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Nearlattices☆

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Abstract

By a nearlattice is meant a join-semilattice where every principal filter is a lattice with respect to the induced order. Alternatively, a nearlattice can be described as an algebra with one ternary operation satisfying eight simple identities. Hence, the class of nearlattices is a variety. We characterize nearlattices every sublattice of which is distributive. Then we introduce the so-called section pseudocomplementation on nearlattices which can also be characterized by identities.

MSC: 06B05; 06B25; 06A12; 08B10

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To describe algebraic properties of the connective of implication in logics, we use the so-called implication algebra. For a classical logic it was done by Abbott [1], for various logics of quantum mechanic, see e.g. [2–6]. It turns out that the corresponding algebraic structures are join-semilattices where each principal filter is a lattice with respect to the induced order and, moreover, they are equipped with an antitone involution which is a complementation. However, this approach cannot be generalized for implication in intuitionistic logic because the connective implication is identified with the relative pseudocomplementation which is not an involution; moreover, in intuitionistic logic the connectives conjunction, disjunction and negation are independent. This motivates us to use a quite general approach: we will treat join-semilattices where each principal filter is a lattice and, when axiomatizing them, we will introduce a section pseudocomplementation (similarly as for lattices in [3–6]) to reach another kind of intuitionistic logic.

Hence, we can introduce the following:

Definition 1. By a *nearlattice* we mean a semilattice $\mathscr{S} = (A; \vee)$ where for each $a \in A$ the principal filter $[a) = \{x \in A; a \leqslant x\}$ is a lattice with respect to the induced order $\leqslant$ of $\mathscr{S}$.

Remark 1. Since the operation meet is defined only in a corresponding principal filter, we will indicate this fact by indices, i.e. $\wedge_x$ denotes the meet in $[x)$. On the other hand, if $a, b \in [x)$ and $y \leqslant x$ then $a, b \in [y)$ and $a \wedge_x b = a \wedge_y b$ since both are considered with respect to the same (induced) order $\leqslant$.

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Our first aim is to show that nearlattices can be considered equivalently as algebras with one ternary operation. Hence, they have two faces and we will switch in our investigations between them depending on the fact which one will be more convenient.

We start with the assertion saying that they form an equational class when considered as algebras with one ternary operation. It is apparent that for all x, y, z of a nearlattice $\mathscr{S}$, $(x \vee z)\wedge_z(y \vee z)$ is correctly defined since both $x \vee z, y \vee z \in [z)$.

Theorem 1. *Let* $\mathscr{S} = (A; \vee)$ *be a nearlattice. Define a ternary operation by* $m(x, y, z) = (x \vee z)\wedge_z(y \vee z)$ *on* A. *Then* $m(x, y, z)$ *is an everywhere defined operation and the following identities are satisfied*:

(P1) $m(x, y, x) = x$;
(P2) $m(x, x, y) = m(y, y, x)$;
(P3) $m(m(x, x, y), m(x, x, y), z) = m(x, x, m(y, y, z))$;
(P4) $m(x, y, p) = m(y, x, p)$;
(P5) $m(m(x, y, p), z, p) = m(x, m(y, z, p), p)$;
(P6) $m(x, m(y, y, x), p) = m(x, x, p)$;
(P7) $m(m(x, x, p), m(x, x, p), m(y, x, p)) = m(x, x, p)$;
(P8) $m(m(x, x, z), m(y, y, z), z) = m(x, y, z)$.

Proof. Clearly $x \vee z, y \vee z \in [z)$, $[z)$ is a lattice, thus $m(x, y, z)$ is everywhere defined on A. Prove the identities (P1)–(P8):

(P1) $m(x, y, x) = (x \vee x)\wedge_x(y \vee x) = x\wedge_x(y \vee x) = x$;

(P2) $m(x, x, y) = (x \vee y)\wedge_y(x \vee y) = x \vee y = y \vee x = (y \vee x)\wedge_x(y \vee x) = m(y, y, x)$;

(P3) $m(m(x, x, y), m(x, x, y), z)=m(x\vee y, x\vee y, z)=(x\vee y)\vee z=x\vee(y\vee z)=m(x, x, y\vee z)=m(x, x, m(y, y, z))$;

(P4) $m(x, y, p) = (x \vee p)\wedge_p(y \vee p) = (y \vee p)\wedge_p(x \vee p) = m(y, x, p)$;

(P5) $m(m(x, y, p), z, p)=m((x\vee p)\wedge_p(y\vee p), z, p)=(((x\vee p)\wedge_p(y\vee p))\vee p)\wedge_p(z\vee p)=((x\vee p)\wedge_p(y\vee p))\wedge_p(z\vee p)=(x\vee p)\wedge_p((y\vee p)\wedge_p(z\vee p))=(x\vee p)\wedge_p(((y\vee p)\wedge_p(z\vee p))\vee p)=m(x, (y\vee p)\wedge_p(z\vee p), p)=m(x, m(y, z, p), p)$;

(P6) $m(x, m(y, y, x), p)=m(x, y\vee x, p)=(x \vee p)\wedge_p((y\vee x)\vee p)=(x \vee p)\wedge_p(y\vee (x\vee p))=x \vee p=m(x, x, p)$;

(P7) $m(m(x, x, p), m(x, x, p), m(y, x, p))=m(x\vee p, x\vee p, m(y, x, p))=(x \vee p)\vee m(y, x, p)=(x \vee p)\vee ((y\vee p)\wedge_p(x \vee p)) = x \vee p = m(x, x, p)$;

(P8) $m(m(x, x, z), m(y, y, z), z) = m(x \vee z, y \vee z, z) = ((x \vee z) \vee z)\wedge_z((y \vee z) \vee z) = (x \vee z)\wedge_z(y \vee z) = m(x, y, z)$. □

We are going to prove the converse. For this, let us state the following:

Lemma 1. *Let* $\mathscr{A} = (A; m)$ *be an algebra of type* (3) *satisfying the identities* (*P1*), (*P2*) *and* (*P3*). *Define* $x \vee y = m(x, x, y)$. *Then* $(A; \vee)$ *is a join-semilattice.*

Proof. Idempotency: By (P1), we have $x \vee x = m(x, x, x) = x$.

Commutativity: By (P2), $x \vee y = m(x, x, y) = m(y, y, x) = y \vee x$.

Associativity: Applying (P3), we infer

$$\begin{aligned}(x \vee y) \vee z &= m(x, x, y) \vee z = m(m(x, x, y), m(x, x, y), z)\\ &= m(x, x, m(y, y, z)) = x \vee m(y, y, z) = x \vee (y \vee z).\end{aligned}$$ □

Due to Lemma 1, we can introduce an order $\leqslant$ on the algebra $\mathscr{A} = (A; m)$ as follows:

$$x \leqslant y \quad \text{if and only if } m(x, x, y) = y.$$

It is clear that $\leqslant$ is an order on the set A which coincides with the induced order of the assigned semilattice $(A; \vee)$. In what follows, $\leqslant$ will be called the *induced order* of $\mathscr{A} = (A; m)$. By the symbol $[p)$ for $p \in A$ we denote the principal filter generated by p, with respect to $\leqslant$, i.e. $[p) = \{x \in A; m(p, p, x) = x\}$. We show that if $\mathscr{A} = (A; m)$ satisfies also (P4)–(P7), then for each $p \in A$ the filter $[p)$ is a lattice. Moreover, if $\mathscr{A} = (A, m)$ satisfies also (P8) then the correspondence between nearlattices and these algebras is one-to-one.

Theorem 2. *Let $\mathscr{A} = (A; m)$ be an algebra of type* (3) *satisfying* (P1)–(P7), *and let $\leqslant$ be the induced order. Then for $x \vee y = m(x, x, y)$, $(A; \vee)$ is a join-semilattice and for each $p \in A$, $([p), \leqslant)$ is a lattice, where for $x, y \in [p)$ their infimum is $x \wedge_p y = m(x, y, p)$. Hence $(A, \vee)$ is a nearlattice. If $\mathscr{A} = (A; m)$ satisfies moreover (P8), then the correspondence between nearlattices and algebras $(A; m)$ satisfying (P1)–(P8) is one-to-one.*

Proof. By Lemma 1, $(A; \vee)$ is a join-semilattice. Further let $x, y, z \in [p)$. Then

$$x \wedge_p x = m(x, x, p) = x \vee p = x.$$

By (P4) we get

$$x \wedge_p y = m(x, y, p) = m(y, x, p) = y \wedge_p x$$

and by (P5) also

$$\begin{aligned}(x \wedge_p y) \wedge_p z &= m(x, y, p) \wedge_p z = m(m(x, y, p), z, p)\\ &= m(x, m(y, z, p), p) = x \wedge_p m(y, z, p) = x \wedge_p (y \wedge_p z).\end{aligned}$$

It remains to show absorption laws. By (P6) we have

$$\begin{aligned}x \wedge_p (y \vee x) &= x \wedge_p m(y, y, x) = m(x, m(y, y, x), p)\\ &= m(x, x, p) = x \wedge_p x = x\end{aligned}$$

and, applying (P7),

$$\begin{aligned}x \vee (y \wedge_p x) &= x \vee m(y, x, p) = (x \vee p) \vee m(y, x, p) = m(x, x, p) \vee m(y, x, p)\\ &= m(m(x, x, p), m(x, x, p), m(y, x, p)) = m(x, x, p) = x \wedge_p x = x.\end{aligned}$$

Hence, $([p), \vee, \wedge_p)$ is a lattice.

If $(A, \vee)$ is a nearlattice and (A, m) denotes the corresponding algebra of type (3) then $m(x, x, y) = (x \vee y) \wedge_y (x \vee y) = x \vee y$ for all $x, y \in A$. If, conversely, (A, m) is an algebra of type (3) satisfying (P1)–(P8), $x \vee y := m(x, x, y)$ for all $x, y \in A$ and $x \wedge_p y := m(x, y, p)$ for all $p \in A$ and all $x, y \in A$ with $x, y \geqslant p$ then $(x \vee z) \wedge_z (y \vee z) = m(m(x, x, z), m(y, y, z), z) = m(x, y, z)$ for all $x, y, z \in A$

Thus the correspondence between nearlattices and induced algebras $\mathscr{A} = (A, m)$ is one-to-one. □

Remark 2. Because of Theorems 1 and 2, nearlattices will be alternatively considered as algebras $\mathscr{A} = (A, m)$ of type (3) satisfying (P1)–(P8) and $\leqslant$ will be referred to as the induced order of $\mathscr{A} = (A, m)$. The nearlattice $(A, \vee)$ constructed in Theorem 2 will be called *associated* to $\mathscr{A} = (A, m)$.

Corollary 1. *The class of all nearlattices* (*considered as ternary algebras*) *is a variety.*

Theorem 3. *The variety of all nearlattices is congruence distributive.*

Proof. By Jónsson's Theorem, we need to only verify the existence of ternary Jónsson terms. We can take $n = 4$, and

$$p_0(x, y, z) = x;$$
$$p_1(x, y, z) = m(z, y, x);$$
$$p_2(x, y, z) = m(x, x, z);$$
$$p_3(x, y, z) = m(x, y, z);$$
$$p_4(x, y, z) = z.$$

Then we need to show that

$$p_0(x, y, z) = x; \quad p_4(x, y, z) = z;$$

$$p_0(x, y, x) = p_1(x, y, x) = p_2(x, y, x) = p_3(x, y, x) = p_4(x, y, x) = x;$$

$$p_0(x, x, y) = p_1(x, x, y); \quad p_2(x, x, y) = p_3(x, x, y);$$

$$p_1(x, y, y) = p_2(x, y, y); \quad p_3(x, y, y) = p_4(x, y, y).$$

Evidently $p_0(x, y, z) = x$ and $p_4(x, y, z) = z$ hold. Clearly

$$m(x, y, x) \overset{(P1)}{=} x, \quad m(x, x, x) \overset{(P1)}{=} x,$$

so we obtain $p_0(x, y, x)=x$, $p_1(x, y, x)=m(x, y, x)=x$, $p_2(x, y, x)=m(x, x, x)=x$, $p_3(x, y, x)=m(x, y, x)=x$, $p_4(x, y, x) = x$.

Further, for i even we have

$$p_0(x, x, y) = x \overset{(P1)}{=} m(x, y, x) \overset{(P4)}{=} m(y, x, x) = p_1(x, x, y),$$

$$p_2(x, x, y) = m(x, x, y) = p_3(x, x, y).$$

Finally, for i odd we compute

$$p_1(x, y, y) = m(y, y, x) \overset{(P2)}{=} m(x, x, y) = p_2(x, y, y),$$

$$p_3(x, y, y) = m(x, y, y) \overset{(P4)}{=} m(y, x, y) \overset{(P1)}{=} y = p_4(x, y, y). \qquad \square$$

Now we introduce two more identities for nearlattices:
[(D1)] $m(x, m(y, y, z), p) = m(m(x, y, p), m(x, y, p), m(x, z, p))$ for $x, y, z \in [p)$;
[(D2)] $m(x, x, m(y, z, p)) = m(m(x, x, y), m(x, x, z), p)$ for $x, y, z \in [p)$.

Definition 2. Let $\mathscr{A} = (A; m)$ be an algebra of type (3). We call $\mathscr{A}$ distributive if it satisfies identity (D1). Further, we call $\mathscr{A}$ dually distributive if it satisfies identity (D2).

Lemma 2. *Let $\mathscr{A} = (A; m)$ be an algebra of type* (3) *satisfying* (*P1*), (*P2*), (*P3*), (*P4*), (*P6*), (*P7*). *If $\mathscr{A}$ is distributive then $\mathscr{A}$ is also dually distributive*.

Proof. Let $x, y, z \in [p)$, and assume (D1) holds. We prove (D2):

$$
\begin{aligned}
&m(m(x,x,y), m(x,x,z), p)\\
&\overset{(D1)}{=} m(m(m(x,x,y),x,p), m(m(x,x,y),x,p), m(m(x,x,y),z,p))\\
&\overset{(P2)}{=} m(m(m(y,y,x),x,p), m(m(y,y,x),x,p), m(m(x,x,y),z,p))\\
&\overset{(P4)}{=} m(m(x,m(y,y,x),p), m(x,m(y,y,x),p), m(z,m(x,x,y),p))\\
&\overset{(P6)}{=} m(m(x,x,p), m(x,x,p), m(z,m(x,x,y),p))\\
&\overset{(D1)}{=} m(m(x,x,p), m(x,x,p), m(m(z,x,p), m(z,x,p), m(z,y,p)))\\
&\overset{(P3)}{=} m(m(m(x,x,p),m(x,x,p),m(z,x,p)), m(m(x,x,p),m(x,x,p),m(z,x,p)), m(z,y,p))\\
&\overset{(P7),(P4)}{=} m(m(x,x,p), m(x,x,p), m(y,z,p)) = m(x,x,m(y,z,p)),
\end{aligned}
$$

because for $x \in [p)$ it holds $m(x,x,p) = x$. □

Lemma 3. *Let $\mathscr{A} = (A; m)$ be an algebra of type* (3) *satisfying* $(P1)$–$(P7)$. *If $\mathscr{A}$ is dually distributive then $\mathscr{A}$ is also distributive.*

Proof. Let $x, y, z \in [p)$, and assume (D2) holds. We show (D1). Clearly we have $m(x,x,p) = x$. Further,

$$
\begin{aligned}
&m(m(x,y,p), m(x,y,p), m(x,z,p))\\
&\overset{(D2)}{=} m(m(m(x,y,p),m(x,y,p),x), m(m(x,y,p),m(x,y,p),z), p)\\
&\overset{(P2)}{=} m(m(x,x,m(x,y,p)), m(z,z,m(x,y,p)), p)\\
&\overset{(P4)}{=} m(m(x,x,m(y,x,p)), m(z,z,m(x,y,p)), p)\\
&= m(m(m(x,x,p),m(x,x,p),m(y,x,p)), m(z,z,m(x,y,p)), p)\\
&\overset{(P7)}{=} m(m(x,x,p), m(z,z,m(x,y,p)), p) = m(x, m(z,z,m(x,y,p)), p)\\
&\overset{(D2)}{=} m(x, m(m(z,z,x), m(z,z,y), p), p)\\
&\overset{(P5)}{=} m(m(x, m(z,z,x), p), m(z,z,y), p)\\
&\overset{(P6)}{=} m(m(x,x,p), m(z,z,y), p) = m(x, m(z,z,y), p)\\
&\overset{(P2)}{=} m(x, m(y,y,z), p). \qquad \square
\end{aligned}
$$

Theorem 4. *Let $\mathscr{A} = (A; m)$ be an algebra of type* (3) *satisfying* $(P1)$–$(P8)$. *The following conditions are equivalent*:

(1) *$\mathscr{A}$ is distributive*;
(2) *$\mathscr{A}$ is dually distributive*;
(3) *in the associated semilattice, every principal filter is a distributive lattice*.

Proof. From Lemmas 2 and 3 it follows that (1) is equivalent to (2). We show that (1) is equivalent to (3). Let $\mathscr{A} = (A; m)$ satisfy (P1)–(P8) and let $p \in A$.

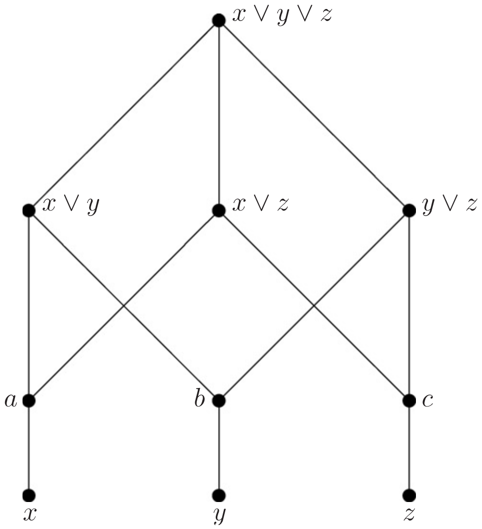

Fig. 1.

Suppose $x, y, z \in [p)$. First we assume (D1). Then

$$\begin{aligned} x\wedge_p(y \vee z) &= x\wedge_p m(y, y, z) = m(x, m(y, y, z), p) \\ &\overset{(D1)}{=} m(m(x, y, p), m(x, y, p), m(x, z, p)) = m(x, y, p) \vee m(x, z, p) \\ &= (x\wedge_p y) \vee (x\wedge_p z). \end{aligned}$$

Conversely, if $[p)$ is distributive, we derive

$$\begin{aligned} m(x, m(y, y, z), p) &= m(x, y \vee z, p) = (x \vee p)\wedge_p((y \vee z) \vee p) \\ &= (x \vee p)\wedge_p((y \vee p) \vee (z \vee p)) = ((x \vee p)\wedge_p(y \vee p)) \vee ((x \vee p)\wedge_p(z \vee p)) \\ &= m(x, y, p) \vee m(x, z, p) = m(m(x, y, p), m(x, y, p), m(x, z, p)). \qquad \Box \end{aligned}$$

We have shown that the class $\mathscr{D}$ of all distributive nearlattices forms a variety (when considered of type (3)). In what follows we present free distributive nearlattices with one, two and three free generators.

(a) If $X = \{x\}$ then $\mathscr{F}_{\mathscr{D}}(X)$ is evidently a 1-element lattice $\{x\}$.
(b) If $X = \{x, y\}$ then $\mathscr{F}_{\mathscr{D}}(X)$ is clearly a 3-element join-semilattice $\{x, y, x \vee y\}$.
(c) If $X = \{x, y, z\}$ then $\mathscr{F}_{\mathscr{D}}(X)$ is a 10-element join-semilattice $\{x, y, z, a, b, c, x \vee y, x \vee z, y \vee z, x \vee y \vee z\}$, where $a = (x \vee y)\wedge_x(x \vee z)$, $b = (x \vee y)\wedge_y(y \vee z)$, $c = (x \vee z)\wedge_z(y \vee z)$, see Fig. 1.

It is almost evident that for a finite set $X = \{x_1, \ldots, x_n\}$ of free generators the free distributive nearlattice $\mathscr{F}_D(X)$ has at most as many elements as the free distributive lattice with this generating set because we can involve all possible joins but not all meets which are generated in lattices. Since the variety of distributive lattices is locally finite, we infer immediately

Corollary. *The variety of distributive nearlattices is locally finite.*

Of course, $\mathscr{F}_D(X)$ is at least of a cardinality of the free semilattice with n free generators, i.e. $2^n - 1$. In fact, it is equal only for $n = 1, 2$ as mentioned above.

In the remaining part, we are concentrated on nearlattices whose principal filters are pseudocomplemented lattices. A nearlattice whose principal filters are pseudocomplemented lattices will be called a *sectionally pseudocomplemented nearlattice*.

Theorem 5. *Let $\mathscr{A} = (A; \vee, \mathbf{1})$ be a nearlattice with greatest element $\mathbf{1}$. Let $d(x, y)$ be a binary operation on A. Then $d(x, y)$ is a pseudocomplement of the element $x \vee y$ in $[y)$ if and only if it satisfies the following identities*:

(0) $d(x, y) = d(x \vee y, y)$, $d(x, x) = \mathbf{1}$,
(a) $d(\mathbf{1}, y) = y$,
(b) $d(d(x, y), y) \vee (x \vee y) = d(d(x, y), y)$,
(c) $d((x \vee y)\wedge_y(z \vee y), y)\wedge_y d((x \vee y)\wedge_y d(z, y), y) = d(x, y)$.

Proof. Let $d(x, y)$ be a pseudocomplement of the element $x \vee y$ in $[y)$. Then $d(x, y)$ clearly satisfies (0) and (a), and by the definition of pseudocomplement, (b) follows immediately. We prove (c): Let $x, y, z \in A$. Then $x \vee y, z \vee y \in [y)$, and thus $(x \vee y)\wedge_y(z \vee y)$ exists. Similarly, $d(z, y) \in [y)$, hence $(x \vee y)\wedge_y d(z, y)$ also exists. Clearly

$$(x \vee y)\wedge_y(z \vee y) \leqslant x \vee y,$$

$$(x \vee y)\wedge_y d(z, y) \leqslant x \vee y,$$

i.e. also

$$d((x \vee y)\wedge_y(z \vee y), y) \geqslant d(x \vee y, y) = d(x, y),$$

$$d((x \vee y)\wedge_y d(z, y), y) \geqslant d(x \vee y, y) = d(x, y)$$

(because pseudocomplementation reverses order). Thus

$$d((x \vee y)\wedge_y(z \vee y), y)\wedge_y d((x \vee y)\wedge_y d(z, y), y) \geqslant d(x, y).$$

Since $d((x \vee y)\wedge_y(z \vee y), y)$ is a pseudocomplement of $(x \vee y)\wedge_y(z \vee y)$ in $[y)$, we have

$$(x \vee y)\wedge_y(z \vee y)\wedge_y d((x \vee y)\wedge_y(z \vee y), y) = y,$$

i.e.

$$(x \vee y)\wedge_y d((x \vee y)\wedge_y(z \vee y), y) \leqslant d(z, y).$$

Hence

$$\begin{aligned}(x \vee y)\wedge_y d((x \vee y)\wedge_y(z \vee y), y) &\leqslant (x \vee y)\wedge_y d(z, y)\\ &\leqslant d(d((x \vee y)\wedge_y d(z, y), y), y),\end{aligned}$$

since the double pseudocomplementation is increasing, i.e.

$$(x \vee y)\wedge_y d((x \vee y)\wedge_y(z \vee y), y)\wedge_y d((x \vee y)\wedge_y d(z, y), y) = y.$$

Thus

$$d((x \vee y)\wedge_y(z \vee y), y)\wedge_y d((x \vee y)\wedge_y d(z, y), y) \leqslant d(x, y).$$

Altogether, we have shown (c).

Conversely, let $d(x, y)$ be a binary operation on A which satisfies (0), (a), (b), (c). From (b) it is evident that

(b′) $d(d(x, y), y) \geqslant x \vee y$.

According to (c) we immediately obtain $d(x, y) \leqslant d((x \vee y)\wedge_y(z \vee y), y)$, and thus

(A) $\quad z \leqslant x \Rightarrow d(x, y) \leqslant d(z \vee y, y) = d(z, y)$

(i.e. the operation $d(x, y)$ is antitone in the first variable). Putting $x = \mathbf{1}$ in (c) and observing (0), we obtain

$$d(z, y)\wedge_y d(d(z, y), y) = d(\mathbf{1}, y) = y,$$

and, according to (b′),

$$d(z, y)\wedge_y(z \vee y) \leqslant d(z, y)\wedge_y d(d(z, y), y) = y.$$

Since $z \leqslant \mathbf{1}$, (A) and (a) yield $d(z, y) \geqslant d(\mathbf{1}, y) = y$. Clearly $z \vee y \geqslant y$, and thus $d(z, y)\wedge_y(z \vee y)$ exists and $d(z, y)\wedge_y(z \vee y) \geqslant y$. Altogether we have shown

$$\text{(B)} \quad d(z, y)\wedge_y(z \vee y) = y.$$

Substituting now $x = d(z, y)$ in (b′) we obtain

$$d(d(d(z, y), y), y) \geqslant d(z, y) \vee y = d(z, y).$$

Conversely (according to (b′), $z \vee y \leqslant d(d(z, y), y)$, thus by (0) and (A) we obtain $d(z, y) = d(z \vee y, y) \geqslant d(d(d(z, y), y), y)$. We have proved

$$\text{(C)} \quad d(d(d(z, y), y), y) = d(z, y).$$

If $(x \vee y)\wedge_y(z \vee y) = y$, then, according to (c) and (0), it holds $d((x \vee y)\wedge_y d(z, y), y) = d(x, y)$, and hence by (b′) and (A) also

$$\begin{aligned} x \vee y &\leqslant d(d(x, y), y) = d(d((x \vee y)\wedge_y d(z, y), y), y) \\ &\leqslant d(d(d(z, y), y), y) = d(z, y). \end{aligned}$$

This together with (B) yields that $d(z, y)$ is the greatest element in $[y)$ such that $d(z, y)\wedge_y(z \vee y) = y$, i.e. it is a pseudocomplement of $z \vee y$ in $[y)$. □

Remark 3. Our identities (0), (a), (b), (c) of Theorem 5 can be easily transferred into the operation $m(x, y, z)$ for nearlattices as follows:

(0) $d(x, y) = d(m(x, x, y), y), \quad d(x, x) = \mathbf{1}$,
(a) $d(\mathbf{1}, y) = y$,
(b) $m(d(d(x, y), y), d(d(x, y), y), m(x, x, y)) = d(d(x, y), y)$,
(c) $m(d(m(x, z, y), d(m(m(x, x, y), d(z, y), y), y), y) = d(x, y)$.

Corollary. *The class of all sectionally pseudocomplemented nearlattices* (*considered as algebras of type* (3,2)) *is a variety.*

Corollary. *Every finite distributive nearlattice is sectionally pseudocomplemented.*

Proof. Suppose $x \in [y)$ and $Z = \{z \in [y); x\wedge_y z = y\}$. Since $y \in Z$, Z is non-void and finite, i.e. $Z = \{z_1, \ldots, z_n\}$. Take $w = z_1 \vee z_2 \vee \cdots \vee z_n$. Due to distributivity of $[y)$, we have $x\wedge_y w = (x\wedge_y z_1) \vee \cdots \vee (x\wedge_y z_n) = y \vee \cdots \vee y = y$ and w is the greatest element of $[y)$ such that $x\wedge_y w = y$. Hence, w is the pseudocomplement of x in $[y)$. □

References

[1] J.C. Abbott, Semi-Boolean algebra, Mat. Vesnik 4 (1967) 177–198.
[2] J.C. Abbott, Orthoimplication algebras, Studia Logica 35 (1976) 173–177.
[3] I. Chajda, An extension of relative pseudocomplementation to non-distributive lattices, Acta Sci. Math. (Szeged) 69 (2003) 491–496.
[4] I. Chajda, R. Halaš, H. Länger, Orthomodular implication algebras, Intern. J. Theor. Phys. 40 (2001) 1875–1884.
[5] I. Chajda, R. Halaš, H. Länger, Simple axioms for orthomodular implication algebras, Intern. J. Theor. Phys. 43 (2004) 911–914.
[6] I. Chajda, S. Radeleczki, On varieties defined by pseudocomplemented nondistributive lattices, Publ. Math. Debrecen 63 (4) (2003) 737–750.