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On Double Basic Algebras and Pseudo-effect Algebras

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Abstract Double basic algebras are a counterpart of bounded lattices with order-antiautomorphisms on principal filters. In the paper, an independent axiomatization of double basic algebras is given and lattice pseudo-effect algebras are characterized in the setting of double basic algebras.

Keywords Basic algebra · Double basic algebra · Pseudo-MV-algebra · Pseudo-effect algebra · Compatibility

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In [1], the first author introduced the so-called double basic algebras, these being algebras $\mathcal{A} = (A, \boxplus, \oplus, ^-, ^\sim, 0, 1)$ with the property that the rule $x \leq y$ iff $x^- \boxplus y = 1$ (which is the same as $x^\sim \oplus y = 1$) defines a bounded lattice in which $x \vee y = (x^- \boxplus y)^\sim \oplus y = (x^\sim \oplus y)^- \boxplus y$ and $x \wedge y = (x^- \vee y^-)^\sim = (x^\sim \vee y^\sim)^-$ and where, for every $a \in A$, the maps $x \mapsto x^- \boxplus a$ and $x \mapsto x^\sim \oplus a$ are mutually inverse order-antiautomorphisms on the interval $[a, 1]$. In the case when the two 'additions' and the 'negations' coincide, double basic algebras reduce to basic algebras that were

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defined in [3] in an attempt to generalize orthomodular lattices in a similar way in which MV-algebras generalize Boolean algebras.

The present paper has two parts. The former is a revision of [1] in the sense that we provide a new independent axiomatic system and enlighten the relations between double basic algebras and pseudo-MV-algebras (GMV-algebras). In the latter part, lattice pseudo-effect algebras are characterized as a subvariety of double basic algebras, and pairs of compatible elements of lattice pseudo-effect algebras are described in terms of double basic algebras.

1 Double Basic Algebras

We first explain some basic concepts. By a **lattice with sectional antiautomorphisms** we mean a structure $\mathcal{L} = (L, \vee, \wedge, 0, 1, (\beta_a)_{a \in L})$ where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and, for each $a \in L$, β_a is an order-antiautomorphism on the interval $[a, 1]$, i.e., β_a is a bijection from $[a, 1]$ onto itself such that $x \leq y$ iff $\beta_a(x) \geq \beta_a(y)$ for all $x, y \in [a, 1]$. If all β_a's are involutive, then we say that $\mathcal{L}$ is a **lattice with sectional antitone involutions**.

A **basic algebra** [2, 3, 6] is an algebra $\mathcal{A} = (A, \oplus, {}^-, 0, 1)$ of type $\langle 2, 1, 0, 0\rangle$ satisfying the identities

$$x \oplus 0 = x,$$

$$x^{--} = x,$$

$$(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x,$$

$$(((x \oplus y)^- \oplus y)^- \oplus z)^- \oplus (x \oplus z) = 1.$$

As shown in [3], there is a one-one correspondence between lattices with sectional antitone involutions and basic algebras. Specifically, given $\mathcal{L} = (L, \vee, \wedge, 0, 1, (\beta_a)_{a \in L})$ a lattice with sectional antitone involutions, the associated basic algebra $\mathcal{L}^B = (L, \oplus, {}^-, 0, 1)$ is defined by

$$x^- = \beta_0(x) \quad \text{and} \quad x \oplus y = \beta_y(x^- \vee y),$$

and on the other hand, if $\mathcal{A} = (A, \oplus, {}^-, 0, 1)$ is a basic algebra, then the stipulation $x \leq y$ iff $x^- \oplus y = 1$ defines a bounded lattice in which

$$x \vee y = (x^- \oplus y)^- \oplus y \quad \text{and} \quad x \wedge y = (x^- \vee y^-)^-,$$

and where, for every $a \in A$, $\beta_a \colon x \mapsto x^- \oplus a$ is an antitone involution on $[a, 1]$; thus $\mathcal{A}^L = (A, \vee, \wedge, 0, 1, (\beta_a)_{a \in A})$ is a lattice with sectional antitone involutions. The assignments are mutually inverse, i.e., we have $(\mathcal{L}^B)^L = \mathcal{L}$ and $(\mathcal{A}^L)^B = \mathcal{A}$.

Examples of basic algebras include MV-algebras, which are precisely the associative basic algebras, and orthomodular lattices, which may be described as basic algebras satisfying the identity $x \oplus (x \wedge y) = x$. Indeed, in an orthomodular lattice $(L, \vee, \wedge, {}^\perp, 0, 1)$, the maps $x \mapsto x^\perp \vee a$ are antitone involutions on the sections $[a, 1]$, hence the addition $\oplus$ is defined by $x \oplus y = (x \wedge y^\perp) \vee y$ and the identity obviously captures the orthomodular law.

As we have already mentioned, double basic algebras were invented as a generalization of basic algebras corresponding to lattices with sectional antiautomorphisms.

According to [1], a **double basic algebra** is an algebra $\mathcal{A} = (A, \boxplus, \oplus, ^-, ^\sim, 0, 1)$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ that satisfies the identities

$$
\begin{aligned}
&x \boxplus 0 = x = x \oplus 0,\\
&x^{-\sim} = x = x^{\sim -},\\
&(x^- \boxplus y)^\sim \oplus y = (y^- \boxplus x)^\sim \oplus x = (y^\sim \oplus x)^- \boxplus x = (x^\sim \oplus y)^- \boxplus y,\\
&(((x \boxplus y)^\sim \oplus y)^- \boxplus z)^\sim \oplus (x \boxplus z) = 1,\\
&(((x \oplus y)^- \boxplus y)^\sim \oplus z)^- \boxplus (x \oplus z) = 1,\\
&0^- = 1 = 0^\sim.
\end{aligned}
$$

It can easily be shown that if $\mathcal{L} = (L, \vee, \wedge, 0, 1, (\beta_a)_{a\in L})$ is a lattice with sectional antiautomorphisms and if we define

$$
\begin{aligned}
x^- = \beta_0(x), \quad x^\sim = \beta_0^{-1}(x),\\
x \boxplus y = \beta_y(x^\sim \vee y), \quad x \oplus y = \beta_y^{-1}(x^- \vee y),
\end{aligned} \tag{1}
$$

then $\mathcal{L}^D = (L, \boxplus, \oplus, ^-, ^\sim, 0, 1)$ is a double basic algebra, and all double basic algebras arise in this way. Indeed, if $\mathcal{A} = (A, \boxplus, \oplus, ^-, ^\sim, 0, 1)$ is a double basic algebra, then letting

$$x \le y \quad \text{iff} \quad x^- \boxplus y = 1 \quad (\text{or, equivalently, } x^\sim \oplus y = 1)$$

we obtain a bounded lattice with

$$
\begin{aligned}
x \vee y = (x^- \boxplus y)^\sim \oplus y = (x^\sim \oplus y)^- \boxplus y,\\
x \wedge y = (x^- \vee y^-)^\sim = (x^\sim \vee y^\sim)^-,
\end{aligned}
$$

such that for each $a \in A$, the map $\beta_a\colon x \mapsto x^- \boxplus a$ is an antiautomorphism on $[a, 1]$ the inverse of which is $\beta_a^{-1}\colon x \mapsto x^\sim \oplus a$. Thus $\mathcal{A}^L = (A, \vee, \wedge, 0, 1, (\beta_a)_{a\in A})$ is a lattice with sectional antiautomorphisms from which, using Eq. 1, we can recover the initial $\mathcal{A}$, i.e., $(\mathcal{A}^L)^D = \mathcal{A}$. We also have $(\mathcal{L}^D)^L = \mathcal{L}$.

The connections between basic and double basic algebras are obvious. If $\mathcal{A} = (A, \oplus, ^-, 0, 1)$ is a basic algebra, then $(\mathcal{A}^L)^D = (A, \oplus, \oplus, ^-, ^-, 0, 1)$ is a double basic algebra, and conversely, if we are given a double basic algebra in which $\boxplus$ coincides with $\oplus$, then the 'negations' $^-$ and $^\sim$ coincide too, so that the double basic algebra becomes a basic algebra. More precisely, we have

Theorem 1 *Let $\mathcal{A} = (A, \boxplus, \oplus, ^-, ^\sim, 0, 1)$ be a double basic algebra. Then the reduct $\mathcal{A}_1 = (A, \boxplus, ^-, 0, 1)$ is a basic algebra if and only if $\mathcal{A}$ satisfies the identity*

$$x \boxplus y = x \oplus y. \tag{2}$$

Proof Let $\mathcal{A}_1$ be a basic algebra. Then $\mathcal{A}^L = (\mathcal{A}_1)^L$ is a lattice with sectional antitone involutions, hence for every $a \in A$, the map $\beta_a\colon x \mapsto x^- \boxplus a$ is an antitone involution which coincides with its inverse $\beta_a^{-1}\colon x \mapsto x^\sim \oplus a$. Thus $x^- = \beta_0(x) = \beta_0^{-1}(x) = x^\sim$ and $x \boxplus y = \beta_y(x^\sim \vee y) = \beta_y^{-1}(x^- \vee y) = x \oplus y$ for all $x, y \in A$.

Conversely, assume that $\mathcal{A}$ satisfies Eq. 2. Since $\boxplus$ and $\oplus$ coincide, for all $x, y \in A$ we have the following equivalences: $y \le x^-$ iff $y^- \boxplus x^- = 1$ iff $y^- \oplus x^- = 1$ iff

$y^{--\sim} \oplus x^- = 1$ iff $y^{--} \leq x^-$ iff $y^- = y^{--\sim} \geq x^{-\sim} = x$ iff $y = y^{-\sim} \leq x^\sim$. Thus $x^- = x^\sim$ for all $x \in A$, and it follows that $\mathcal{A}_1$ is a basic algebra. □

Axiomatization In what follows, we aim at proving that double basic algebras can be axiomatized by the identities

$$x \boxplus 0 = x, \tag{D1}$$

$$x \oplus 0 = x, \tag{D2}$$

$$x^{-\sim} = x, \tag{D3}$$

$$(x^- \boxplus y)^\sim \oplus y = (y^\sim \oplus x)^- \boxplus x, \tag{D4}$$

$$(((x \boxplus y)^\sim \oplus y)^- \boxplus z)^\sim \oplus (x \boxplus z) = 1, \tag{D5}$$

$$(((x \oplus y)^- \boxplus y)^\sim \oplus z)^- \boxplus (x \oplus z) = 1. \tag{D6}$$

Thus some identities from the original axioms can be omitted.

Lemma 2 *Every algebra satisfying Eqs. D1–D6 satisfies the following identities:*

$$x^- \boxplus x = 1 = x^\sim \oplus x, \tag{3}$$

$$0^- = 1 = 0^\sim, \tag{4}$$

$$1^{\sim-} = 1, \tag{5}$$

$$1^- = 0 = 1^\sim, \tag{6}$$

$$1 \boxplus x = 1 = 1 \oplus x, \tag{7}$$

$$x^{\sim-} = x, \tag{8}$$

$$0 \boxplus x = x = 0 \oplus x, \tag{9}$$

$$x \boxplus 1 = 1 = x \oplus 1, \tag{10}$$

$$x^- \boxplus (y \oplus x) = 1 = x^\sim \oplus (y \boxplus x). \tag{11}$$

Proof By Eq. D5 we have $1 = (((x \boxplus 0)^\sim \oplus 0)^- \boxplus 0)^\sim \oplus (x \boxplus 0) = x^{\sim-\sim} \oplus x = x^\sim \oplus x$ and, analogously, $1 = (((x \oplus 0)^- \boxplus 0)^\sim \oplus 0)^- \boxplus (x \oplus 0) = x^{-\sim-} \boxplus x = x^- \boxplus x$ by Eq. D6. As an immediate consequence of Eq. 3 we get $1 = 0^- \boxplus 0 = 0^-$ and $1 = 0^\sim \oplus 0 = 0^\sim$, which is Eq. 4. Now, $1^{\sim-} = 0^{-\sim-} = 0^- = 1$, $1^\sim = 0^{-\sim} = 0$ and $1^- = 1^- \boxplus 0 = (0^\sim \oplus 0)^- \boxplus 0 = (0^- \boxplus 0)^\sim \oplus 0 = 0^{-\sim} = 0$ by Eq. D4, proving Eqs. 5 and 6.

Further, $1 = (((1 \oplus y)^- \boxplus y)^\sim \oplus 0)^- \boxplus (1 \oplus 0) = ((0^\sim \oplus y)^- \boxplus y)^{\sim-} \boxplus 1 = ((y^- \boxplus 0)^\sim \oplus 0)^{\sim-} \boxplus 1 = y^{-\sim\sim-} \boxplus 1 = y^{\sim-} \boxplus 1$. When replacing y with $(x \oplus 1)^-$, we have $(x \oplus 1)^- \boxplus 1 = 1$, whence $1 = (((x \oplus 1)^- \boxplus 1)^\sim \oplus 0)^- \boxplus (x \oplus 0) = 1^{\sim-} \boxplus x = 1 \boxplus x$.

Before proving $1 \oplus x = 1$, we notice that

$$x^{\sim-} = 0 \boxplus x = 0 \oplus x, \tag{12}$$

because $x^{\sim-} = (x^\sim \oplus 0)^- \boxplus 0 = (0^- \boxplus x)^\sim \oplus x = (1 \boxplus x)^\sim \oplus x = 1^\sim \oplus x = 0 \oplus x$ and $1^\sim \oplus x = (x^- \boxplus x)^\sim \oplus x = (x^\sim \oplus x)^- \boxplus x = 1^- \boxplus x = 0 \boxplus x$.

Now we have $1 = (((1 \boxplus 1)^\sim \oplus 1)^- \boxplus x^-)^\sim \oplus (1 \boxplus x^-) = ((1^\sim \oplus 1)^- \boxplus x^-)^\sim \oplus 1 = (1^- \boxplus x^-)^\sim \oplus 1 = (0 \boxplus x^-)^\sim \oplus 1 = x^{-\sim-\sim} \oplus 1 = x \oplus 1$, which is the second part of

Eq. 10. It follows that $1 = (((x \boxplus 1)^{\sim} \oplus 1)^{-} \boxplus 0)^{\sim} \oplus (x \boxplus 0) = 1^{-\sim} \oplus x = 1 \oplus x$. This completes the proof of Eq. 7.

Using Eqs. 12 and 7, we get $x = x^{-\sim} = (x^{-} \boxplus 0)^{\sim} \oplus 0 = (0^{\sim} \oplus x)^{-} \boxplus x = (1 \oplus x)^{\sim} \boxplus x = 1^{\sim} \boxplus x = 0 \boxplus x = x^{\sim -}$, which proves Eq. 8 as well as Eq. 9.

We have shown above that $y^{\sim -} \boxplus 1 = 1$ for all y, which together with Eq. 8 implies Eq. 10.

Finally, we have $1 = (((y \boxplus 1)^{\sim} \oplus 1)^{-} \boxplus x)^{\sim} \oplus (y \boxplus x) = x^{\sim} \oplus (y \boxplus x)$ and $1 = (((y \oplus 1)^{-} \boxplus 1)^{\sim} \oplus x)^{-} \boxplus (y \oplus x) = x^{-} \boxplus (y \oplus x)$, which is Eq. 11. □

Theorem 3 *An algebra $\mathcal{A} = (A, \boxplus, \oplus, ^{-}, ^{\sim}, 0, 1)$ is a double basic algebra if and only if $\mathcal{A}$ satisfies the identities Eqs. D1–D6.*

Proof Applying Lemma 2, we only have to show that if $\mathcal{A}$ satisfies Eqs. D1–D6, then $\mathfrak{a} = \mathfrak{b}$ where

$$\mathfrak{a} = (x^{-} \boxplus y)^{\sim} \oplus y = (y^{\sim} \oplus x)^{-} \boxplus x,$$

$$\mathfrak{b} = (y^{-} \boxplus x)^{\sim} \oplus x = (x^{\sim} \oplus y)^{-} \boxplus y.$$

Owing to Eq. 11 we have $x^{-} \boxplus \mathfrak{b} = 1 = y^{\sim} \oplus \mathfrak{b}$ which yields $\mathfrak{b} = 1^{\sim} \oplus \mathfrak{b} = (x^{-} \boxplus \mathfrak{b})^{\sim} \oplus \mathfrak{b} = (\mathfrak{b}^{\sim} \oplus x)^{-} \boxplus x$ and $(\mathfrak{b}^{\sim} \oplus x)^{-} \boxplus (y^{\sim} \oplus x) = ((1^{-} \boxplus \mathfrak{b})^{\sim} \oplus x)^{-} \boxplus (y^{\sim} \oplus x) = (((y^{\sim} \oplus \mathfrak{b})^{-} \boxplus \mathfrak{b})^{\sim} \oplus x)^{-} \boxplus (y^{\sim} \oplus x) = 1$ by Eq. D6, whence

$$\mathfrak{a} = (1^{\sim} \oplus (y^{\sim} \oplus x))^{-} \boxplus x = [((\mathfrak{b}^{\sim} \oplus x)^{-} \boxplus (y^{\sim} \oplus x))^{\sim} \oplus (y^{\sim} \oplus x)]^{-} \boxplus x$$

and thus

$$\mathfrak{a}^{\sim} \oplus \mathfrak{b} = \big([((\mathfrak{b}^{\sim} \oplus x)^{-} \boxplus (y^{\sim} \oplus x))^{\sim} \oplus (y^{\sim} \oplus x)]^{-} \boxplus x\big)^{\sim} \oplus ((\mathfrak{b}^{\sim} \oplus x)^{-} \boxplus x) = 1$$

by Eq. D5.

Analogously, $x^{\sim} \oplus \mathfrak{a} = 1 = y^{-} \boxplus \mathfrak{a}$ by Eq. 11, so $\mathfrak{a} = 1^{-} \boxplus \mathfrak{a} = (x^{\sim} \oplus \mathfrak{a})^{-} \boxplus \mathfrak{a} = (\mathfrak{a}^{-} \boxplus x)^{\sim} \oplus x$ and $(\mathfrak{a}^{-} \boxplus x)^{\sim} \oplus (y^{-} \boxplus x) = ((1^{\sim} \oplus \mathfrak{a})^{-} \boxplus x)^{\sim} \oplus (y^{-} \boxplus x) = (((y^{-} \boxplus \mathfrak{a})^{\sim} \oplus \mathfrak{a})^{-} \boxplus x)^{\sim} \oplus (y^{-} \boxplus x) = 1$. Then

$$\mathfrak{b} = (1^{-} \boxplus (y^{-} \boxplus x))^{\sim} \oplus x = [((\mathfrak{a}^{-} \boxplus x)^{\sim} \oplus (y^{-} \boxplus x))^{-} \boxplus (y^{-} \boxplus x)]^{\sim} \oplus x$$

and

$$\mathfrak{b}^{-} \boxplus \mathfrak{a} = \big([((\mathfrak{a}^{-} \boxplus x)^{\sim} \oplus (y^{-} \boxplus x))^{-} \boxplus (y^{-} \boxplus x)]^{\sim} \oplus x\big)^{-} \boxplus ((\mathfrak{a}^{-} \boxplus x)^{\sim} \oplus x) = 1$$

by Eq. D6.

Now we conclude

$$\mathfrak{a} = 1^{\sim} \oplus \mathfrak{a} = (\mathfrak{b}^{-} \boxplus \mathfrak{a})^{\sim} \oplus \mathfrak{a} = (\mathfrak{a}^{\sim} \oplus \mathfrak{b})^{-} \boxplus \mathfrak{b} = 1^{-} \boxplus \mathfrak{b} = \mathfrak{b}$$

as desired. □

The examples below show that this simplified axiomatization of double basic algebras is independent.

(a) The following algebra obviously does not satisfy Eq. D1 since $0 \boxplus 0 = 1$, but it satisfies Eqs. D2–D6:

$\boxplus$	0	1
0	1	1
1	1	1

$\oplus$	0	1
0	0	1
1	1	1

x	0	1
$x^{-} = x^{\sim}$	0	1

(b) If we switch $\boxplus$ and $\oplus$, we obtain an algebra that does not satisfy Eq. D2.

(c) This algebra does not satisfy Eq. D3 since $0^{-\sim} = 1$:

$\boxplus = \oplus$	0	1
0	0	0
1	1	1

x	0	1
x^-	1	1
$x^\sim$	0	1

(d) In the following algebra we have $(1^- \boxplus 0)^\sim \oplus 0 = 1 \neq 0 = (0^\sim \oplus 1)^- \boxplus 1$, so it does not fulfill Eq. D4:

$\boxplus = \oplus$	0	1
0	0	1
1	1	0

x	0	1
$x^- = x^\sim$	1	0

(e) This algebra does not satisfy Eq. D5 since $(((a \boxplus 1)^\sim \oplus 1)^- \boxplus b)^\sim \oplus (a \boxplus b) = b$:

$\boxplus$	0	a	b	1
0	0	a	b	1
a	a	1	0	1
b	b	0	1	1
1	1	1	1	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	1	b	1
b	b	a	1	1
1	1	1	1	1

x	0	a	b	1
$x^- = x^\sim$	1	a	b	0

(f) Finally, when interchanging $\boxplus$ and $\oplus$ we get an algebra in which Eq. D6 fails to be true since $(((a \oplus 1)^- \boxplus 1)^\sim \oplus b)^- \boxplus (a \oplus b) = b$.

Pseudo-MV-algebras Besides double basic algebras, the so-called double MV-algebras were defined in [1]. The motivation was to have a particular class of double basic algebras that stand to MV-algebras as double basic algebras stand to basic algebras. Though it is not the original definition, we may say a **double MV-algebra** is a double basic algebra satisfying the identity

$$x \boxplus (y \oplus z) = y \oplus (x \boxplus z). \tag{13}$$

In this paragraph we show that these double MV-algebras are in fact pseudo-MV-algebras (also called GMV-algebras).

Let us recall that pseudo-MV-algebras were introduced by Georgescu and Iorgulescu [12], and independently by Rachůnek [13] under the name 'GMV-algebras', as a non-commutative counterpart of well-known MV-algebras (see [7]):

A **pseudo-MV-algebra** is an algebra $\mathcal{A} = (A, \oplus, ^-, ^\sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0\rangle$ such that $(A, \oplus, 0)$ is a monoid and the following identities are satisfied:

$$x \oplus 1 = 1 = 1 \oplus x, \quad 1^- = 0 = 1^\sim, \quad x^{-\sim} = x,$$

$$(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-,$$

$$x \oplus (y^- \oplus x)^\sim = y \oplus (x^- \oplus y)^\sim = (y \oplus x^\sim)^- \oplus y = (x \oplus y^\sim)^- \oplus x,$$

$$x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y,$$

where the term operation $\odot$ is defined by $x \odot y = (y^- \oplus x^-)^\sim$.

If we put $x \vee y = x \oplus (y^- \oplus x)^\sim$ and $x \wedge y = x \odot (x^- \oplus y)$, then we obtain a bounded distributive lattice whose underlying order is given by $x \leq y$ iff $x^- \oplus y = 1$

iff $y \oplus x^{\sim} = 1$, and where, for each $a \in A$, $x \mapsto x^{-} \oplus a$ is an order-antiautomorphism on $[a, 1]$ whose inverse is $x \mapsto a \oplus x^{\sim}$. Thus every pseudo-MV-algebra is a lattice with sectional antiautomorphisms and hence can be regarded as a double basic algebra. Specifically, given a pseudo-MV-algebra $\mathcal{A} = (A, \oplus, ^{-}, ^{\sim}, 0, 1)$, the corresponding double basic algebra $\mathcal{A}^{\dagger} = (A, \boxplus^{\dagger}, \oplus^{\dagger}, ^{-}, ^{\sim}, 0, 1)$ is defined by

$$x \boxplus^{\dagger} y = x \oplus y \quad \text{and} \quad x \oplus^{\dagger} y = y \oplus x.$$

Since $\oplus$ is associative, it is plain that $\mathcal{A}^{\dagger}$ satisfies Eq. 13.

Theorem 4 *Let $\mathcal{A} = (A, \boxplus, \oplus, ^{-}, ^{\sim}, 0, 1)$ be a double basic algebra. Then the reduct $\mathcal{A}_2 = (A, \boxplus, ^{-}, ^{\sim}, 0, 1)$ is a pseudo-MV-algebra if and only if $\mathcal{A}$ satisfies the identity 13, i.e., $\mathcal{A}$ is a double MV-algebra.*

Proof If $\mathcal{A}_2$ is a pseudo-MV-algebra, then $\mathcal{A}$ fulfills Eq. 13 because $\mathcal{A} = (\mathcal{A}_2)^{\dagger}$. The converse follows from some considerations in [4]. Roughly speaking, a lattice with sectional antiautomorphisms is derived from a pseudo-MV-algebra if and only if

$$\beta_{\beta_z^{-1}(y \vee z)}(x \vee \beta_z^{-1}(y \vee z)) = \beta^{-1}_{\beta_z(x \vee z)}(y \vee \beta_z(x \vee z))$$

for all x, y, z. Recalling Eq. 1, this condition becomes

$$x^{-} \boxplus (y^{\sim} \oplus z) = y^{\sim} \oplus (x^{-} \boxplus z),$$

which is clearly equivalent to Eq. 13. Thus if $\mathcal{A}$ satisfies Eq. 13, then the above condition holds in $\mathcal{A}^{L}$, and so $\mathcal{A}_2$ is a pseudo-MV-algebra.

In the next section we give another proof of Theorem 4. □

We can therefore identify pseudo-MV-algebras with double MV-algebras, i.e. with double basic algebras satisfying Eq. 13. Accordingly, if $\mathcal{A}$ is a double basic algebra and $\mathcal{B}$ is a subalgebra that fulfills Eq. 13, then we shall say that $\mathcal{B}$ is a **sub-pseudo-MV-algebra** of $\mathcal{A}$.

2 Pseudo-effect Algebras

Pseudo-effect algebras, introduced by Dvurečenskij and Vetterlein in [9, 10], are a non-commutative generalization of effect algebras (see [14] or [8]):

A **pseudo-effect algebra** [9] is a structure $\mathcal{E} = (E, +, 0, 1)$, where $+$ is a partial binary operation on E and $0, 1$ are distinguished elements of E, satisfying the following conditions:

- (PE1) $+$ is associative, in the sense that $(a + b) + c$ is defined if and only if $a + (b + c)$ is defined, and in this case $(a + b) + c = a + (b + c)$;
- (PE2) for every $a \in E$ there exist unique $a^{-}, a^{\sim} \in E$ such that $a^{-} + a = 1 = a + a^{\sim}$;
- (PE3) if $a + b$ is defined, then $a + b = x + a = b + y$ for some $x, y \in E$;
- (PE4) if $a + 1$ or $1 + a$ is defined, then $a = 0$.

Every pseudo-effect algebra $\mathcal{E} = (E, +, 0, 1)$ has a natural underlying order which is defined by stipulating that

$$x \leq y \quad \text{iff} \quad y = x + z \text{ for some } z \in E,$$

which is the same as $y = z + x$ for some $z \in E$. If the poset $(E, \leq)$ thus obtained is a lattice, $\mathcal{E}$ is called a **lattice pseudo-effect algebra**.

It is worth observing that

$$x + y = z \quad \text{iff} \quad x^{\sim} = y + z^{\sim} \quad \text{iff} \quad y^{-} = z^{-} + x, \tag{14}$$

in other words,

$$x + y \text{ is defined} \quad \text{iff} \quad y \leq x^{\sim} \quad \text{iff} \quad x \leq y^{-}.$$

Furthermore, for every $a \in E$, the maps $x \mapsto x^{-} + a$ and $x \mapsto a + x^{\sim}$ are order-antiautomorphisms on $[a, 1]$ which are inverses of each other, so it is obvious that every lattice pseudo-effect algebra is a lattice with sectional antiautomorphism and hence a double basic algebra:

Proposition 5 *Let $\mathcal{E} = (E, +, 0, 1)$ be a lattice pseudo-effect algebra. Upon defining*

$$x \boxplus y = (x \wedge y^{-}) + y \quad \text{and} \quad x \oplus y = y + (x \wedge y^{\sim}),$$

the algebra $\mathcal{E}^D = (E, \boxplus, \oplus, ^{-}, ^{\sim}, 0, 1)$ is a double basic algebra whose underlying order coincides with that of $\mathcal{E}$. If $x + y$ exists in $\mathcal{E}$, then $x + y = x \boxplus y = y \oplus x$. Moreover, $\mathcal{E}^D$ satisfies the quasi-identity

$$x \leq y^{-} \quad \& \quad x \boxplus y \leq z^{-} \quad \Rightarrow \quad x \boxplus (z \oplus y) = z \oplus (x \boxplus y). \tag{15}$$

Proof We know that $\beta_a \colon x \mapsto x^{-} + a$ is an order-antiautomorphism on $[a, 1]$ (and that its inverse is $\beta_a^{-1} \colon x \mapsto a + x^{\sim}$), so that $\mathcal{E}^L = (E, \vee, \wedge, 0, 1, (\beta_a)_{a \in E})$ is a lattice with sectional antiautomorphisms. The double basic algebra $(\mathcal{E}^L)^D$ associated to $\mathcal{E}^L$ by Eq. 1 is then defined as follows:

$$x \boxplus y = \beta_y(x^{\sim} \vee y) = (x^{\sim} \vee y)^{-} + y = (x \wedge y^{-}) + y,$$

$$x \oplus y = \beta_y^{-1}(x^{-} \vee y) = y + (x^{-} \vee y)^{\sim} = y + (x \wedge y^{\sim}).$$

Thus $\mathcal{E}^D = (\mathcal{E}^L)^D$ is a double basic algebra, and its underlying order is just that of $\mathcal{E}^L$, i.e. that of $\mathcal{E}$. It is also evident that $x + y = x \boxplus y = y \oplus x$ since $x + y$ is defined iff $x \leq y^{-}$ iff $x^{\sim} \geq y$.

As for the last claim, if $x \leq y^{-}$ and $x \boxplus y \leq z^{-}$, then by (PE1) both $(x + y) + z$ and $x + (y + z)$ are defined and equal in $\mathcal{E}$. Hence we have $x \boxplus (z \oplus y) = x + (y + z) = (x + y) + z = z \oplus (x \boxplus y)$, proving that $\mathcal{E}^D$ fulfills Eq. 15. □

Remark Instead of $\oplus$ we could have defined $\oplus'$ by $x \oplus' y = \beta_x^{-1}(x \vee y^{-}) = x + (x^{\sim} \wedge y)$, i.e. $x \oplus' y = y \oplus x$. This might seem more natural in the context of pseudo-effect algebras, because when $x + y$ exists in $\mathcal{E}$, then $x + y = x \boxplus y = x \oplus' y$, while with our definition we have $x + y = x \boxplus y = y \oplus x$ (see [5]).

Lemma 6 *Every double basic algebra satisfies the identities*

$$(x \wedge y) \boxplus z = (x \boxplus z) \wedge (y \boxplus z) \quad \text{and} \quad (x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z).$$

Proof Recalling Eq. 1 we have $(x \wedge y) \boxplus z = \beta_z((x \wedge y)^{\sim} \vee z) = \beta_z(x^{\sim} \vee y^{\sim} \vee z) = \beta_z((x^{\sim} \vee z) \vee (y^{\sim} \vee z)) = \beta_z(x^{\sim} \vee z) \wedge \beta_z(y^{\sim} \vee z) = (x \boxplus z) \wedge (y \boxplus z)$. The proof of the other identity is analogous. □

Theorem 7 *Let $\mathcal{A} = (A, \boxplus, \oplus, ^-, ^\sim, 0, 1)$ be a double basic algebra and define the partial algebra $\mathcal{A}^E = (A, +, 0, 1)$ as follows:*

$$x + y = x \boxplus y \quad iff \quad x \leq y^-.$$

Then $\mathcal{A}^E$ is a pseudo-effect algebra if $\mathcal{A}$ satisfies the quasi-identity 15. In this case the underlying orders of $\mathcal{A}$ and $\mathcal{A}^E$ coincide.

Proof Let us assume that Eq. 15 is true in $\mathcal{A}$. We first notice that

$$x + z \text{ is defined iff } x^\sim \geq z, \text{ and } x + z = z \oplus x.$$

Indeed, letting $y = 0$ we get: $x \leq 0^- = 1$ and $x = x \boxplus 0 \leq z^-$ imply $x \boxplus z = x \boxplus (z \oplus 0) = z \oplus (x \boxplus 0) = z \oplus x$. Thus if $x + z$ exists in $\mathcal{A}^E$, then $x + z = x \boxplus z = z \oplus x$. Now we can verify the conditions (PE1)–(PE4).

(PE1) Let $(a + b) + c$ be defined, i.e. $a \leq b^-$ and $a + b \leq c^-$. Since $b \leq a \boxplus b = a + b \leq c^-$, also $b + c$ is defined. Since $\oplus$ is monotone in the first argument (Lemma 6), $a^\sim \geq b$ and $(a \boxplus b)^\sim \geq c$ yield $a^\sim = a^\sim \vee b = (a \boxplus b)^\sim \oplus b \geq c \oplus b = b + c$, which means that $a + (b + c)$ is defined. Similar arguments show that if $a + (b + c)$ is defined, then $(a + b) + c$ is defined, too. Hence $(a + b) + c$ is defined iff so is $a + (b + c)$, in which case, by Eq. 15, $a \leq b^-$ and $a \boxplus b \leq c^-$ imply $a + (b + c) = a \boxplus (c \oplus b) = c \oplus (a \boxplus b) = (a+b)+c$.

(PE2) If $x + a = 1$ for some $x \in A$, then $x \leq a^-$ and $x \vee a^- = (a^{-\sim} \oplus x)^- \boxplus x = (x + a)^- \boxplus x = 1^- \boxplus x = x$, so $x \geq a^-$, proving $x = a^-$. Certainly, $a^- + a = a^- \boxplus a = 1$, and hence a^- is the only element x such that $x + a = 1$. Analogously, $a^\sim$ is the only element y such that $a + y = 1$.

(PE3) Assuming $a + b$ is defined, we put $x = ((a + b)^\sim \oplus a)^-$ and $y = ((a + b)^- \boxplus b)^\sim$. Then $x^\sim \geq a$ and $y^- \geq b$, so $x + a$ and $b + y$ are defined, and we have $x + a = x \boxplus a = ((a + b)^\sim \oplus a)^- \boxplus a = (a + b) \vee a = a + b$ and $b + y = y \oplus b = ((a + b)^- \boxplus b)^\sim \oplus b = (a + b) \vee b = a + b$ since $a + b = b \oplus a \geq a$ and $a + b = a \boxplus b \geq b$.

(PE4) If $a + 1$ is defined, then $a \leq 1^- = 0$. If $1 + a$ is defined, then $0 = 1^\sim \geq a$. Thus $a = 0$ in either case.

We have proved that $\mathcal{A}^E$ is a pseudo-effect algebra and there remains to show that the underlying orders are the same. Let us denote by $\sqsubseteq$ the order in $\mathcal{A}^E$. If $a \leq b$ in $\mathcal{A}$, then we may write $b = a \vee b = (b^\sim \oplus a)^- \boxplus a = (b^\sim \oplus a)^- + a$ as $b^\sim \oplus a \geq a$; hence $a \sqsubseteq b$ in $\mathcal{A}^E$. Conversely, if $a \sqsubseteq b$, then $b = x + a$ for some $x \in E$, so that $b = x \boxplus a \geq a$ in $\mathcal{A}$. □

Combining Proposition 5 and Theorem 7 we conclude that there is a one-one correspondence between lattice pseudo-effect algebras and double basic algebras that satisfy Eq. 15. Indeed, it is apparent that for every lattice pseudo-effect algebra $\mathcal{E}$ we have $(\mathcal{E}^D)^E = \mathcal{E}$. On the other hand, given $\mathcal{A} = (A, \boxplus, \oplus, ^-, ^\sim, 0, 1)$ a double basic algebra satisfying Eq. 15, the additions in $(\mathcal{A}^E)^D = (A, \boxplus^\sharp, \oplus^\sharp, ^-, ^\sim, 0, 1)$ are defined by means of $+$ (which is inherited from $\boxplus$ and $\oplus$) as follows: $x \boxplus^\sharp y = (x \wedge y^-) + y = (x \wedge y^-) \boxplus y = x \boxplus y$ and $x \oplus^\sharp y = y + (x \wedge y^\sim) = (x \wedge y^\sim) \oplus y = x \oplus y$ (in both cases the last equality follows from Lemma 6). Therefore $(\mathcal{A}^E)^D = \mathcal{A}$.

Compatibility in Lattice Pseudo-effect Algebras In [11], Dvurečenskij and Vetterlein introduced five types of compatibilities between elements of pseudo-effect algebras: besides 'pure' compatibility these are ultra strong, strong, weak and ultra weak compatibility. In general they differ from one another, but it turns out that in lattice pseudo-effect algebras, except for ultra weak compatibility, all of them coincide. Rather than giving the original definition, we use one of the alternative characterizations presented in [11].

We must define two partial subtractions $\setminus, /$ that are naturally determined by the underlying order: $x \setminus y$ and y / x exist iff $y \leq x$, and they are unique elements such that

$$(x \setminus y) + y = x = y + (y / x).$$

We should notice that in view of Eq. 14 we have

$$x \setminus y = (y + x^{\sim})^{-} \quad \text{and} \quad y / x = (x^{-} + y)^{\sim}. \tag{16}$$

Now, we can say that x and y are **compatible** and write $x \leftrightarrow y$ iff

$$(x \vee y) \setminus y = x \setminus (x \wedge y) \quad \text{and} \quad (x \vee y) \setminus x = y \setminus (x \wedge y).$$

By [11], Proposition 3.6, we could equivalently use $/$ instead of $\setminus$, i.e., $x \leftrightarrow y$ iff

$$y / (x \vee y) = (x \wedge y) / x \quad \text{and} \quad x / (x \vee y) = (x \wedge y) / y.$$

The concept of ultra weak compatibility is obtained by replacing 'and' with 'or', that is, x and y are called **ultra weakly compatible**, in symbols $x \overset{\mathrm{uw}}{\longleftrightarrow} y$, if $(x \vee y) \setminus y = x \setminus (x \wedge y)$ or $(x \vee y) \setminus x = y \setminus (x \wedge y)$, or equivalently, if $y / (x \vee y) = (x \wedge y) / x$ or $x / (x \vee y) = (x \wedge y) / y$.

We can easily describe pairs of compatible elements in the setting of double basic algebras associated to lattice pseudo-effect algebras:

Theorem 8 *Let $\mathcal{E}$ be a lattice pseudo-effect algebra and $\mathcal{E}^D$ the corresponding double basic algebra. Then, for all $x, y \in E$, the following are equivalent:*

(i) $x \leftrightarrow y$ *in* $\mathcal{E}$,
(ii) $x^{\sim} \oplus y = y \boxplus x^{\sim}$ *and* $y^{\sim} \oplus x = x \boxplus y^{\sim}$ *in* $\mathcal{E}^D$,
(iii) $x^{-} \boxplus y = y \oplus x^{-}$ *and* $y^{-} \boxplus x = x \oplus y^{-}$ *in* $\mathcal{E}^D$.

Consequently, $x \leftrightarrow y$ *iff* $x^{-} \leftrightarrow y^{-}$ *iff* $x^{\sim} \leftrightarrow y^{\sim}$.

Proof Recalling Proposition 5 and Eq. 16 we have

$$(x^{\sim} \oplus y)^{-} = (y + (x^{\sim} \wedge y^{\sim}))^{-} = (y + (x \vee y)^{\sim})^{-} = (x \vee y) \setminus y$$

and

$$(y \boxplus x^{\sim})^{-} = ((y \wedge x^{\sim -}) + x^{\sim})^{-} = ((x \wedge y) + x^{\sim})^{-} = x \setminus (x \wedge y),$$

whence $x \leftrightarrow y$ iff $x^{\sim} \oplus y = y \boxplus x^{\sim}$ and $y^{\sim} \oplus x = x \boxplus y^{\sim}$, so (i) is equivalent to (ii). That (i) and (iii) are equivalent is verified by observing that $(x^{-} \boxplus y)^{\sim} = y / (x \vee y)$ and $(y \oplus x^{-})^{\sim} = (x \wedge y) / x$.

The last assertion is a direct corollary. (This is also proved in [11], Proposition 3.6.) □

Remark It is obvious that an analogous statement with ‘or’ in place of ‘and’ holds true for ultra weak compatibility.

We have to make a remark on pseudo-MV-algebras here. It is easy to show that if we are given a pseudo-MV-algebra, then by restricting the addition to $\{(x, y) : x \leq y^{-}\}$ we obtain a lattice pseudo-effect algebra. On the other hand, owing to [10], Theorem 8.7, (ultra weak) compatibility can characterize pseudo-MV-algebras within lattice pseudo-effect algebras. Namely, Theorem 8.7 in [10] essentially states that if $\mathcal{E} = (E, +, 0, 1)$ is a lattice pseudo-effect algebra, then—with the notation of our Theorem 4—the algebra $(\mathcal{E}^D)_2 = (E, \boxplus, ^{-}, ^{\sim}, 0, 1)$ is a pseudo-MV-algebra if and only if $x \overset{\text{uw}}{\longleftrightarrow} y$ for all $x, y \in E$. Thus pseudo-MV-algebras are equivalent to lattice pseudo-effect algebras where $x \overset{\text{uw}}{\longleftrightarrow} y$ for all x, y.

Corollary 9 *Let $\mathcal{A}$ be a double basic algebra satisfying Eq. 15. Then $\mathcal{A}_2$ is a pseudo-MV-algebra if and only if $\mathcal{A}$ satisfies the identity*

$$x \boxplus y = y \oplus x. \tag{17}$$

Proof By Theorem 4 we know that if $\mathcal{A}_2$ is a pseudo-MV-algebra, then $\mathcal{A}$ satisfies the identity 13, and letting $z = 0$ we get Eq. 17.

Conversely, if $\mathcal{A}$ fulfills Eq. 17, then Theorem 8 entails that in the pseudo-effect algebra $\mathcal{A}^E$ we have $x \leftrightarrow y$ for all $x, y \in A$. Hence, by [10], Theorem 8.7, $\mathcal{A}_2$ is a pseudo-MV-algebra. □

Next, let us assume that $\mathcal{E} = (E, +, 0, 1)$ is a lattice pseudo-effect algebra satisfying the following condition which is referred to as **complement compatibility property** in [11]:

$$(\forall x, y \in E) \quad x \leftrightarrow y \quad \Rightarrow \quad x^{-} \leftrightarrow y. \tag{18}$$

In this case, if $x \leftrightarrow y$, then $x^{\sim} \leftrightarrow y^{\sim}$ whence $x^{\sim} \leftrightarrow y^{\sim -} = y$ by Eq. 18, and the condition actually means

$$x \leftrightarrow y \quad \text{iff} \quad x^{-} \leftrightarrow y \quad \text{iff} \quad x^{\sim} \leftrightarrow y.$$

Moreover, by [11], Proposition 4.8, this complement compatibility property entails that ultra weak compatibility coincides with compatibility, i.e., for all $x, y \in E$, $x \leftrightarrow y$ iff $x \overset{\text{uw}}{\longleftrightarrow} y$.

Theorem 10 *Let $\mathcal{E}$ and $\mathcal{E}^D$ be as before. If $\mathcal{E}$ satisfies the condition 18, then $x \leftrightarrow y$ in $\mathcal{E}$ if and only if $x \boxplus y = y \oplus x$ in $\mathcal{E}^D$.*

Proof Using Eq. 18 and (iii) of the previous theorem, if $x \leftrightarrow y$, then $x^{\sim} \leftrightarrow y$ and $x \boxplus y = y \oplus x$. Conversely, if $x \boxplus y = y \oplus x$, then $x^{\sim} \overset{\text{uw}}{\longleftrightarrow} y$, which yields $x^{\sim} \leftrightarrow y$, and so $x \leftrightarrow y$ by Eq. 18. □

By [11], a **block** in a lattice pseudo-effect algebra is a maximal subset of mutually compatible elements. Theorem 4.9 in [11] says that a lattice pseudo-effect algebra satisfying Eq. 18 is the union of its blocks, which are pseudo-MV-algebras where the total addition is defined as our $\boxplus$, i.e., $x \boxplus y = (x \wedge y^{-}) + y$.

We can use this result in characterizing double basic algebras derived from lattice pseudo-effect algebras with Eq. 18. First, in order to find a suitable definition of a block for double basic algebras, we need a better understanding of what compatibility in pseudo-effect algebras means in terms of sectional antiautomorphisms.

By Eq. 16 we have $y \,/\, (x \vee y) = ((x \vee y)^{-} + y)^{\sim} = (\beta_y(x \vee y))^{\sim}$ and $(x \wedge y) \,/\, x = (x^{-} + (x \wedge y))^{\sim} = (\beta_{x \wedge y}(x))^{\sim}$, and analogously, $(x \vee y) \setminus y = (\beta_y^{-1}(x \vee y))^{-}$ and $x \setminus (x \wedge y) = (\beta_{x \wedge y}^{-1}(x))^{-}$. Thus, in a lattice pseudo-effect algebra, we have

$$\begin{aligned} x \leftrightarrow y \quad &\text{iff} \quad \beta_y(x \vee y) = \beta_{x \wedge y}(x) \quad \text{and} \quad \beta_x(x \vee y) = \beta_{x \wedge y}(y) \\ &\text{iff} \quad \beta_y^{-1}(x \vee y) = \beta_{x \wedge y}^{-1}(x) \quad \text{and} \quad \beta_x^{-1}(x \vee y) = \beta_{x \wedge y}^{-1}(y). \end{aligned}$$

Therefore, in double basic algebras, we shall write

$$x \leftrightarrow y \quad \text{iff} \quad \beta_y(x \vee y) = \beta_{x \wedge y}(x) \quad \text{and} \quad \beta_x(x \vee y) = \beta_{x \wedge y}(y),$$

and

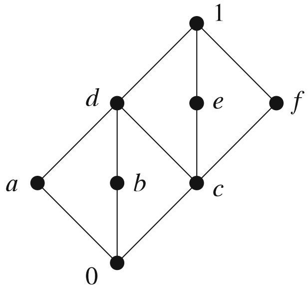
$$x \leftrightsquigarrow y \quad \text{iff} \quad \beta_y^{-1}(x \vee y) = \beta_{x \wedge y}^{-1}(x) \quad \text{and} \quad \beta_x^{-1}(x \vee y) = \beta_{x \wedge y}^{-1}(y).$$

Using the total operations $\boxplus$ and $\oplus$, we have

$$\begin{aligned} x \leftrightarrow y \quad &\text{iff} \quad x^{-} \boxplus y = x^{-} \boxplus (x \wedge y) \quad \text{and} \quad y^{-} \boxplus x = y^{-} \boxplus (x \wedge y), \\ x \leftrightsquigarrow y \quad &\text{iff} \quad x^{\sim} \oplus y = x^{\sim} \oplus (x \wedge y) \quad \text{and} \quad y^{\sim} \oplus x = y^{\sim} \oplus (x \wedge y). \end{aligned}$$

By a **left block** [respectively, a **right block**] in a double basic algebra we shall mean a maximal subset such that $x \leftrightarrow y$ [respectively, $x \leftrightsquigarrow y$] for all x, y in the subset.

The relations $\leftrightarrow$ and $\leftrightsquigarrow$, and hence the left and right blocks, are distinct in general:



Example 11 Let $\mathcal{A}$ be the double basic algebra whose underlying lattice is shown in the above image, where the antiautomorphisms β_0 and β_c are given as follows:

x	0	a	b	c	d	e	f	1
$\beta_0(x)$	1	f	e	d	c	b	a	0

x	c	d	e	f	1
$\beta_c(x)$	1	e	f	d	c

The other sections are finite chains and hence admit unique antitone involutions.

We have $\beta_b(b \vee c) = \beta_b(d) = d = \beta_0(c) = \beta_{b \wedge c}(c)$ and $\beta_c(b \vee c) = \beta_c(d) = e = \beta_0(b) = \beta_{b \wedge c}(b)$, thus $b \leftrightarrow c$, while $b \not\leftrightsquigarrow c$ because $\beta_c^{-1}(b \vee c) = \beta_c^{-1}(d) = f$ and $\beta_{b \wedge c}^{-1}(b) = \beta_0^{-1}(b) = e$.

At the same time, this example shows that other seemingly natural definitions of blocks need not work. For instance, we cannot define $x \leftrightarrow y$ by $x \boxplus y = y \oplus$

x since it might happen that $x \boxplus x \neq x \oplus x$ and $x \boxplus y = y \oplus x$, but $y \boxplus x \neq x \oplus y$. Indeed, $c \boxplus c = \beta_c(c^\sim \vee c) = \beta_c(d \vee c) = \beta_c(d) = e$, while $c \oplus c = \beta_c^{-1}(c^- \vee d) = \beta^{-1}(d \vee c) = \beta_c^{-1}(d) = f$, and $a \boxplus c = \beta_c(a^\sim \vee c) = \beta_c(f) = d = \beta_a^{-1}(d) = \beta_a^{-1}(c^- \vee a) = c \oplus a$, but $a \oplus c = \beta_c^{-1}(a^- \vee c) = \beta_c^{-1}(f) = e$ and $c \boxplus a = \beta_a(c^\sim \vee a) = \beta_a(d) = d$.

Theorem 12 *Let* $\mathcal{A} = (A, \boxplus, \oplus, ^-, ^\sim, 0, 1)$ *be a double basic algebra and* $\mathcal{A}^E = (A, +, 0, 1)$ *the partial algebra as in Theorem 7. The following statements are equivalent:*

(i) *$\mathcal{A}^E$ is a lattice pseudo-effect algebra satisfying Eq. 18.*
(ii) *Every left block of $\mathcal{A}$ is (the carrier of) a sub-pseudo-MV-algebra of $\mathcal{A}$.*
(iii) *Every right block of $\mathcal{A}$ is (the carrier of) a sub-pseudo-MV-algebra of $\mathcal{A}$.*

Proof

(i) ⇒ (ii)/(iii) In the light of Theorem 10 and the above discussion, the blocks of $\mathcal{A}^E$ are precisely the left/right blocks of $\mathcal{A}$. As we have already mentioned, by [11], Theorem 4.9, A is the union of the blocks, which are pseudo-MV-algebras. More precisely, every block B is a sublattice and a subalgebra of $\mathcal{A}^E$, in the sense that $x^-, x^\sim \in B$ for all $x \in B$, and if $x, y \in B$ and $x + y$ is defined, then $x + y \in B$. Therefore, if we equip B with the operations $\boxplus$ and $\oplus$ as in Proposition 5, we obtain a sub-pseudo-MV-algebra of $\mathcal{A}$.

(ii)/(iii) ⇒ (i) We first observe that if $x, y \in A$ are comparable, then $x \leftrightarrow y$ as well as $x \leftrightsquigarrow y$. In order to show that $\mathcal{A}$ satisfies Eq. 15, let $x \leq y^-$ and $x \boxplus y \leq z^-$. Then $x \leftrightarrow y^-$ and so x, y^- belong to some left block B. Since blocks are sub-pseudo-MV-algebras, also $y \in B$ and $x \boxplus y \in B$. Further, $x \leftrightarrow z^-$ and $y \leftrightarrow z^-$ because both x and y are less than or equal to $x \boxplus y = y \oplus x$. Hence $z^- \in B$, which yields $z \in B$. Now, since $x, y, z \in B$ and since blocks are sub-pseudo-MV-algebras, it follows that $x \boxplus (z \oplus y) = x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z = z \oplus (x \boxplus y)$. □

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