_2) \leq (-n_1 + m_1, e_{12})$, then $x_1 = 0$ and evidently, $(x_1, x_2) + (y_1, y_2) = (y_1, y_2) + (x_1, x_2)$. □

The following result has been proved in (Dvurečenskij and Krňávek 2013, Thm 3.7).

Theorem 3.3 *A po-group $\mathbb{Z} \overrightarrow{\times} G$ satisfies RDP₁ if and only if G is a directed po-group with RDP₁.*

In addition, there is a categorical equivalence of the category of pseudo effect algebras with RDP₁ with the category of unital po-groups (G, u) with RDP₁.

Let $n \geq 1$ be an integer. We define $\frac{1}{n}\mathbb{Z}$ as the ℓ -group of rational numbers of the form $\{\frac{i}{n} : i \in \mathbb{Z}\}$ which is an ℓ -group isomorphic to the group of integers \mathbb{Z} . The number 1 is a strong unit for $\frac{1}{n}\mathbb{Z}$.

Proposition 3.4 *Let $n \geq 1$ be an integer and G a po-group. The following statements are equivalent:*

- (1) *The po-group $\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G$ satisfies RDP₁.*
- (2) *The po-group G is directed and with RDP₁.*
- (3) *The po-group G is directed and the pseudo effect algebra $\Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ satisfies RDP₁.*

Proof (1) \Leftrightarrow (2) The po-group $E_n := \frac{1}{n}\mathbb{Z} \overrightarrow{\times} G$ is isomorphic to the po-group $\mathbb{Z} \overrightarrow{\times} G$. By (Dvurečenskij and Krňávek 2013, Thm 3.7), $\mathbb{Z} \overrightarrow{\times} G$ satisfies RDP₁ iff G is a directed po-group with RDP₁. Consequently, the po-group $\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G$ satisfies RDP₁ iff G is a directed po-group with RDP₁. (1) \Rightarrow (3) is trivial.

Finally, let (3) hold. Assume $g_1, g_2 \leq h_1, h_2$ for some $g_1, g_2, h_1, h_2 \in G$. Directness of G entails the existence of $g_0 \in G$ such that $g_0 \leq g_1, g_2$. Then $0 \leq g_1 - g_0, g_2 - g_0 \leq h_1 - g_0, h_2 - g_0$ which yields $(0, g_1 - g_0), (0, g_2 - g_0) \leq (0, h_1 - g_0), (0, h_2 - g_0) \in E_n$ and RDP₁ of E_n implies the existence of $g \in G^+$ such that $(0, g_1 - g_0), (0, g_2 - g_0) \leq (0, g) \leq (0, h_1 - g_0), (0, h_2 - g_0)$. Hence, $g_1 - g_0, g_2 - g_0 \leq g \leq h_1 - g_0, h_2 - g_0$ and finally, $g_1, g_2 \leq g + g_0 \leq h_1, h_2$ which implies G satisfies RIP. By (Dvurečenskij and Krňávek 2013, Prop 3.1(1)), $\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G$ satisfies RIP, and by (Dvurečenskij and Krňávek 2013, Thm 3.6), $\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G$ satisfies RDP₁. \square

Theorem 3.5 *Let G be a po-group. Consider the following statements:*

- (1) *The po-group $\mathbb{Q} \overrightarrow{\times} G$ satisfies RDP₁.*
- (2) *The po-group G satisfies RDP₁.*
- (2') *The po-group G is directed and satisfies RDP₁.*
- (3) *The pseudo effect algebra $E := \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0))$ satisfies RDP₁.*

Then (1) implies (2), and (1) and (3) are equivalent. If G is directed, then (1) and (2') are equivalent, too.

Proof (1) \Rightarrow (2). By (Dvurečenskij and Krňávek 2013, Prop 3.2 (2)), G is a po-group with RDP₁. (1) \Rightarrow (3). It is trivial. (3) \Rightarrow (1). Since G satisfies also RIP, by (Dvurečenskij and Krňávek 2013, Prop 3.2(1)), $\mathbb{Q} \overrightarrow{\times} G$ satisfies RIP. Applying (Dvurečenskij and Krňávek 2013, Thm 3.6), $\mathbb{Q} \overrightarrow{\times} G$ satisfies RDP₁.

Now let G be directed. Due to the implication (1) \Rightarrow (2), (1) implies (2'). Conversely, let G satisfy RDP₁, and let for $(r_1, g_1), (r_2, g_2), (q_1, h_1), (q_2, h_2) \in (\mathbb{Q} \overrightarrow{\times} G)^+$, we have $(r_1, g_1) + (r_2, g_2) = (q_1, h_1) + (q_2, h_2)$. There is an integer $n \geq 2$ such that $r_1 = \frac{i_1}{n}, r_2 = \frac{i_2}{n}, q_1 = \frac{j_1}{n}$ and $q_2 = \frac{j_2}{n}$. Applying Proposition 3.4(1), $\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G$ satisfies RDP₁. Hence, we can find the necessary RDP₁ decomposition for $(\frac{i_1}{n}, g_1) + (\frac{i_2}{n}, g_2) = (\frac{j_1}{n}, h_1) + (\frac{j_2}{n}, h_2)$, which implies that $\mathbb{Q} \overrightarrow{\times} G$ satisfies RDP₁. \square

We note that in contrast to (2) of Proposition 3.4, G is not necessarily directed because \mathbb{Q} satisfies SRIP.

The following result can be proved in the same way as Theorem 3.5.

Theorem 3.6 *Let G be a po-group. Define the following statements:*

- (1) *The po-group $\mathbb{R} \overrightarrow{\times} G$ satisfies RDP₁.*
- (2) *The po-group G is with RDP₁.*
- (3) *The pseudo effect algebra $E := \Gamma(\mathbb{R} \overrightarrow{\times} G, (1, 0))$ satisfies RDP₁.*

Then (1) and (3) are equivalent, and (1) implies (2).

Now we show when (2) in Theorem 3.6 implies (1).

Theorem 3.7 *Let G be a directed po-group. The following statements are equivalent:*

- (1) *The po-group $\mathbb{R} \overrightarrow{\times} G$ satisfies RDP₁.*
- (2) *The po-group G is with RDP₁.*
- (3) *The pseudo effect algebra $E := \Gamma(\mathbb{R} \overrightarrow{\times} G, (1, 0))$ satisfies RDP₁.*

Proof According to Theorem 3.6, it is necessary to show that (2) implies (1). Assume that $(s_1, a_1) + (s_2, a_2) = (t_1, b_1) + (t_2, b_2)$ holds in $(\mathbb{R} \overrightarrow{\times} G)^+$.

If we use the RDP decomposition tables from the proof of Theorem 3.1 which were used there for two first cases, we see that for these particular cases we have RDP₁ decompositions. According to that proof, we can assume that all s_1, s_2, t_1, t_2 are strictly positive. Since \mathbb{Q} is dense in \mathbb{R} , we can find two rational numbers r_1 and r_2 in \mathbb{Q}^+ such that $0 < r_i < s_i$ if s_i is irrational, otherwise, $r_i = s_i$. We can choose also two additional rational numbers $q_1, q_2 \in \mathbb{Q}$ such that $q_1 + q_2 = r_1 + r_2$ and $0 < q_i < t_i$ if $t_i > 0$ otherwise $q_i = t_i$. By Theorem 3.5, $\mathbb{Q} \overrightarrow{\times} G$ satisfies RDP₁, therefore, we can find the following RDP₁ decomposition table

$$\begin{array}{c|cc} (r_1, a_1) & (r_{11}, e_{11}) & (r_{12}, e_{12}) \\ (r_2, a_2) & (r_{21}, e_{21}) & (r_{22}, e_{22}) \\ \hline & (q_1, b_1) & (q_2, b_2) \end{array},$$

where $(r_{ij}, e_{ij}) \in (\mathbb{Q} \overrightarrow{\times} G)^+$ for $i, j = 1, 2$. Without loss of generality, we can assume that all r_{ij} 's are strictly positive.

Since $(s_1 - r_1) + (s_2 - r_2) = (t_1 - q_1) + (t_2 - q_2)$ and all summands are nonnegative, using RDP holding in \mathbb{R} , we have the RDP decomposition table

$$\begin{array}{c|cc} (s_1 - r_1) & s_{11} & s_{12} \\ (s_2 - r_2) & s_{21} & s_{22} \\ \hline & (t_1 - q_1) & (t_2 - q_2) \end{array},$$

where $s_{11}, s_{12}, s_{21}, s_{22} \in [0, \infty)$. Then we have the following RDP decomposition

$$\begin{array}{c|cc} (s_1, a_1) & (r_{11} + s_{11}, e_{11}) & (r_{12} + s_{12}, e_{12}) \\ (s_2, a_2) & (r_{21} + s_{21}, e_{21}) & (r_{22} + s_{22}, e_{22}) \\ \hline & (t_1, b_1) & (t_2, b_2) \end{array}.$$

Finally, we have to show that $(r_{12} + s_{12}, e_{12}) \mathbf{com} (r_{21} + s_{21}, e_{21})$. Choose $(0, 0) \leq (x_1, x_2) \leq (r_{12} + s_{12}, e_{12})$ and $(0, 0) \leq (y_1, y_2) \leq (r_{21} + s_{21}, e_{21})$.

We have the following four cases.

- (1) $x_1 > 0, y_1 > 0$. Choose two rational numbers r' and r'' such that $0 < r' < x_1$ and $0 < r'' < y_1$. Then $(0, 0) < (r', x_2) < (x_1, x_2)$, $(0, 0) < (r'', y_2) < (y_1, y_2)$, and $(0, 0) \leq (r', x_2) < (r_{12}, e_{12})$, $(0, 0) \leq (r'', y_2) < (r_{21}, e_{21})$. By the RDP_1 table used above and holding in $\mathbb{Q} \overrightarrow{\times} G$, (r', x_2) and (r'', y_2) commute, so that x_2 and y_2 commute, which yields $(x_1, x_2) + (y_1, y_2) = (y_1, y_2) + (x_1, x_2)$.
- (2) $x_1 = 0, y_1 = 0$. Then $(0, 0) \leq (0, x_2) < (r_{12} + s_{12}, e_{12})$ which entails $(0, x_2) < (r_{12}, e_{12})$ and similarly, $(0, 0) \leq (0, y_2) < (r_{21}, e_{21})$. Therefore, we have $(0, x_2)$ and $(0, y_2)$ commute, so that x_2 and y_2 commute, too, and finally (x_1, x_2) and (y_1, y_2) commute.
- (3) $x_1 > 0, y_1 = 0$. Choose a rational number $r > 0$ such $0 < r < x_1$. Then $(0, 0) \leq (r, x_2) \leq (x_1, x_2) \leq (r_{12} + s_{12}, e_{12})$. Then $(0, 0) \leq (r, x_2) < (r_{12}, e_{12})$. Similarly as in the case (ii), we have $(0, 0) \leq (0, y_2) < (r_{21}, e_{21})$. Therefore, (x, x_2) and $(0, y_2)$ commute, which entails x_2 and y_2 also commute, and finally (x_1, x_2) and (y_1, y_2) commute.
- (4) $x_1 = 0, y_1 > 0$. It follows the same steps as the proof of the case (3).

Combining all cases (1)–(4), we see that RDP_1 holds as was claimed. \square

The last two theorems can be generalized as follows.

Theorem 3.8 *Let G be a directed po-group and \mathbb{H} be a subgroup of the group \mathbb{R} such that $1 \in \mathbb{H}$. The following statements are equivalent:*

- (1) *The po-group $\mathbb{H} \overrightarrow{\times} G$ satisfies RDP_1 .*
- (2) *The po-group G is with RDP_1 .*
- (3) *The pseudo effect algebra $E := \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$ satisfies RDP_1 .*

Proof (1) \Rightarrow (2). It follows from (Dvurečenskij and Krňávek 2013, Prop 3.2(2)). (1) \Rightarrow (3). It is trivial. (3) \Rightarrow (1). Since G satisfies also RIP, by (Dvurečenskij and Krňávek 2013, Prop 3.2 (1)), $\mathbb{H} \overrightarrow{\times} G$ satisfies RIP. Applying (Dvurečenskij and Krňávek 2013, Thm 3.6), $\mathbb{H} \overrightarrow{\times} G$ satisfies RDP_1 . (2) \Rightarrow (1). By the assumption, G is directed. Due to (Goodearl 1986, Lem 4.21), \mathbb{H} is either a cyclic subgroup of \mathbb{R} or a dense subset of \mathbb{R} .

In the first case, $\mathbb{H} = \frac{1}{n}\mathbb{Z}$ for some integer $n \geq 1$. Therefore, we can apply Proposition 3.4, and whence, $\mathbb{H} \overrightarrow{\times} G$ has RDP_1 .

In the second case, we can literally apply the proof of the implication (2) \Rightarrow (1) of Theorem 3.7 changing \mathbb{R} to \mathbb{H} . \square

Problem 1. Since both \mathbb{Q}, \mathbb{R} as well as any non-discrete \mathbb{H} satisfy SRIP, according to (Goodearl 1986, Cor 2.12) or (Dvurečenskij and Krňávek 2013, Prop 3.2(1)), $\mathbb{Q} \overrightarrow{\times} G,$

$\mathbb{R} \overrightarrow{\times} G$ and $\mathbb{H} \overrightarrow{\times} G$ satisfy RDP_1 whenever G is an Abelian po-group, not necessarily directed. It is unknown whether is this true also for any po-group G not necessarily Abelian.

4 \mathbb{H} -decompositions of pseudo effect algebras

In the rest of the paper we will assume that \mathbb{H} is a subgroup of the group of real numbers \mathbb{R} such that $1 \in \mathbb{H}$ and we denote $[0, 1]_{\mathbb{H}} := [0, 1] \cap \mathbb{H}$. We define two types of decompositions of a pseudo effect algebra indexed by elements of $[0, 1]_{\mathbb{H}}$.

We say that a decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ of a pseudo effect algebra E , i.e. a system $(E_t : t \in [0, 1]_{\mathbb{H}})$ of nonempty subsets of E such that $E_s \cap E_t = \emptyset$ for $s < t, s, t \in [0, 1]_{\mathbb{H}}$ and $\bigcup_{t \in [0, 1]_{\mathbb{H}}} E_t = E$ is

- (I) *an \mathbb{H} -decomposition of type I* if
 - (a) $E_s + E_t$ exists if $s + t < 1$,
 - (b) E_0 is a unique maximal ideal of E ;
- (2) *an \mathbb{H} -decomposition* if
 - (a) $E_t^- = E_t^{\sim} = E_{1-t}$ for any $t \in [0, 1]_{\mathbb{H}}$,
 - (b) if $x \in E_s, y \in E_t$ and $x + y$ exists in E , then $s + t \leq 1$ and $x + y \in E_{s+t}$ for $s, t \in [0, 1]_{\mathbb{H}}$.

For example, if $E = \Gamma(\mathbb{R} \overrightarrow{\times} G, (1, 0))$, then $(E_t : t \in [0, 1])$, where $E_0 := \{(0, g) : g \in G^+\}, E_t = \{(t, g) : g \in G\}$ for $0 < t < 1$ and $E_1 = \{(1, -g) : g \in G^+\}$, and G is a po-group, is an \mathbb{R} -decomposition of the pseudo effect algebra E of both types. In fact, we have $E_s + E_t = E_{s+t}$, whenever $s + t < 1$. To show that E_0 is a maximal ideal, take $(t, g) \in E \setminus E_0$, and let I be the ideal of E generated by E_0 and (t, g) . Then, for any $s < t, E_s \subseteq I$. In particular, there is an integer n such that $1/n < t$. Then $(1/n, 0) \in I$ but $(1, 0) = n(1/n, 0) \in I$ proving $I = E$. In addition, it is easy to verify that E_0 is also a normal ideal, $E_0 = \text{Infinit}(E)$ and $E_t^- = E_{1-t} = E_t^{\sim}$ for any $t \in [0, 1]$, and E_0 is a unique maximal ideal of E .

We say that a state s on a pseudo effect algebra E is an \mathbb{H} -valued state if $s(E) = [0, 1]_{\mathbb{H}}$, where $[0, 1]_{\mathbb{H}} := \Gamma(\mathbb{H}, 1) = [0, 1] \cap \mathbb{H}$. If $s(E) \subseteq [0, 1]_{\mathbb{H}}$, we say that s is an \mathbb{H} -state. In particular, if $\mathbb{H} = \frac{1}{n}\mathbb{Z}$, a $\frac{1}{n}\mathbb{Z}$ -valued state is also said to be an $(n + 1)$ -valued discrete state, (Dvurečenskij et al. (2013).

Theorem 4.1 *Let E be a pseudo effect algebra and \mathbb{H} be a subgroup of \mathbb{R} containing 1. The following two statements are equivalent:*

- (1) *There exists an \mathbb{H} -valued state on E .*
- (2) *There exists an \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ of nonempty subsets of E .*

Proof Let s be an \mathbb{H} -valued state. Given $t \in [0, 1]_{\mathbb{H}}$, we define $E_t := s^{-1}(\{t\})$. It is evident that, the system $(E_t : t \in [0, 1]_{\mathbb{H}})$, is a decomposition of E . For (a), let $x \in E_t$. Then $s(x) = t$ and $s(x^-) = 1 - s(x) = s(x^{\sim})$, which proves $x^-, x^{\sim} \in E_{1-t}$. Conversely, if $y \in E_{1-t}$, then $y^-, y^{\sim} \in E_t$. For (b), assume $x \in E_s$ and $y \in E_t$ and let $x + y$ be defined in E . Then $s(x + y) = s(x) + s(y) = s + t \leq 1$, which implies $x + y \in E_{s+t}$. Then $(E_t : t \in [0, 1]_{\mathbb{H}})$ is an \mathbb{H} -decomposition of E .

Conversely, let (ii) hold. Define a mapping $s : E \rightarrow [0, 1]$ by $s(x) = t$ iff $x \in E_t$. Take $x, y \in E$ such that $s(x) = t_1$ and $s(y) = t_2$. Since $x \in E_{t_1}$ and $y \in E_{t_2}$, due to (b), we have $t_1 + t_2 \leq 1$. Hence, $s(x + y) = s(x) + s(y)$. There exists a unique $t \in [0, 1]_{\mathbb{H}}$ such that $0 \in E_t$. For every $x \in E_1$, $x + 0 = x = 0 + x$, thus by (b), $t + 1 \leq 1$ which yields $t = 0$ and therefore, $s(0) = 0$ and $s(1) = 1$. In other words, s is an \mathbb{H} -valued state. \square

We denote by $\mathcal{S}_{\mathbb{H}}(E)$ and $\mathcal{D}_{\mathbb{H}}(E)$ the set of all \mathbb{H} -decompositions and the set of all \mathbb{H} -valued states on H , respectively.

Theorem 4.2 *There exists a one-to-one correspondence between $\mathcal{D}_{\mathbb{H}}(E)$ and $\mathcal{S}_{\mathbb{H}}(E)$.*

Proof By Theorem 4.1, $\mathcal{D}_{\mathbb{H}}(E)$ is nonempty iff so is $\mathcal{S}_{\mathbb{H}}(E)$. Assume that $D = (E_t : t \in [0, 1]_{\mathbb{H}})$ is an \mathbb{H} -decomposition of E . We define a mapping $f : \mathcal{D}_{\mathbb{H}}(E) \rightarrow \mathcal{S}_{\mathbb{H}}(E)$ by $f(D) = s$, where $s(x) = t$ iff $x \in E_t$ for $t \in [0, 1]_{\mathbb{H}}$. By the proof of Theorem 4.1, f is bijective. \square

Remark 4.3 If in the definition of an \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ of E we do not suppose that every E_t has to be nonempty, then in the same way as it was proved in Theorem 4.2 we can prove a one-to-one correspondence between these new \mathbb{H} -decompositions and \mathbb{H} -states, i.e. we suppose only that $s(E) \subseteq [0, 1]_{\mathbb{H}}$.

In what follows, we will assume that in any decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ every E_t is nonempty.

Corollary 4.4 *Let $(E_t : t \in [0, 1]_{\mathbb{H}})$ be an \mathbb{H} -decomposition of a pseudo effect algebra E . Then E_0 is a normal ideal of E and $E_0 = \text{Infinit}(E)$.*

Proof By Theorem 4.1, there exists a unique \mathbb{H} -valued state s on E such that $s(x) = t$ iff $x \in E_t$, $t \in [0, 1]_{\mathbb{H}}$. Hence, $E_0 = \text{Ker}(s)$ and $\text{Ker}(s)$ is always a normal ideal. Since $E_0 + E_0$ is defined in E , we have $E_0 + E_0 = E_0 \subseteq \text{Infinit}(E)$. Let now $x \in \text{Infinit}(E)$, then mx is defined in E for any integer $m \geq 1$. Hence, $s(mx) = ms(x) \leq 1$, so that $s(x) = 0$ and $x \in E_0$. \square

We say that an \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ of E is ordered if, for $s < t$, $s, t \in [0, 1]_{\mathbb{H}}$, we have $E_s \leq E_t$.

Theorem 4.5 *Let $(E_t : t \in [0, 1]_{\mathbb{H}})$ be an \mathbb{H} -decomposition of a pseudo effect algebra E . Then $(E_t : t \in [0, 1]_{\mathbb{H}})$ is ordered if and only if $E_u + E_v$ exists in E whenever $u + v < 1$ for $u, v \in [0, 1]_{\mathbb{H}}$.*

In such a case,

- (1) $E_0 = \text{Infinit}(E)$ and $\text{Infinit}(E)$ is a normal ideal.
- (2) $E_u + E_v = E_{u+v}$ whenever $u + v < 1$.
- (3) If $u + v > 1$, for any $x \in E_u$ and $y \in E_v$, neither $x + y$ nor $y + x$ exists.

Proof By Theorem 4.1, there is a unique \mathbb{H} -valued state s such that $s(E_t) = t$ for each $t \in [0, 1]_{\mathbb{H}}$.

Assume $(E_t : t \in [0, 1]_{\mathbb{H}})$ is ordered. Choose $u, v \in [0, 1]_{\mathbb{H}}$ with $u + v < 1$. We have that $u < 1 - v$, and $E_u \leq E_{1-v} = E_v^-$, which implies that $E_u + E_v$ exists and we show $E_u + E_v = E_{u+v}$. Indeed, for any $a \in E_u$, and any $b \in E_v$, we have $s(a + b) = u + v$, which implies that $a + b \in E_{u+v}$. Conversely, let $c \in E_{u+v}$. For any $a \in E_u$, we have that $a \leq c$. Then there exists an element $b \in E$ such that $a + b = c$. Hence, $s(a + b) = s(a) + s(b) = u + v$, then $s(b) = v$, which implies that $b \in E_v$.

Now let $E_t + E_w$ exist in E for $t + w < 1$, $t, w \in [0, 1]_{\mathbb{H}}$. Choose $u < v$, $u, v \in [0, 1]_{\mathbb{H}}$. Then $u + (1 - v) < 1$ so that $E_u + E_{1-v} = E_u + E_v^-$ exists in E which yields $E_u \leq E_v$.

- (1) For any $x, y \in E_0$, we have that $x + y$ exists in E . Then $s(x + y) = s(x) + s(y) = 0$ and $x + y \in E_0$ which implies $E_0 \subseteq \text{Infinit}(E)$. Conversely, let $x \in \text{Infinit}(E)$, we have that mx is defined in E for each integer $m \geq 1$. Then $s(mx) = ms(x) \leq 1$ which implies $s(x) = 0$ and $x \in \text{Ker}(s)$, and so $x \in E_0$.
- (2) Assume that $a \in E_u$ and $b \in E_v$ for $u + v < 1$. Then $a + b$ exists and $s(a + b) = u + v$, and so $a + b \in E_{u+v}$. Conversely, let $z \in E_{u+v}$, then for any $x \in E_u$, $x \leq z$, so that $z = x + (x/z)$, by $s(z) = s(x) + s(x/z)$, which implies that $x/z \in E_v$.
- (3) Assume that $u + v > 1$, $x \in E_u$, $y \in E_v$, either $x + y$ or $y + x$ exists, then $s(x + y) > 1$ or $s(y + x) > 1$, which is absurd. \square

We recall the following two definitions of radicals used in Dvurečenskiĭ (2008). Let E be a pseudo effect algebra. We denote by $\mathcal{M}(E)$ and $\mathcal{N}(E)$ the set of maximal ideals and the set of normal ideals of E , respectively. We define (1) the radical of E , $\text{Rad}(E)$, as the set

$$\text{Rad}(E) = \bigcap \{I : I \in \mathcal{M}(E)\},$$

and (2) the normal radical of E , via

$$\text{Rad}_n(E) = \bigcap \{I : I \in \mathcal{M}(E) \cap \mathcal{N}(E)\}.$$

It is obvious that $\text{Rad}(E) \subseteq \text{Rad}_n(E)$ holds in any pseudo effect algebra E .

Lemma 4.6 (1) Let $(E_t : t \in [0, 1]_{\mathbb{H}})$ be an ordered \mathbb{H} -decomposition of a pseudo effect algebra E . Then E_0 is a unique maximal ideal of E , and it is a normal ideal such that $E_0 = \text{Infin}(E) = \text{Rad}(E) = \text{Rad}_n(E)$.

(2) If $(E_t : t \in [0, 1]_{\mathbb{H}})$ is any \mathbb{H} -decomposition of E of type I , then $E_0 + E_0 = E_0$.

Proof (1) By Corollary 4.4, E_0 is a normal ideal, such that $E_0 = \text{Infin}(E)$. Now we show that E_0 is a maximal ideal. Take $x \in E_t \setminus E_0$, where $0 < t < 1$, $t \in [0, 1]_{\mathbb{H}}$. Let I be an ideal of E generated by E_0 and x . Then, for every $v < t$, $v \in [0, 1]_{\mathbb{H}}$, we have $E_v \leq E_t$, whence $E_v \subseteq I$. There are two cases: (1) there is no $v \in [0, 1]_{\mathbb{H}}$ such that $0 < v < t$. Then $t = 1/n$ for some integer $n \geq 1$ and $\mathbb{H} = \frac{1}{n}\mathbb{Z}$. If $n = 1$, then $s(x) = 1$, $s(x^-) = 0$, and $x^- \in E_0$. Hence, $1 \in I$.

If $n \geq 2$, then $y := (n-1)x$ is defined in E , and $y \in I$. For the element y^- we have $s(y^-) = 1/n$, so that $y^- \in I$ which means $1 \in I$.

(2) \mathbb{H} is not a cyclic subgroup of \mathbb{R} , so that it is dense in \mathbb{R} . There is a strictly decreasing sequence $\{t_i\}$ of non-zero elements of $[0, 1]_{\mathbb{H}}$ such that $t_i \searrow 0$. For every t_i , there is a maximal integer m_i such that $y_i := m_i t_i$ is defined in E . Hence, for enough small t_i , $s(y_i^-) < t$ so that $y_i^- \in I$ which again proves $I = E$, and E_0 is a maximal ideal.

Uniqueness. Assume that I is another maximal ideal of E . If there is $x \in E_t \cap I$ for some $t \in [0, 1]_{\mathbb{H}}$, $t > 0$, then, for every $z \in M_0$, we have $z \leq x$ and $z \in I$, so that $E_0 \subseteq I$. The maximality of E_0 yields $E_0 = I$.

Finally, we have $\text{Rad}(E) = \text{Rad}_n(E)$.

(2) By (b), $E_0 + E_0$ is defined in E , and by (c), E_0 is an ideal. Hence, $E_0 = E_0 + \{0\} \subseteq E_0 + E_0 \subseteq E_0$. \square

Proposition 4.7 Let G be a directed po-group with RDP_1 and let \mathbb{H} be a subgroup of \mathbb{R} such that $1 \in \mathbb{H}$. Choose $g_0 \in G$ and set $E_0 = \{(0, g) : g \in G^+\}$, $E_1 = \{(1, g) : g \leq g_0\}$ and $M_t = \{(t, g) : g \in G\}$ for $t \in [0, 1]_{\mathbb{H}}$, $0 < t < 1$. Then $E := \Gamma(\mathbb{Z} \times G, (1, g_0))$ is a pseudo effect algebra which is \mathbb{H} -perfect, $(E_t : t \in [0, 1]_{\mathbb{H}})$ is an ordered \mathbb{H} -decomposition. E is symmetric if and only if $g_0 \in C(G)$.

In addition, E has a unique state, it is an \mathbb{H} -valued state, and E has a unique \mathbb{H} -decomposition.

Proof It is clear that $(E_t : t \in [0, 1]_{\mathbb{H}})$ is an ordered \mathbb{H} -decomposition of E , and E is a symmetric \mathbb{H} -perfect pseudo effect algebra iff $g_0 \in C(G)$, see (Dvurečenskij 2004, p. 98).

By Theorem 4.1, E has an \mathbb{H} -valued state, say s , namely $s(E_t) = \{t\}$, $t \in [0, 1]_{\mathbb{H}}$. Assume that s_1 is any state on E . Then $E_0 = \text{Infin}(E) \subseteq \text{Ker}(s_1)$, and since E_0 is a maximal ideal, $E_0 = \text{Ker}(s_1)$.

Since \mathbb{H} is in fact a linearly ordered ℓ -group, we can define the MV-algebra $M := \Gamma(\mathbb{H}, 1)$, for more details on MV-algebras, see e.g. Cignoli et al. (2000). Let \hat{s}_1 be a mapping on M defined by $\hat{s}_1(t) = s_1(t, 0)$ if $t \in [0, 1]_{\mathbb{H}} \setminus \{1\}$ and $s_1(1) = 1 = s_1(1, g_0)$. It is straightforward to verify that \hat{s}_1

is a state on M . In the same way we can define also a state \hat{s} on M . Moreover, M being linearly ordered, M has a unique state, see (Dvurečenskij 2001, Thm 5.5), and $\hat{s}_1 = \hat{s}$.

By Theorem 3.8, $\mathbb{H} \times G$ has RDP_1 , therefore, s_1 can be uniquely extended to a state \overline{s}_1 on the unital ℓ -group $(\mathbb{H} \times G, (1, g_0))$.

We have $s_1(t, 0) = t$ for $t \in [0, 1]_{\mathbb{H}} \setminus \{1\}$ and $s_1(1, g_0)$. Therefore, $s_1(0, g) = 0$ for any $g \in G^+$ and since G is directed, $\overline{s}_1(0, g) = 0$ for any $g \in G$. If $0 < t < 1$, then $s_1(t, g) = \overline{s}_1(t, g) = s_1(t, 0) + \overline{s}_1(0, g) = t$. On the other hand, if $(1, g) \in E_1$, then $(1, g)^- \in E$. Therefore, $0 = s_1((1, g)^-)$ which yields $s_1(1, g) = 1$. In other words, $s = s_1$.

Applying Theorem 4.2, we see E has a unique \mathbb{H} -decomposition. \square

5 \mathbb{H} -perfect, \mathbb{R} -perfect and \mathbb{Q} -perfect pseudo effect algebras

We say that a pseudo effect algebra E is \mathbb{H} -perfect if it has an ordered \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$.

In particular, if $\mathbb{H} = \mathbb{R}$ or $\mathbb{H} = \mathbb{Q}$, we are saying that E is \mathbb{R} -perfect and \mathbb{Q} -perfect, respectively. If $\mathbb{H} = \frac{1}{n}\mathbb{Z}$, a $\frac{1}{n}\mathbb{Z}$ -perfect pseudo effect algebra is said to be also an n -perfect pseudo effect algebra, see Dvurečenskij et al. (2013).

The following notion of a cyclic element was defined for a special class of pseudo effect algebras, called GMV-algebras, in (Dvurečenskij 2008, 2010) and for pseudo effect algebras in Dvurečenskij et al. (2013).

Let $n \geq 1$ be an integer. An element a of a pseudo effect algebra E is said to be cyclic of order n or simply cyclic if na exists in E and $na = 1$. If a is a cyclic element of order n , then $a^- = a^{\sim}$, indeed, $a^- = (n-1)a = a^{\sim}$. It is clear that 1 is a cyclic element of order 1 .

We note that a pseudo effect algebra E has a cyclic element of order n iff E has a pseudo effect subalgebra of E isomorphic to $\Gamma(\frac{1}{n}\mathbb{Z}, 1)$.

Proposition 5.1 If c is a cyclic element, then, for any $x \in E$, $x + c$ exists in E if and only if $c + x$ exists in E .

Proof There is an integer $n \geq 1$ such that $nc = 1$. We have that $c^{\sim} = c^-$. Then $x + c$ exists if and only if $x \leq c^-$ if and only if $x \leq c^{\sim}$ if and only if $c + x$ exists. \square

We say that a group G is torsion-free if $ng \neq 0$ for any $g \neq 0$ and every nonzero integer n . For example, every ℓ -group is torsion-free, see (Glass 1999, Cor 2.1.3). We recall that a po-group G is torsion-free iff $\mathbb{H} \times G$ is torsion-free, where \mathbb{H} is a subgroup of the group \mathbb{R} with $1 \in \mathbb{H}$.

We recall that according to (Glass 1999, p. 16), a group G enjoys unique extraction of roots if, for all positive integers n and all $g, h \in G$, $ng = nh$ implies $g = h$. We note that every

linearly ordered group, or a representable ℓ -group (i.e. it is a subdirect product of linearly ordered groups), in particular every Abelian ℓ -group, enjoys unique extraction of roots, see (Glass 1999, Lem. 2.1.4). On the other hand, there is a unital divisible ℓ -group (i.e. g/n is defined in G for every $g \in G$ and every integer $n \geq 1$), but the unique extraction of roots fails, see (Glass 1999, p. 16).

The following results have been originally proved in Dvurečenskij et al. (2013) for strong n -perfect pseudo effect algebras. In what follows, we generalize them for \mathbb{H} -perfect effect algebras.

Let $E = \Gamma(G, u)$ for some unital po-group (G, u) . An element $c \in E$ such that (a) $nc = u$ for some integer $n \geq 1$, and (b) $c \in C(G)$, is said to be a *strong cyclic element of order n* .

For example, if $E = \Gamma(\mathbb{R} \overrightarrow{\times} G, (1, 0))$, then any element of the form $(1/n, 0)$, $n \geq 1$, is a strong cyclic element of order n .

We note that this notion was used in Dvurečenskij (2008); Dvurečenskij et al. (2013) to show when a pseudo effect algebra is of the form $\Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, (1, 0))$.

As a matter of interest, we recall that if $\mathbb{H} = \mathbb{H}(\alpha)$ is a subgroup of \mathbb{R} generated by $\alpha \in [0, 1]$ and 1, then $\mathbb{H} = \frac{1}{n}\mathbb{Z}$ for some integer $n \geq 1$ if α is a rational number. Otherwise, $\mathbb{H}(\alpha)$ is countable and dense in \mathbb{R} , and $E(\alpha) := \Gamma(\mathbb{H}(\alpha), 1) = \{m + n\alpha : m, n \in \mathbb{Z}, 0 \leq m + n\alpha \leq 1\}$, see (Cignoli et al. 2000, p. 149). If α is irrational, then $E(\alpha)$ and $\Gamma(\mathbb{H}(\alpha) \overrightarrow{\times} G, (1, 0))$ have only one (strong) cyclic element, namely 1 which is of order 1.

Now let G be a non Abelian po-group and choose an element $g_0 \in G \setminus C(G)$. Set $E_t := \{(t, g) \in \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, g_0))\}$, $t \in [0, 1]_{\mathbb{H}}$. Then $E := \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, g_0))$ is \mathbb{H} -perfect, $(E_t : t \in [0, 1]_{\mathbb{H}})$ is an ordered \mathbb{H} -decomposition of E but E does not have any strong cyclic element. We note that E is not symmetric.

Lemma 5.2 *Let H be a torsion-free po-group with a strong unit u . Let $c \in E = \Gamma(H, u)$ be a strong cyclic element of order n . If $d \in E$ is any cyclic element of order n , then $c = d$.*

Proof Since $c \in C(H)$ and $d \in H$, we have $c + d = d + c$ in the group H . Then $n(c - d) = nc - nd = 0$ so that $c = d$. □

Proposition 5.3 *Let E be a pseudo effect algebra with an ordered \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$. Then there exists a unique directed po-group G such that $G^+ = E_0$.*

Proof By Lemma 4.6(1), E_0 is a maximal ideal of E , and $E_0 + E_0$ is defined, so that $E_0 + E_0 = E_0$. Hence, $(E_0; +, 0)$ is a semigroup. For any $x, y \in E_0$, the equation $x + y = 0$, implies that $x = y = 0$. For any $x, y, z \in E_0$, the equation $x + y = x + z$ implies that $y = z$, and equation

$y + x = z + x$ implies that $y = z$. Then $(E_0; +, 0)$ is a cancellative semigroup satisfying the conditions of the Birkhoff Theorem, (Fuchs 1963, Thm 2.4), which guarantees that E_0 is the positive cone of a unique (up to isomorphism) po-group G . Without loss of generality, we can assume that G is generated by the positive cone E_0 , so that G is directed. □

We say that a pseudo effect algebra E enjoys the *1-divisibility property* if, given integer $n \geq 1$, there is an element $a_n \in E$ such that $na_n = 1$. We see that a_n is a cyclic element of order n . We recall that a_n is not necessarily unique, and we denote $a_n = \frac{1}{n}1$ if a_n is unique. For example by (Glass 1999, p. 16), there is a unital ℓ -group (G, u) such that in $\Gamma(G, u)$ there are two different elements $a \neq b$ with $2a = u = 2b$.

Let $E = \Gamma(G, u)$ be an interval pseudo effect algebra. We say that E enjoys the *strong 1-divisibility property* if, given integer $n \geq 1$, there is an element $a_n \in C(G) \cap E$ such that $na_n = 1$. We see that a_n is a strong cyclic element of order n . If G is e.g. torsion-free, then due to Lemma 5.2, a_n is unique.

We say that a pseudo effect algebra E enjoys *unique extraction of roots of 1* if $a, b \in E$ and na, nb exist in E , and $na = 1 = nb$, then $a = b$. Then every $\Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$ enjoys unique extraction of roots of 1 for any torsion-free directed po-group G . Indeed, let $k(s, g) = (1, 0) = k(t, h)$ for some $s, t \in [0, 1]_{\mathbb{H}}$, $g, h \in G, k \geq 1$. Then $ks = 1 = kt$ which yields $s = t > 0$, and $kg = 0 = kh$ implies $g = 0 = h$.

We say that a nonempty subset A of a pseudo effect algebra E is (1) *downwards directed* if given $a_1, a_2 \in A$, there is an element $a \in A$ such that $a \leq a_1, a_2$; (2) *upwards directed* if given $a_1, a_2 \in A$, there is an element $b \in A$ such that $b \geq a_1, a_2$; and (3) *directed* if it is both upwards and downwards directed.

Proposition 5.4 (1) *Let $(E_t : t \in [0, 1]_{\mathbb{H}})$ be an ordered \mathbb{H} -decomposition of a pseudo effect algebra E . Then E_0 and E_1 are directed.*

(2) *If $(E_0, E_{1/n}, \dots, E_{n/n})$ is an ordered $\frac{1}{n}\mathbb{Z}$ -decomposition of a pseudo effect algebra E satisfying RDP_0 , then every E_i is directed, $i = 0, 1, \dots, n$.*

Proof (1) It is clear that E_0 is downwards directed. Let $a, b \in E_0$. Since $E_0 = \text{Infin}(E)$, $a + b$ is defined in E and $a + b \in E_0$. Then $a, b \leq a + b$. Since $E_1 = E_0^-$, we see that E_1 is directed.

(2) It follows from (Dvurečenskij et al. 2013, Prop 5.12). □

We recall that if we set $E = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$ for some po-group G , then E_t for $t \in [0, 1] \setminus \{0, 1\}$ is neither downwards nor upwards directed whenever G is not directed, and in general, E_t is directed iff G is directed.

Inspiring by this, we say that an \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ of E has the *directness property* if every E_t is directed.

6 Representation of strong \mathbb{Q} -perfect pseudo effect algebras

In this section, we define a strong \mathbb{Q} -perfect pseudo effect algebra and we show when it is an interval in the lexicographic product $\mathbb{Q} \overrightarrow{\times} G$. In addition, we derive a categorical equivalence of the category of strong \mathbb{Q} -perfect pseudo effect algebras with the category of directed torsion-free po-groups.

Definition 6.1 We say that a pseudo effect algebra E with RDP_1 is *strong \mathbb{Q} -perfect* if

- (1) E enjoys the strong 1-divisibility property,
- (2) E has an ordered \mathbb{Q} -decomposition having the directness property,
- (3) the unital po-group (G, u) such that $E = \Gamma(G, u)$ is torsion free.

In what follows, we present a representation of strong \mathbb{Q} -perfect pseudo effect algebras. We start with a preparatory statement.

Proposition 6.2 *Let G be a directed torsion-free po-group with RDP_1 . Then the pseudo effect algebra*

$$\mathcal{Q}(G) := \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0)) \tag{6.1}$$

is a strong \mathbb{Q} -perfect pseudo effect algebra.

Proof By Theorem 3.5, $\mathcal{Q}(G)$ has RDP_1 , it has an ordered \mathbb{Q} -decomposition $(E_t : t \in [0, 1]_{\mathbb{Q}})$, where $E_0 = \{(0, g) : g \in G^+\}$, $E_1 = \{(1, -g) : g \in G^+\}$, and $E_t = \{(t, g) : g \in G\}$, for $t \in [0, 1]_{\mathbb{Q}} \setminus \{0, 1\}$, and each element $a_n := (1/n, 0)$ is a strong cyclic element of order n . Thus E enjoys the strong 1-divisibility property, and G being directed, every E_t is directed. \square

Theorem 6.3 *Let E be a strong \mathbb{Q} -perfect pseudo effect algebra with RDP_1 . Then there is a unique (up to isomorphism) torsion-free directed po-group G with RDP_1 such that $E \cong \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0))$.*

Proof Since E has RDP_1 , due to (Dvurečenskij and Vetterlein 2001b, Thm 5.7), there is a unique unital (up to isomorphism of unital po-groups) po-group (H, u) with RDP_1 such that $E = \Gamma(H, u)$. Assume $(E_t : t \in [0, 1]_{\mathbb{Q}})$ is an ordered \mathbb{Q} -decomposition of E with the directness property; due to Theorem 4.2, it is unique. By Theorem 4.1, there is a unique \mathbb{Q} -valued state s . By Proposition 5.3, there is a unique directed po-group G such that $E_0 = G^+$. Since

H is torsion-free and with RDP_1 , $G = G^+ - G^+$ is also torsion-free, directed and with RDP_1 . For any integer $n \geq 1$, there is a unique element $a_n = \frac{1}{n}1$. It is clear that, for any integer $1 \leq m \leq n$, the element $m\frac{1}{n}1$ is defined and we denote it by $\frac{m}{n}1$. Define $\mathcal{Q}(G)$ by (4.1) and define a mapping $\phi : E \rightarrow \mathcal{Q}(G)$ by

$$\phi(x) = (\frac{i}{n}, x - \frac{i}{n}1) \tag{6.2}$$

whenever $x \in E_{\frac{i}{n}}$. We note that “ $-$ ” in the right-hand side of the formula means the group subtraction in the po-group H . Since $E_{\frac{i}{n}}$ is directed, there is an element $x_0 \in E_{\frac{i}{n}}$ such that $x_0 \leq x, \frac{i}{n}1$. Then $x - x_0 = x \setminus x_0 \in E_0$ and $\frac{i}{n}1 - x_0 = \frac{i}{n}1 \setminus x_0 \in E_0$ because $s(x \setminus x_0) = 0 = s(\frac{i}{n}1 \setminus x_0)$. Therefore, $x - \frac{i}{n}1 = (x - x_0) - (\frac{i}{n}1 - x_0) \in G$, which means that ϕ is a well-defined mapping.

Claim: The mapping ϕ is an injective and surjective homomorphism of pseudo effect algebras.

We have $\phi(0) = (0, 0)$ and $\phi(1) = (1, 0)$. Let $x \in E_{\frac{i}{n}}$. Then $x^- \in E_{\frac{n-i}{n}}$, and $\phi(x^-) = (\frac{n-i}{n}, x^- - \frac{n-i}{n}1) = (1, 0) - (\frac{i}{n}, x - \frac{i}{n}1) = \phi(x)^-$. In an analogous way, $\phi(x^{\sim}) = \phi(x)^{\sim}$.

Now let $x, y \in E$ and let $x + y$ be defined in E . Then $x \in E_{\frac{i}{n_1}}$ and $y \in E_{\frac{j}{n_2}}$. Without loss of generality, we can assume that $n_1 = n_2 = n$. Since $x \leq y^-$, we have $\frac{i}{n} \leq \frac{n-j}{n}$. So that $\phi(x) \leq \phi(y^-) = \phi(y)^-$ which means $\phi(x) + \phi(y)$ is defined in $\mathcal{Q}(G)$. Then $\phi(x + y) = (\frac{i+j}{n}, x + y - \frac{i+j}{n}1) = (\frac{i+j}{n}, x + y - (\frac{i}{n}1 + \frac{j}{n}1)) = (\frac{i}{n}, x - \frac{i}{n}1) + (\frac{j}{n}, y - \frac{j}{n}1) = \phi(x) + \phi(y)$.

Assume $\phi(x) \leq \phi(y)$ for some $x \in E_{\frac{i}{n}}$ and $y \in E_{\frac{j}{n}}$. Then $(\frac{i}{n}, x - \frac{i}{n}1) \leq (\frac{j}{n}, y - \frac{j}{n}1)$. If $i = j$, then $x - \frac{i}{n}1 \leq y - \frac{i}{n}1$ so that $x \leq y$. If $i < j$, then $x \in E_{\frac{i}{n}}$ and $y \in E_{\frac{j}{n}}$ so that $x < y$. Therefore, ϕ is injective.

To prove that ϕ is surjective, assume two cases: (1) Take $g \in G^+ = E_0$. Then $\phi(g) = (0, g)$. In addition $g^- \in E_1$ so that $\phi(g^-) = \phi(g)^- = (0, g)^- = (1, 0) - (0, g) = (1, -g)$. (2) Let $g \in G$ and $\frac{i}{n}$ with $1 < i < n$ be given. Then $g = g_1 - g_2$, where $g_1, g_2 \in G^+ = E_0$. Since $\frac{i}{n}1 \in E_{\frac{i}{n}}$, $g_1 + \frac{i}{n}1$ exists in E and it belongs to $E_{\frac{i}{n}}$, and $g_2 \leq g_1 + \frac{i}{n}1$ which yields $(g_1 + \frac{i}{n}1) - g_2 = (g_1 + \frac{i}{n}1) \setminus g_2 \in E_{\frac{i}{n}}$. Hence, $g + \frac{i}{n}1 = (g_1 + \frac{i}{n}1) \setminus g_2 \in E_{\frac{i}{n}}$ which entails $\phi(g + \frac{i}{n}1) = (\frac{i}{n}, g)$.

Consequently, E is isomorphic to $\mathcal{Q}(G)$.

If $E \cong \Gamma(\mathbb{Q} \overrightarrow{\times} G', (1, 0))$, then G and G' are isomorphic po-groups. \square

We note that the notion of the directness property of a pseudo effect algebra E was introduced in order to guarantee the existence of the element $x - \frac{i}{n}1$ in the po-group G , and consequently, to show that the mapping ϕ in (6.2) is well defined. If E satisfies RDP_2 , then by Dvurečenskij and Vetterlein (2001a,b), E is in fact a pseudo MV-algebra, and by

Dvurečenskij (2002), there is a unital ℓ -group (H, u) such that $E = \Gamma(H, u)$. At any rate, any ℓ -group is torsion-free and directed. Hence, let for this case a pseudo effect algebra E satisfying RDP_2 is strong \mathbb{Q} -perfect if it enjoys the strong 1-divisibility property and it has an ordered \mathbb{Q} -decomposition. We assert that ϕ is well defined. Indeed, E_0 in this case is in fact a positive cone of an ℓ -group G . The unique state s on E corresponding to the given ordered \mathbb{H} -decomposition is an extremal state, therefore, by (Dvurečenskij 2001, Prop 4.7), $s(a \wedge b) = \min\{s(a), s(b)\}$ for all $a, b \in E$, and the same is true for its extension \hat{s} onto (H, u) and all $a, b \in H$. Let $x \in E_{\frac{i}{n}}$. For the element $x - \frac{i}{n}1 \in H$, we define $(x - \frac{i}{n}1)^+ := (x - \frac{i}{n}1) \vee 0 = (x \vee \frac{i}{n}1) - \frac{i}{n}1 \in E_0$ while $s((x \vee \frac{i}{n}1) - \frac{i}{n}1) = s(x \vee \frac{i}{n}1) - s(\frac{i}{n}1) = \frac{i}{n} - \frac{i}{n} = 0$ and similarly $(x - \frac{i}{n}1)^- := -((x - \frac{i}{n}1) \wedge 0) = \frac{i}{n}1 - (x \wedge \frac{i}{n}1) \in E_0$. This implies that $x - \frac{i}{n}1 = (x - \frac{i}{n}1)^+ - (x - \frac{i}{n}1)^- \in G$. It proves the following result.

Theorem 6.4 *Let E be a strong \mathbb{Q} -perfect pseudo effect algebra in the sense of the last note and with RDP_2 . Then there is a unique ℓ -group G such that $E \cong \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0))$.*

Proof It follows from the last note and the proof of Theorem 6.3. \square

Problem 2. Can we relax the notion of directness in the definition of a strong \mathbb{Q} -perfect pseudo effect algebra in order to prove Theorem 6.3?

In what follows, we show the categorical equivalence of the category of strong \mathbb{Q} -perfect pseudo effect algebras with the category of directed torsion-free po-groups with RDP_1 .

Let \mathcal{SQPPEA} be the category of strong \mathbb{Q} -perfect pseudo effect algebras whose objects are strong \mathbb{Q} -perfect pseudo effect algebras and morphisms are homomorphisms of pseudo effect algebras. Similarly, let \mathcal{G} be the category whose objects are directed torsion-free po-groups with RDP_1 , and morphisms are homomorphisms of unital po-groups.

Define a functor $\mathcal{E}_{\mathbb{Q}} : \mathcal{G} \rightarrow \mathcal{SQPPEA}$ as follows: for $G \in \mathcal{G}$, let

$$\mathcal{E}_{\mathbb{Q}}(G) := \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0))$$

and if $h : G \rightarrow G_1$ is a po-group homomorphism, then

$$\mathcal{E}_{\mathbb{Q}}(h)\left(\frac{i}{n}, g\right) = \left(\frac{i}{n}, h(g)\right), \quad \left(\frac{i}{n}, g\right) \in \Gamma\left(\mathbb{Q} \overrightarrow{\times} G, (1, 0)\right).$$

By Proposition 6.2, $\mathcal{E}_{\mathbb{Q}}$ is a well-defined functor.

Proposition 6.5 *$\mathcal{E}_{\mathbb{Q}}$ is a faithful and full functor from the category \mathcal{G} of directed torsion-free po-groups with RDP_1 into the category \mathcal{SQPPEA} of strong \mathbb{Q} -perfect pseudo effect algebras.*

Proof Let h_1 and h_2 be two morphisms from G into G' such that $\mathcal{E}_{\mathbb{Q}}(h_1) = \mathcal{E}_{\mathbb{Q}}(h_2)$. Then $(0, h_1(g)) = (0, h_2(g))$ for any $g \in G^+$, consequently $h_1 = h_2$.

To prove that $\mathcal{E}_{\mathbb{Q}}$ is a full functor, suppose that f is a morphism from a strong \mathbb{Q} -perfect pseudo effect algebra $\Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0))$ into another one $\Gamma(\mathbb{Q} \overrightarrow{\times} G_1, (1, 0))$. Then $f(0, g) = (0, g')$ for a unique $g' \in G'^+$. Define a mapping $h : G^+ \rightarrow G'^+$ by $h(g) = g'$ iff $f(0, g) = (0, g')$. Then $h(g_1 + g_2) = h(g_1) + h(g_2)$ if $g_1, g_2 \in G^+$. Assume now that $g \in G$ is arbitrary. Then $g = g_1 - g_2 = g'_1 - g'_2$, where $g_1, g_2, g'_1, g'_2 \in G^+$, which gives $g_1 + g'_2 = g'_1 + g_2$, i.e., $h(g) = h(g_1) - h(g_2)$ is a well-defined extension of h from G^+ onto G .

Let $0 \leq g_1 \leq g_2$. Then $(0, g_1) \leq (0, g_2)$, which means h is a homomorphism of po-groups, and $\mathcal{E}_{\mathbb{Q}}(h) = f$ as desired. \square

We recall that by a *universal group* for a pseudo effect algebra E we mean a pair (G, γ) consisting of a po-group G and a G -valued measure $\gamma : E \rightarrow G^+$ (i.e., $\gamma(a + b) = \gamma(a) + \gamma(b)$ whenever $a + b$ is defined in E) such that the following conditions hold: (1) $\gamma(E)$ generates G , and (2) if H is a group and $\phi : E \rightarrow H$ is an H -valued measure, then there is a group homomorphism $\phi^* : G \rightarrow H$ such that $\phi = \phi^* \circ \gamma$.

Due to (Dvurečenskij and Vetterlein 2001b, Thm 7.2), every pseudo algebra with RDP_1 admits a universal group, which is unique up to isomorphism, and ϕ^* is unique. The universal group for $E = \Gamma(G, u)$ is (G, id) where id is the embedding of E into G , and G satisfies RDP_1 .

Let \mathcal{A} and \mathcal{B} be two categories and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism. Suppose that g, h be two morphisms from \mathcal{B} to \mathcal{A} such that $g \circ f = id_{\mathcal{A}}$ and $f \circ h = id_{\mathcal{B}}$, then g is a *left-adjoint* of f and h is a *right-adjoint* of f .

Proposition 6.6 *The functor $\mathcal{E}_{\mathbb{Q}}$ from the category \mathcal{G} into the category \mathcal{SQPPEA} has a left-adjoint.*

Proof We show that given a strong \mathbb{Q} -perfect pseudo effect algebra E with an ordered \mathbb{Q} -decomposition $(E_t : t \in [0, 1]_{\mathbb{Q}})$, there is a universal arrow (G, f) , i.e., G is an object in \mathcal{G} and f is a homomorphism from E into $\mathcal{E}_{\mathbb{Q}}(G)$ such that if G' is an object from \mathcal{G} and f' is a homomorphism from E into $\mathcal{E}_{\mathbb{Q}}(G')$, then there exists a unique morphism $f^* : G \rightarrow G'$ such that $\mathcal{E}_{\mathbb{Q}}(f^*) \circ f = f'$.

By Theorem 6.3, there is a unique directed torsion-free po-group G with RDP_1 such that $E \cong \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0))$. By Theorem 3.7, $\mathbb{Q} \overrightarrow{\times} G$ is a directed po-group with RDP_1 , so that by (Dvurečenskij and Vetterlein 2001b, Thm 7.2), $(\mathbb{Q} \overrightarrow{\times} G, \gamma)$ is a universal group for E , where $\gamma : E \rightarrow \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0))$ is defined by $\gamma(a) = (\frac{i}{n}, a - \frac{i}{n}1)$, if $a \in E_{\frac{i}{n}}$. By the proof of Theorem 6.3, γ is an isomorphism. \square

Define a mapping $\mathcal{P} : \mathcal{SQPPEA} \rightarrow \mathcal{G}$ via $\mathcal{P}(E) := G$ whenever $(\mathbb{Q} \overrightarrow{\times} G, f)$ is a universal group for E . It is clear that if f_0 is a morphism from E into F , then f_0 can be uniquely extended to a homomorphism $\mathcal{P}(f_0)$ from G into

G_1 , where $(\mathbb{Q} \overrightarrow{\times} G_1, f_1)$ is a universal group for the strong \mathbb{Q} -perfect pseudo effect algebra F .

Proposition 6.7 *The mapping \mathcal{P} is a functor from the category \mathcal{SQPPEA} into the category \mathcal{G} which is a left-adjoint of the functor $\mathcal{E}_{\mathbb{Q}}$.*

Proof It follows from the construction of the universal group. □

Now we present the main result on categorical equivalence of the category of strong \mathbb{Q} -perfect pseudo effect algebras and the category \mathcal{G} .

Theorem 6.8 *The functor $\mathcal{E}_{\mathbb{Q}}$ defines a categorical equivalence of the category \mathcal{G} and the category \mathcal{SQPPEA} of strong \mathbb{Q} -perfect pseudo effect algebras.*

In addition, suppose that $h : \mathcal{E}_{\mathbb{H}}(G) \rightarrow \mathcal{E}_{\mathbb{H}}(H)$ is a homomorphism of pseudo effect algebras, then there is a unique homomorphism $f : G \rightarrow H$ of unital po-groups such that $h = \mathcal{E}_{\mathbb{H}}(f)$, and

- (1) *if h is surjective, so is f ;*
- (2) *if h is injective, so is f .*

Proof According to (Mac Lane 1971, Thm IV.4.1), it is necessary to show that, for a strong \mathbb{Q} -perfect pseudo effect algebra E , there is an object G in \mathcal{G} such that $\mathcal{E}_{\mathbb{Q}}(G)$ is isomorphic to E . To show that, we take a universal group $(\mathbb{Q} \overrightarrow{\times} G, f)$. Then $\mathcal{E}_{\mathbb{Q}}(G)$ and E are isomorphic. □

7 Strong \mathbb{H} -perfect pseudo effect algebras and their representation

In this section, we extend the results of the previous section to the most general case, namely for strong \mathbb{H} -perfect pseudo effect algebras. Here we use Theorem 3.8.

We say that an \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ of E has the *cyclic property* if there is a system of elements $(c_t \in E : t \in [0, 1]_{\mathbb{H}})$ such that (1) $c_t \in E_t$ for any $t \in [0, 1]_{\mathbb{H}}$, (2) if $s + t \leq 1$, $s, t \in [0, 1]_{\mathbb{H}}$, then $c_s + c_t = c_{s+t}$, and (3) $c_1 = 1$. Properties: (a) $c_0 = 0$; indeed, by (2) we have $c_0 + c_0 = c_0$, so that $c_0 = 0$. (b) If $t = 1/n$, then $c_{\frac{1}{n}}$ is a cyclic element of order n .

Let $E = \Gamma(G, u)$. An \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ of E has the *strong cyclic property* if there is a system of elements $(c_t \in E : t \in [0, 1]_{\mathbb{H}})$ such that (1) $c_t \in E_t \cap C(G)$ for any $t \in [0, 1]_{\mathbb{H}}$, (2) if $s + t \leq 1$, $s, t \in [0, 1]_{\mathbb{H}}$, then $c_s + c_t = c_{s+t}$, and (3) $c_1 = 1$. We recall that if $t = 1/n$, $c_{\frac{1}{n}}$ is a strong cyclic element of order n .

For example, let $E = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$ and $E_t = \{(t, g) : (t, g) \in E\}$ for $t \in [0, 1]_{\mathbb{H}}$. If we set $c_t = (t, 0)$,

$t \in [0, 1]_{\mathbb{H}}$, then the system $(c_t : t \in [0, 1]_{\mathbb{H}})$ satisfies (1)–(3) of the strong cyclic property, and $(E_t : t \in [0, 1]_{\mathbb{H}})$ is an \mathbb{H} -decomposition of E with the strong cyclic property.

Finally, we say that an interval pseudo effect algebra E has the *\mathbb{H} -strong cyclic property* if there is an \mathbb{H} -decomposition $(E_t : t \in [0, 1]_{\mathbb{H}})$ of E with the strong cyclic property.

Proposition 7.1 *An interval pseudo effect algebra $E = \Gamma(G, u)$, where G is torsion-free, has the \mathbb{Q} -strong cyclic property if and only if E has the strong 1-divisibility property.*

Proof If a \mathbb{Q} -decomposition $(E_{\frac{i}{n}} : \frac{i}{n} \in [0, 1]_{\mathbb{Q}})$ of E has the strong cyclic property, there is a system of elements $(c_{\frac{i}{n}} : \frac{i}{n} \in [0, 1]_{\mathbb{Q}})$ satisfying (1)–(3) of the strong cyclic property. Then every $c_{\frac{1}{n}}$ is a strong cyclic element of order n which by Lemma 5.2 means that it is a unique strong cyclic element of order n . Hence, $c_{\frac{1}{n}} = \frac{1}{n}1$.

Conversely, let E have the strong 1-divisibility property, then the elements $c_{\frac{i}{n}} := \frac{i}{n}1 \in E_{\frac{i}{n}}$, and the system $(c_{\frac{i}{n}} : \frac{i}{n} \in [0, 1]_{\mathbb{Q}})$ satisfies the conditions (1)–(3) of the strong cyclic property. □

Definition 6.1 for strong \mathbb{Q} -perfect pseudo effect algebras will be change for strong \mathbb{H} -perfect pseudo effect algebras as follows.

Definition 7.2 We say that a pseudo effect algebra E with RDP_1 is *strong \mathbb{H} -perfect* if

- (1) E has an ordered \mathbb{H} -decomposition having both the directness property and the strong cyclic property,
- (2) the unital po-group (G, u) such that $E = \Gamma(G, u)$ is torsion free.

Proposition 6.2 has the following counter part.

Proposition 7.3 *Let G be a directed torsion-free po-group with RDP_1 . Then the pseudo effect algebra*

$$\mathcal{E}_{\mathbb{H}}(G) := \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0)) \tag{7.1}$$

is a strong \mathbb{H} -perfect pseudo effect algebra.

The representation theorem for strong \mathbb{H} -perfect pseudo effect algebras by (7.1) is the following result; compare it with Theorem 6.3.

Theorem 7.4 *Let E be a strong \mathbb{H} -perfect pseudo effect algebra with RDP_1 . Then there is a unique (up to isomorphism) torsion-free directed po-group G with RDP_1 such that $E \cong \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$.*

Proof The proof follows the basic steps of the proof of Theorem 6.3. Let $(E_t : t \in [0, 1]_{\mathbb{H}})$ be an \mathbb{H} -decomposition with the strong cyclic property with a given system of elements

$(c_t : t \in [0, 1]_{\mathbb{H}})$ satisfying (1)–(3) of the strong cyclic property.

Hence, we derive a directed po-group G such that $G^+ = E_0$. Take the \mathbb{H} -strong cyclic pseudo effect algebra $\mathcal{E}_{\mathbb{H}}(G)$ defined by (7.1), and define a mapping $\phi : E \rightarrow \mathcal{E}_{\mathbb{H}}(G)$ by

$$\phi(x) := (t, x - c_t) \quad (7.2)$$

whenever $x \in E_t$ for some $t \in [0, 1]_{\mathbb{H}}$, where $x - c_t$ denotes the difference taken in the ℓ -group H . In the same way as in the proof of Theorem 6.3, we can prove that ϕ is a well-defined injective and surjective homomorphism of pseudo effect algebras. \square

Finally, let $SPP\mathcal{E}A_{\mathbb{H}}$ be the category of strong \mathbb{H} -perfect pseudo effect algebras whose objects are strong \mathbb{H} -perfect pseudo effect algebras and morphisms are homomorphisms of pseudo effect algebras. Again, let \mathcal{G} be the category whose objects are directed torsion-free po-groups with RDP_1 , and morphisms are homomorphisms of unital po-groups.

Define a functor $\mathcal{E}_{\mathbb{H}} : \mathcal{G} \rightarrow SPP\mathcal{E}A_{\mathbb{H}}$ as follows: for $G \in \mathcal{G}$, let

$$\mathcal{E}_{\mathbb{H}}(G) := \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$$

and if $h : G \rightarrow G_1$ is a po-group homomorphism, then

$$\mathcal{E}_{\mathbb{H}}(h)(t, g) = (t, h(g)), \quad (t, g) \in \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0)).$$

We note that the $SQPPEA = SPP\mathcal{E}A_{\mathbb{Q}}$.

Using the same ideas as for the categorical equivalence in Theorem 6.8, we can prove a new categorical equivalence.

Theorem 7.5 *The functor $\mathcal{E}_{\mathbb{H}}$ defines a categorical equivalence of the category \mathcal{G} and the category $SPP\mathcal{E}A_{\mathbb{H}}$ of strong \mathbb{H} -perfect pseudo effect algebras.*

Consequently, if \mathbb{H} is cyclic or the group of rational numbers or an arbitrary subgroup of reals containing the number 1, all the studied categories of \mathbb{H} -strong perfect pseudo effect algebras are mutually categorically equivalent, and categorically equivalent to the category \mathcal{G} .

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References

- Cignoli R, D'Ottaviano IML, Mundici D (2000) Algebraic foundations of many-valued reasoning. Kluwer Academic Publ, Dordrecht
- Dvurečenskij A (2001) States on pseudo MV-algebras. *Studia Logica* 68:301–327
- Dvurečenskij A (2002) Pseudo MV-algebras are intervals in ℓ -groups. *J Austral Math Soc* 72:427–445
- Dvurečenskij A (2004) States and radicals of pseudo-effect algebras. *Atti Sem Mat Fis Univ Modena Reggio Emilia* 52:85–103
- Dvurečenskij A (2007) Perfect effect algebras are categorically equivalent with Abelian interpolation po-groups. *J Austral Math Soc* 82:183–207
- Dvurečenskij A (2008) On n -perfect GMV-algebras. *J Algebra* 319:4921–4946
- Dvurečenskij A (2010) Cyclic elements and subalgebras of GMV-algebras. *Soft Comput* 14:257–264. doi:10.1007/s00500-009-0400-x
- Dvurečenskij A, Krňávek J (2013) The lexicographic product of po-groups and n -perfect pseudo effect algebras. *Inter J Theor Phys* 52:2760–2772. doi:10.1007/s10773-013-1568-5
- Dvurečenskij A, Pulmannová S (2000) New trends in quantum structures. Kluwer Acad. Publ, Dordrecht
- Dvurečenskij A, Vetterlein T (2001a) Pseudoeffect algebras. I. Basic properties. *Inter J Theor Phys* 40:685–701
- Dvurečenskij A, Vetterlein T (2001b) Pseudoeffect algebras. II. Group representation. *Inter J Theor Phys* 40:703–726
- Dvurečenskij A, Xie Y, Yang A (2013) Discrete $(n+1)$ -valued states and n -perfect pseudo-effect algebras. *Soft Comput*. doi:10.1007/s00500-013-1001-2
- Foulis DJ, Bennett MK (1994) Effect algebras and unsharp quantum logics. *Found Phys* 24:1331–1352
- Fuchs L (1963) Partially ordered algebraic systems. Pergamon Press, Oxford
- Glass AMW (1999) Partially Ordered Groups. World Scientific, Singapore
- Goodearl KR (1986) Partially Ordered Abelian Groups with Interpolation. *Math Surveys and Monographs* No. 20, Amer Math Soc Providence, Rhode Island
- Kolmogorov AN (1933) Grundbegriffe der Wahrscheinlichkeitsrechnung. Julius Springer, Berlin
- Kôpka F, Chovanec F (1994) D-posets. *Math Slovaca* 44:21–34
- Mac Lane S (1971) Categories for the working mathematician. Springer-Verlag, New York
- Ravindran K (1996) On a structure theory of effect algebras. PhD thesis, Kansas State Univ., Manhattan, Kansas