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FOUNDATIONS

On some properties of directoids

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Abstract We study some properties of directoids and their expansions by additional signature, including bounded involutive directoids and complemented directoids. Among other results, we provide a shorter proof of the direct decomposition theorem for bounded involutive directoids given in Chajda and Länger (Directoids. An algebraic approach to ordered sets. Heldermann Verlag, Lemgo 2011); we present a description of central elements of complemented directoids; we show that the variety of directoids, as well as its expansions mentioned above, all have the strong amalgamation property.

Keywords Directoid · Partially ordered set · Antitone involution · Church algebra

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1 Introduction

It is superfluous to recall how important partially ordered sets, and in particular directed posets, are for the whole of mathematics. However, unlike other equally fundamental mathematical structures, such as groups or Boolean algebras, posets and directed posets are *relational structures*, not algebras, whence they do not lend themselves to be the objects of common algebraic constructions like quotients, products, subalgebras and the like. In fact, insofar as they exist at all for relational structures, these constructions admit of several competing variants, none of which enjoys a universal acclaim, and are generally recognised as more cumbersome and less efficient than in the algebraic case. In order to enable such algebraic constructions with ordered sets, (Ježek and Quackenbush 1990)—and, independently, (Kopytov and Dimitrov 1988) and (Gardner and Parmenter 1995)—introduced the notion of *directoid*. To every directed poset $\mathbf{A} = \langle A, \leq \rangle$ a groupoid $\mathcal{D}(\mathbf{A}) = \langle A, \sqcup \rangle$ can be associated in such a way that for all $a, b \in A$, $a \leq b$ if and only if $a \sqcup b = b \sqcup a = b$. In the terminology of Ježek and Quackenbush (1990), this groupoid is called a *commutative directoid*. Directoids were investigated in detail by several authors; for a survey, see Chajda and Länger (2011).

Here, we study some properties of directoids and some of their expansions by additional signature. The paper is structured as follows. In Sect. 2, after recapping some preliminary notions, we investigate involutive directoids, that correspond to directed posets with an antitone involution, and some of their notable subclasses, including complemented directoids. In Sect. 3, we focus on some classes of directoids where the binary operation $\sqcup$ has "join-like" properties. In Sect. 4, we improve on the direct decomposition theorem for bounded involutive directoids given in Chajda and Länger (2011), by providing a shorter proof; moreover, we present a compact description of central elements of complemented directoids. Finally, in Sect. 5, we show that the variety of directoids, as well as its expansions mentioned above, all have the strong amalgamation property.

2 Involutive directoids

Recall that a partially ordered set (poset) $\mathbf{A} = \langle A, \leq \rangle$ is said to be *directed* in case any two $a, b \in A$ have a common upper bound, i.e. in case the upper corner $U(a, b) = \{c \in A : a, b \leq c\}$ is nonempty. Of course, if $\mathbf{A}$ has a greatest element 1, then it is directed. An *antitone involution* on a poset $\mathbf{A} = \langle A, \leq \rangle$ is a unary operation $'$ s.t., for any $a \in A$, $(a')' = a$, and if $a \leq b$ in $\mathbf{A}$, then $b' \leq a'$. $(a')'$ will be shortened to a'' hereafter. It is evident that, whenever a poset with antitone involution $\mathbf{D}$ has a greatest element 1, then it contains a smallest element too, namely, $1'$. In place of $1'$, we denote such an element by 0. Furthermore, observe that if $a \vee b$ exists in D, then the infimum $a' \wedge b' = (a \vee b)'$ also exists in D.

A *directoid* (*commutative directoid*, in the usage of Ježek and Quackenbush) is a groupoid $\mathbf{D} = \langle D, \sqcup \rangle$ that satisfies the following axioms:

(D1) $x \sqcup x \approx x$;
(D2) $x \sqcup y \approx y \sqcup x$;
(D3) $x \sqcup ((x \sqcup y) \sqcup z) \approx (x \sqcup y) \sqcup z$.

If $\mathbf{D} = \langle D, \sqcup \rangle$ is a directoid, the partial order relation $\leq$ defined for all $a, b \in D$ by

$a \leq b$ iff $a \sqcup b = b$

will be called *the order induced by* $\sqcup$ on $\mathbf{D}$, or its *induced order*, while the poset $\langle D, \leq \rangle$ will be called the *induced poset* of $\mathbf{D}$.

Any directed poset $\mathbf{A} = (A, \leq)$ can be turned into a directoid as follows:

– if $a \leq b$, then we set $a \sqcup b = b \sqcup a = b$;
– if a and b are incomparable (denoted by $a \parallel b$), then $a \sqcup b = b \sqcup a$ is an arbitrary common upper bound of a, b.

The resulting directoid $\mathcal{D}(\mathbf{A}) = \langle A, \sqcup \rangle$ is such that its induced order coincides with the partial ordering of $\mathbf{A}$. In other words, the directoid fully retrieves the ordering of the original poset. However, it may happen that two incomparable elements $a, b \in A$ have a supremum $a \vee b$ that does not coincide with our choice of $a \sqcup b$. And this is a shortcoming under several respects. It is therefore our aim to prove that, for directed posets $\mathbf{A} = \langle A, \leq \rangle$ that admit an antitone involution, we can get around this difficulty.

An *involutive directoid* is an algebra $\mathbf{D} = \langle D, \sqcup, ' \rangle$ of type (2, 1) s.t. $\langle D, \sqcup \rangle$ is a directoid and $'$ is an antitone involution on the induced poset of $\mathbf{D}$. Observe that:

Proposition 1 *The class of involutive directoids is a variety.*

Proof We only have to prove that the quasi-identity $x \leq y \Rightarrow y' \leq x'$ can be expressed equationally. Indeed, it can be expressed by the single equation $x' \sqcup (x \sqcup y)' \approx x'$. If we assume the quasi-identity, then since $a \leq a \sqcup b$, we have $(a \sqcup b)' \leq a'$, and therefore $a' \sqcup (a \sqcup b)' = a'$. For the other implication, if the equation is valid, and $a \leq b$, then $a \sqcup b = b$, and therefore $a' \sqcup b' = a' \sqcup (a \sqcup b)' = a'$. That is, $b' \leq a'$, as required. □

Recall from Chajda and Länger (2011) that two elements a, b of a directoid $\mathbf{D}$ are said to be *orthogonal* in case $a \leq b'$, or equivalently $b \leq a'$.

Theorem 1 *Let $\mathbf{D} = \langle D, \sqcup, ' \rangle$ be an involutive directoid, and let $\leq$ be its induced order. The following conditions are equivalent:*

(1) *for all $a, b \in D$, if a, b are orthogonal, then $a \sqcup b = a \vee b$;*
(2) $\mathbf{D}$ *satisfies the identity*

$$\text{(D4)} \quad (((x \sqcup z) \sqcup (y \sqcup z)')' \sqcup (y \sqcup z)') \sqcup z' \approx z'.$$

Proof First, notice that item (1) is equivalent to

(A) if $a \leq b'$ and $a, b \leq c$ then $a \sqcup b \leq c$.

For, if $a \leq b'$ and $a, b \leq c$, then $a \sqcup b = a \vee b \leq c$. The converse is obvious, since, by axiom (D3), $a, b \leq a \sqcup b$. Moreover, the identity (D4) is clearly equivalent to

(B)$((x \sqcup z) \sqcup (y \sqcup z)')' \sqcup (y \sqcup z)' \leq z'$,

by the definition of induced order.

Hence, to obtain our claim, it suffices to show the equivalence of (A) and (B). Assume (A). Set $a = ((x \sqcup z) \sqcup (y \sqcup z)')'$ and $b = (y \sqcup z)'$. Clearly, $(x \sqcup z) \sqcup (y \sqcup z)' \geq (y \sqcup z)'$, i.e. $b \leq a'$. Hence, $a \leq b'$. Also, $b \leq z'$. Moreover, $a' \geq x \sqcup z \geq z$. Therefore, $a \leq z'$. Thus, by (A) $((x \sqcup z) \sqcup (y \sqcup z)')' \sqcup (y \sqcup z)' \leq z'$, which is (B). Conversely, assume (B). Let $x \leq y'$ and $x, y \leq z$. Then $y \leq x'$ and $x', y' \geq z'$, i.e. $y \sqcup x' = x'$, $x' \sqcup z' = x'$ and $y' \sqcup z' = y'$. So we obtain:

$$\begin{aligned} x \sqcup y = x'' \sqcup y &= (x' \sqcup y)' \sqcup y = ((x' \sqcup z') \sqcup y)' \sqcup y \\ &= ((x' \sqcup z') \sqcup y'')' \sqcup y = ((x' \sqcup z') \sqcup (y' \sqcup z')')' \sqcup y \\ &= ((x' \sqcup z') \sqcup (y' \sqcup z')')' \sqcup y'' \\ &= ((x' \sqcup z') \sqcup (y' \sqcup z')')' \sqcup (y' \sqcup z')' \\ &\quad \text{(by (B))} \leq z. \end{aligned}$$

□

The previous correspondence assumes a particularly interesting form when the poset in question is bounded, and the type includes two constants denoting the bounds. A case in point is given by *effect algebras*, which play a noteworthy role in quantum logic (see e.g. Dalla Chiara et al. 2004; Dvurečenskij and Pulmannová 2000)—in fact, they can be presented as bounded posets equipped with an antitone involution, such that the supremum $a \vee b$ exists for orthogonal elements a, b. We have that:

Corollary 1 *Let* $\mathbf{A} = \langle A, \leq, ', 0, 1\rangle$ *be a bounded poset with antitone involution. The following conditions are equivalent:*

(1) *For* $a, b \in A$, $a \vee b$ *exists whenever* a, b *are orthogonal.*
(2) $\mathcal{D}(\mathbf{A}) = \langle A, \sqcup, ', 0, 1\rangle$ *satisfies (D2)–(D4) and*

(D5) $x \sqcup 0 \approx x$

Proof (1 $\Rightarrow$ 2) Since **A** is bounded, it follows that **A** is directed. Then, by Theorem 1, **A** satisfies (D2)–(D4).

(2 $\Rightarrow$ 1) First, let us observe that $\mathcal{D}(\mathbf{A})$ is a directoid, since, putting $y = z = 0$ in (D3), we get for any $a \in A$,

$$\begin{aligned} a \sqcup a &= a \sqcup (a \sqcup 0) = a \sqcup ((a \sqcup 0) \sqcup 0) \\ &= (a \sqcup 0) \sqcup 0 = a \sqcup 0 = a. \end{aligned}$$

Our claim, then, follows from Theorem 1. □

Corollary 1 entails that bounded involutive directoids, such that $a \vee b$ exists for orthogonal elements a, b, are completely characterised by the equations (D2)–(D5), and therefore form a variety of type (2, 1, 0, 0).

Given an involutive directoid $\mathbf{D} = \langle D, \sqcup, '\rangle$, we define

$$x \sqcap y := (x' \sqcup y')'.$$

It is not difficult to verify (see e.g. Chajda and Länger 2011, Remark 7.4) that $\langle A, \sqcap\rangle$ is again a directoid whose induced order is dual to the induced order of **D**. Moreover, the absorption laws

$$x \sqcap (x \sqcup y) \approx x \quad \text{and} \quad x \sqcup (x \sqcap y) \approx x,$$

are satisfied. In fact,

$$\begin{aligned} x \leq y &\Leftrightarrow y' \leq x' \Leftrightarrow x' \sqcup y' = x' \Leftrightarrow (x' \sqcup y')' \\ &= x'' \Leftrightarrow x \sqcap y = x. \end{aligned}$$

Therefore, since $x \leq x \sqcup y$, we have $x \sqcap (x \sqcup y) = x$. And since $x \sqcap y \leq x$, we also have $x \sqcup (x \sqcap y) = x$. Thus, we obtain the following theorem (cf. Chajda and Länger 2011, Theorem 7.8).

Theorem 2 *Any variety of involutive directoids is congruence distributive, with* $\frac{2}{3}$ *majority term*

$$M(x, y, z) := ((x \sqcap y) \sqcup (y \sqcap z)) \sqcup (x \sqcap z).$$

Proof We only prove that $M(x, x, z) = x$, the other conditions being just slight modifications thereof.

$$\begin{aligned} M(x, x, z) &= ((x \sqcap x) \sqcup (x \sqcap z)) \sqcup (x \sqcap z) \\ &= (x \sqcup (x \sqcap z)) \sqcup (x \sqcap z) \\ &= x \sqcup (x \sqcap z) \\ &= x. \end{aligned}$$

□

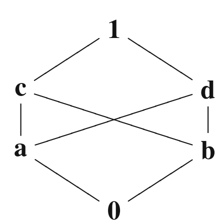
In the absence of the involution, Theorem 2 fails, because semilattices (a subvariety of directoids) satisfy no nontrivial lattice identity (Freese and Nation 1973, Theorem 2).

Let us call a bounded involutive directoid *complemented* in case it satisfies the equation $x \sqcup x' \approx 1$. If this directoid satisfies the equivalent conditions in Theorem 1, we get that $x \sqcup x' = x \vee x'$, because $x \leq x = x''$, i.e. x and x' are orthogonal. Now, all the aforementioned properties are captured by means of identities. That is, the class of complemented directoids satisfying Theorem 1 forms a variety that includes, for example, orthomodular lattices.

3 Saturated directoids

We have seen in the previous section that there are directoids where $a \sqcup b = a \vee b$, at least for orthogonal or comparable elements a, b. In this section, we show that the classes of directoids where $x \sqcup y$ is minimal in the upper corner $U(x, y)$, or where $x \sqcup y = x \vee y$ in case $x \vee y$ exists, have a special significance. To this aim we introduce the following notions. A directoid $\mathbf{D} = \langle D, \sqcup\rangle$ is called *saturated* if $x \sqcup y$ is minimal in $U(x, y)$. **D** is *supremal* if $x \sqcup y = x \vee y$ in case $x \vee y$ exists.

Example 1 Consider the following ordered set:



If we set $a \sqcup b = c$ or $a \sqcup b = d$, and for $\{x, y\} \neq \{a, b\}$ we take $x \sqcup y = x \vee y$, then it is a saturated directoid. However, upon setting $a \sqcup b = 1$, on the same ordered set, the resulting directoid is no longer saturated, since 1 is not minimal in $U(a, b)$, even though it is still trivially supremal, because $x \vee y$ does not exist.

Note that every saturated directoid is supremal. In fact, if $x \vee y$ exists, then it is minimal in $U(x, y)$, whence $x \sqcup y = x \vee y$. The previous example shows that the converse is not true.

Theorem 3 *A directoid* $\mathbf{D} = \langle D, \sqcup\rangle$ *is saturated if and only if it satisfies the quasi-identity:*

(Q) $(x \sqcup z \approx z \approx y \sqcup z)$ & $(z \sqcup (x \sqcup y) \approx x \sqcup y)$
$\Rightarrow z \approx x \sqcup y$.

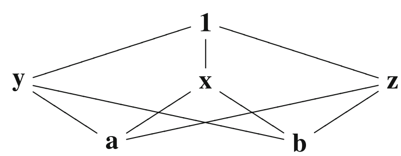
Proof Assume **D** satisfies the quasiequation and $x, y \leq z \leq x \sqcup y$. Then, $x \sqcup z = z = y \sqcup z$ and $z \sqcup (x \sqcup y) = x \sqcup y$. Hence $z = x \sqcup y$. Therefore, $x \sqcup y$ is minimal in $U(x, y)$, i.e. **D** is saturated. Conversely, if **D** is saturated, and $x \sqcup z = z = y \sqcup z$ and $z \sqcup (x \sqcup y) = x \sqcup y$ hold, then $x, y \leq z \leq x \sqcup y$. Since $z \in U(x, y)$ and $x \sqcup y$ is minimal in $U(x, y)$, then $x \sqcup y = z$.

Observe that the quasi-identity (Q) is in fact equivalent to the condition

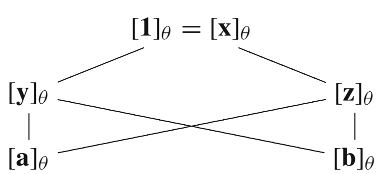
$$x, y \leq z \leq x \sqcup y \Rightarrow z \approx x \sqcup y.$$

By Theorem 3, the class of saturated directoids is a quasivariety. The next example shows that it is not a variety, because it is not closed under quotients.

Example 2 Let **D** be the directoid given by the following diagram:



where $a \sqcup b = x$, and $p \sqcup q = p \vee q$ for the remaining elements. **D** is a saturated directoid. Consider the congruence $\theta(x, 1)$. Then, we obtain the quotient



where $[a]_\theta \sqcup [b]_\theta$ is not minimal in $U([a]_\theta, [b]_\theta)$.

Note that the variety of join semilattices is a nontrivial class strictly included in the quasivariety of saturated directoids. For involutive directoids, we can provide a sufficient condition for saturation formulated in the form of an identity.

Theorem 4 *Let* $\mathbf{D} = \langle D, \sqcup, ' \rangle$ *be an involutive directoid. If* **D** *satisfies*

(D6) $((x \sqcap ((x \sqcup y) \sqcap z)) \sqcup (y \sqcap ((x \sqcup y) \sqcap z))) \sqcup z \approx z$

then **D** *is saturated.*

Proof Suppose that for $a, b, c \in D$ $a \sqcup c = c = b \sqcup c$ and $c \sqcup (a \sqcup b) = a \sqcup b$. Then

$$c = ((a \sqcap ((a \sqcup b) \sqcap c)) \sqcup (b \sqcap ((a \sqcup b) \sqcap c))) \sqcup c = (a \sqcup b) \sqcup c = a \sqcup b,$$

whence we get our conclusion.

Observe that the variety of involutive directoids satisfying (D6) contains all the involutive lattices. We can also characterise the quasivariety of supremal directoids.

Theorem 5 *A directoid is supremal if and only if it satisfies the quasiequation*

$$x, y \leq w \quad \& \quad w \leq x \sqcup y \quad \& \quad x, y, w \leq z \Rightarrow w \approx x \sqcup y.$$

Proof If a directoid **D** satisfies the antecedent of the quasiequation, the w is the smallest element in $U(x, y)$. Therefore, if it is supremal then $w = x \sqcup y = x \vee y$. And if the quasiequation itself is satisfied, then it is clear that **D** is supremal.

Let us note that the quasi-identity of Theorem 5 can be easily expressed as a quasi-identity in the language of directoids.

4 Decomposition of bounded involutive directoids

In Chajda and Länger (2011, Theorem 7.28) the standard direct decomposition theorem for orthomodular lattices (see e.g. Bruns and Harding 2000, Theorem 2.7) is generalised to the effect that an appropriate version of it is shown to hold for bounded involutive directoids. Contextually, a characterisation of central elements of bounded involutive directoids is provided. The aim of this section is giving an alternative proof of this result, as well as a simplified description of central elements in case the directoid is complemented. To this aim, we put to good use the tools developed in the theory of *Church algebras* Salibra et al. (2013).

The key observation motivating the introduction of Church algebras is that many algebras arising in completely different fields of mathematics—including Heyting algebras, rings with unit, or combinatory algebras—have a term operation q satisfying the fundamental properties of the if-then-else connective: $q(1, x, y) \approx x$ and $q(0, x, y) \approx y$. As simple as they may appear, these properties are enough to yield rather strong results. This motivates the next definitions.

An algebra **A** of type ν is a *Church algebra* if there are term-definable elements $0^{\mathbf{A}}, 1^{\mathbf{A}} \in A$ and a term operation $q^{\mathbf{A}}$ s.t., for all $a, b \in A$, $q^{\mathbf{A}}(1^{\mathbf{A}}, a, b) = a$ and $q^{\mathbf{A}}(0^{\mathbf{A}}, a, b) = b$. A variety $\mathcal{V}$ of type ν is a Church variety if every member of $\mathcal{V}$ is a Church algebra with respect to the same term $q(x, y, z)$ and the same constants 0, 1.

Expanding on an idea due to Vaggione (1996), we also say that an element e of a Church algebra **A** is *central* if the pair $(\theta(e, 0), \theta(e, 1))$ is a pair of complementary factor congruences on **A**. A central element e is nontrivial if $e \notin \{0, 1\}$. By $\mathrm{Ce}(\mathbf{A})$ we denote the centre of **A**, i.e. the set of central elements of the algebra **A**.

By defining

$$x \wedge y = q(x, y, 0), x \vee y = q(x, 1, y) \text{ and } x^* = q(x, 0, 1),$$

we get:

Theorem 6 [Salibra et al. (2013)] *Let* $\mathbf{A}$ *be a Church algebra. Then* $\mathrm{Ce}(\mathbf{A}) = \langle \mathrm{Ce}(A), \wedge, \vee, ^*, 0, 1\rangle$ *is a Boolean algebra which is isomorphic to the Boolean algebra of factor congruences of* $\mathbf{A}$.

Hereafter, it will be clear from the context when the symbols $\wedge, \vee$ will denote the previously defined operations on Church algebras instead of defining lattice meet and join, respectively.

If $\mathbf{A}$ is a Church algebra of type ν and $e \in A$ is a central element, then we define $\mathbf{A}_e = (A_e; g_e)_{g\in\nu}$ to be the ν-algebra defined as follows:

$$A_e = \{e \wedge b : b \in A\}; \quad g_e(e \wedge \overline{b}) = e \wedge g(e \wedge \overline{b}).$$

By Ledda et al. (2013, Theorem 4), we have that:

Theorem 7 *Let* $\mathbf{A}$ *be a Church algebra of type* ν *and* e *be a central element. Then we have:*

(1) *For every* n*-ary* $g \in \nu$ *and every sequence of elements* $\overline{b} \in A^n$, $e \wedge g(\overline{b}) = e \wedge g(e \wedge \overline{b})$, *so that the function* $h : A \to A_e$, *defined by* $h(b) = e \wedge b$, *is a homomorphism from* $\mathbf{A}$ *onto* $\mathbf{A}_e$.
(2) $\mathbf{A}_e$ *is isomorphic to* $\mathbf{A}/\theta(e, 1)$. *It follows that* $\mathbf{A} = \mathbf{A}_e \times \mathbf{A}_{e'}$ *for every central element* e*, as in the Boolean case.*

The if-then-else term that makes orthomodular lattices into Church algebras works, more generally, for bounded involutive directoids:

Proposition 2 *Bounded involutive directoids form a Church variety, as witnessed by the term* $q(x, y, z) = (x \sqcup z) \sqcap (x' \sqcup y)$.

Proof If $\mathbf{A}$ is a bounded involutive directoid, and $a, b \in A$, then $q^{\mathbf{A}}(1, a, b) = (1 \sqcup b) \sqcap (1' \sqcup a) = 1 \sqcap (0 \sqcup a) = 1 \sqcap a = a$. Also, $q^{\mathbf{A}}(0, a, b) = (0 \sqcup b) \sqcap (0' \sqcup a) = b \sqcap (1 \sqcup a) = b \sqcap 1 = b$.

In Chajda and Länger (2011, Chapter 7), central elements (in Vaggione's sense) of a bounded involutive directoid $\mathbf{D}$ are described as the members of $C(\mathbf{D}) \cap \mathrm{Is}(\mathbf{D})$, namely, those elements e that satisfy the following conditions for all $a, b \in D$:

$$a = (e \sqcap a) \sqcup (e' \sqcap a)$$
$$(a \sqcap b) \sqcap e = (a \sqcap e) \sqcap (b \sqcap e)$$
$$(a \sqcap b) \sqcap e' = \left(a \sqcap e'\right) \sqcap \left(b \sqcap e'\right)$$
$$(a \sqcup b) \sqcap e = (a \sqcap e) \sqcup (b \sqcap e)$$
$$(a \sqcup b) \sqcap e' = \left(a \sqcap e'\right) \sqcup \left(b \sqcap e'\right)$$

However, according to Salibra et al. (2013, Proposition 3.6), central elements of a Church algebra can also be characterised in a completely general way, as follows.

Proposition 3 *If* $\mathbf{A}$ *is a Church algebra of type* ν *and* $e \in A$, *the following conditions are equivalent:*

(1) *e is central;*
(2) *for all* $a, b, \overline{a}, \overline{b} \in A$:
   (a) $q(e, a, a) = a$,
   (b) $q(e, q(e, a, b), c) = q(e, a, c) = q(e, a, q(e, b, c))$,
   (c) $q(e, f(\overline{a}), f(\overline{b})) = f(q(e, a_1, b_1), \ldots, q(e, a_n, b_n))$, *for every* $f \in \nu$,
   (d) $q(e, 1, 0) = e$.

If $\mathbf{A}$ is a bounded involutive directoid, condition (a) says $a = (e \sqcup a) \sqcap (e' \sqcup a)$, for every $a \in A$, or equivalently $a = (e \sqcap a) \sqcup (e' \sqcap a)$, for every $a \in A$.

The first equality of condition (b) says $(e \sqcup c) \sqcap \left(e' \sqcup ((e \sqcup b) \sqcap (e' \sqcup a))\right) = (e \sqcup c) \sqcap (e' \sqcup a)$, for every $a, b, c \in A$. Taking $c = 1$, it is easy to see that this is equivalent to $e' \sqcup ((e \sqcup b) \sqcap (e' \sqcup a)) = e' \sqcup a$, for every $a, b \in A$. The second equality is analogous, and boils down to $e \sqcup ((e' \sqcup b) \sqcap (e \sqcup c)) = e \sqcup c$, for every $b, c \in A$.

Condition (c) is $q(e, 1, 1) = 1$ and $q(e, 0, 0) = 0$ for the constants. These equalities are trivially satisfied for every element $e \in A$. If f is $'$, then $(e \sqcup b') \sqcap (e' \sqcup a') = ((e \sqcup b) \sqcap (e' \sqcup a))'$, that is to say, $(e \sqcup b') \sqcap (e' \sqcup a') = (e' \sqcap b') \sqcup (e \sqcap a')$, for every $a, b \in A$. But this is equivalent to $(e \sqcup b) \sqcap (e' \sqcup a) = (e' \sqcap b) \sqcup (e \sqcap a)$, for every $a, b \in A$. If f is $\sqcup$, we have that

$$(e \sqcup (b_1 \sqcup b_2)) \sqcap (e' \sqcup (a_1 \sqcup a_2))$$
$$= ((e \sqcup b_1) \sqcap (e' \sqcup a_1)) \sqcup ((e \sqcup b_2) \sqcap (e' \sqcup a_2)).$$

Finally, condition (d) is trivial, since $q(e, 1, 0) = e$ is always true for every element $e \in A$.

We will use one or the other of these two alternative characterisations of central elements, according to convenience.

We now focus for a while on *complemented* directoids, for which we show that the latter set of conditions can be considerably streamlined. For a start, we need to prove the following lemmas.

Lemma 1 *If* $\mathbf{A}$ *is a bounded involutive directoid, then it satisfies:*

(1) $x \sqcup y \approx x \sqcup (x \sqcup y)$,
(2) $x \approx x \sqcup (x \sqcup (x' \sqcup y))'$. *If it is complemented, it also satisfies:*
(3) $(x \sqcup y) \sqcup y' \approx 1$.

Proof (1) This is true, since $x \leqslant x \sqcup y$.
(2) Since $x' \leqslant x' \sqcup y \leqslant x \sqcup (x' \sqcup y)$, we have $(x \sqcup (x' \sqcup y))' \leqslant x$, and therefore $x = x \sqcup (x \sqcup (x' \sqcup y))'$.
(3) Substituting x by $(x \sqcup y) \sqcup y'$ and y by y' in the previous item, we obtain:

$$
\begin{aligned}
(x \sqcup y) \sqcup y' &= ((x \sqcup y) \sqcup y') \sqcup (((x \sqcup y) \sqcup y') \\
&\quad \sqcup (((x \sqcup y) \sqcup y')' \sqcup y'))' \\
&= ((x \sqcup y) \sqcup y') \sqcup (((x \sqcup y) \sqcup y') \sqcup y')' \\
&= ((x \sqcup y) \sqcup y') \sqcup ((x \sqcup y) \sqcup y')' \\
&= 1.
\end{aligned}
$$

Now, consider the equations

(C1) $a = (e \sqcap a) \sqcup (e' \sqcap a)$

(C2) $(e \sqcup (b_1 \sqcup b_2)) \sqcap (e' \sqcup (a_1 \sqcup a_2))$
$= ((e \sqcup b_1) \sqcap (e' \sqcup a_1)) \sqcup ((e \sqcup b_2) \sqcap (e' \sqcup a_2))$

Lemma 2 *If* **A** *is a complemented directoid and $e \in A$ satisfies (C1) and (C2) for every $a, b, a_1, a_2, b_1, b_2 \in A$, then for every $a, b \in A$,*

(1) $e \sqcup (a \sqcup b) = (e \sqcup a) \sqcup (e \sqcup b)$,
(2) $e \sqcap (a \sqcap b) = (e \sqcap a) \sqcap (e \sqcap b)$,
(3) (a) $e \sqcup a = e \sqcup (e' \sqcap a)$,
(b) $e \sqcap a = e \sqcap (e' \sqcup a)$,
(4) $e \sqcup (a \sqcup (e \sqcap b)) = e \sqcup a$,
(5) $e \sqcup (a \sqcup b) = (e \sqcup a) \sqcup b$,
(6) $e \sqcap (a \sqcup b) = (e \sqcap a) \sqcup (e \sqcap b)$,
(7) $(e \sqcup a)' \sqcup b = (e \sqcup (a' \sqcup b)) \sqcap (e' \sqcup b)$,
(8) $(e \sqcup b) \sqcap (e' \sqcup a) = (e' \sqcap b) \sqcup (e \sqcap a)$.

Proof Taking $b_1 = a$, $b_2 = b$, and $a_1 = 1 = a_2$ in (C2), we obtain (1).

For (2), we only have to use the De Morgan laws and the fact that e satisfies (C1)–(C2) if and only e' also satisfies them. In order to prove (3a), observe that:

$$
\begin{aligned}
e \sqcup a &= e \sqcup ((e \sqcap a) \sqcup (e' \sqcap a)) \\
&= (e \sqcup (e \sqcap a)) \sqcup (e \sqcup (e' \sqcap a)) \\
&= e \sqcup (e \sqcup (e' \sqcap a)) = e \sqcup (e' \sqcap a).
\end{aligned}
$$

(3b) is proved dually. (4) is just:

$$
\begin{aligned}
e \sqcup (a \sqcup (e \sqcap b)) &= (e \sqcup a) \sqcup (e \sqcup (e \sqcap b)) \\
&= (e \sqcup a) \sqcup e = e \sqcup a.
\end{aligned}
$$

For (5), we show that

$$
\begin{aligned}
(e \sqcup a) \sqcup b &= e \sqcup (b \sqcup (e \sqcup a)) = (e \sqcup b) \sqcup (e \sqcup (e \sqcup a)) \\
&= (e \sqcup b) \sqcup (e \sqcup a) \\
&= e \sqcup (a \sqcup b).
\end{aligned}
$$

As regards (6), it follows from (C1)–(C2) that for any $a, b \in A$:

$$
\begin{aligned}
e \sqcap (a \sqcup b) &= (e' \sqcap 0) \sqcup (e \sqcap (a \sqcup b)) \\
&= (e \sqcup 0) \sqcap (e' \sqcup (a \sqcup b)) \\
&= (e \sqcup (0 \sqcup 0)) \sqcap (e' \sqcup (a \sqcup b)) \\
&= \big((e \sqcup 0) \sqcap (e' \sqcup a)\big) \sqcup \big((e \sqcup 0) \sqcap (e' \sqcup b)\big) \\
&= (e \sqcap a) \sqcup (e \sqcap b),
\end{aligned}
$$

where the last equality uses (3b).

(7) In fact,

$$
\begin{aligned}
(e \sqcup a)' \sqcup b &= (e \sqcup ((e \sqcup a)' \sqcup b)) \sqcap (e' \sqcup ((e \sqcup a)' \sqcup b)) \\
&= ((e \sqcup (e \sqcup a)') \sqcup b)) \sqcap (e' \sqcup ((e \sqcup a)' \sqcup b)) \\
&= ((e \sqcup a') \sqcup b)) \sqcap (e' \sqcup ((e' \sqcap a') \sqcup b)) \\
&= (e \sqcup (a' \sqcup b)) \sqcap ((e' \sqcup (e' \sqcap a')) \sqcup b) \\
&= (e \sqcup (a' \sqcup b)) \sqcap (e' \sqcup b).
\end{aligned}
$$

(8) We substitute in (7) a by a' and b by $e \sqcap b$ in order to obtain:

$$
(e \sqcup a')' \sqcup (e \sqcap b) = (e \sqcup (a \sqcup (e \sqcap b))) \sqcap (e' \sqcup (e \sqcap b)),
$$

whence, using (3) and (4), we obtain the result. □

We are now ready to obtain our characterisation.

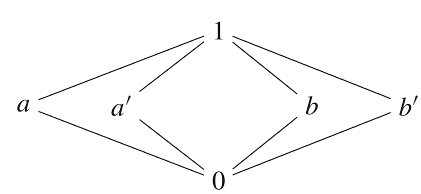
Proposition 4 *An element e of a complemented directoid* **A** *is central if and only if it satisfies (C1) and (C2) for every $a, b, a_1, a_2, b_1, b_2 \in A$.*

Proof In light of the previous lemmas and of the general description of central elements in Church algebras, discussed above, we only have to prove that an element $e \in A$ satisfying (C2) also satisfies $e \sqcup a = e \sqcup ((e \sqcup a) \sqcap (e' \sqcup b))$ and $e' \sqcup a = e' \sqcup ((e' \sqcup a) \sqcap (e \sqcup b))$. Note that if e satisfies (C2) for every $a, b, a_1, a_2, b_1, b_2 \in A$, then the same equation is true replacing e by e'. Therefore it is enough to prove that (C2) implies $e \sqcup a = e \sqcup ((e \sqcup a) \sqcap (e' \sqcup b))$ for every $a, b \in A$. Making $b_1 = a$, $b_2 = e$, $a_1 = b$, $a_2 = e$ in (C2), we have

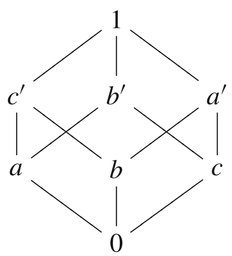
$$
\begin{aligned}
&(e \sqcup (a \sqcup e)) \sqcap (e' \sqcup (b \sqcup e)) \\
&\quad = ((e \sqcup a) \sqcap (e' \sqcup b)) \sqcup ((e \sqcup e) \sqcap (e' \sqcup e)).
\end{aligned}
$$

Using Lemma 1, we have that $e \sqcup (a \sqcup e) = e \sqcup a$ and that $e' \sqcup (b \sqcup e) = 1$, and therefore $(e \sqcup (a \sqcup e)) \sqcap (e' \sqcup (b \sqcup e)) = (e \sqcup a) \sqcap 1 = e \sqcup a$. For the right-hand side of the equation we have $((e \sqcup a) \sqcap (e' \sqcup b)) \sqcup ((e \sqcup e) \sqcap (e' \sqcup e)) = ((e \sqcup a) \sqcap (e' \sqcup b)) \sqcup (e \sqcap 1) = ((e \sqcup a) \sqcap (e' \sqcup b)) \sqcup e$, as we wanted to prove. □

Example 3 Conditions (C1) and (C2) are independent. In fact, in the complemented directoid whose Hasse diagram is hereafter reproduced:



every element satisfies (C2), but only 0 and 1 satisfy (C1). On the other hand, every element of the complemented directoid whose Hasse diagram is hereafter reproduced satisfies (C1), but only 0 and 1 satisfy (C2).



With these results at hand, we can give a more informative version of Theorem 6 above:

Proposition 5 *If* $\mathbf{A}$ *is a complemented directoid, and* $\mathrm{Ce}(A)$ *is the set of the central elements of* $\mathbf{A}$*, then* $\mathrm{Ce}(\mathbf{A}) = \langle \mathrm{Ce}(A), \sqcap, \sqcup, {}', 0, 1\rangle$ *is a Boolean algebra.*

Proof In virtue of Theorem 6, $\langle \mathrm{Ce}(A), \wedge, \vee, {}^*, 0, 1\rangle$ is a Boolean algebra, where $\wedge$, $\vee$ and * are defined as follows:

$$x \wedge y = q(x, y, 0) \quad x \vee y = q(x, 1, y) \quad x^* = q(x, 0, 1).$$

The only thing we need to prove is that $\wedge$, $\vee$, and * coincide with $\sqcup$, $\sqcap$, and $'$, respectively. We note that:

$$\begin{aligned} x \vee y &= q(x, 1, y) = (x \sqcup y) \sqcap (x' \sqcup 1) \\ &= (x \sqcup y) \sqcap 1 = x \sqcup y, \end{aligned}$$

$$x^* = q(x, 0, 1) = (x \sqcup 1) \sqcap (x' \sqcup 0) = 1 \sqcap x' = x'.$$

Therefore, $\vee$ and * coincide with $\sqcup$ and $'$, respectively. And this implies that $\wedge$ and $\sqcap$ must coincide, too, because: $x \wedge y = (x^* \vee y^*)^* = (x' \sqcup y')' = x \sqcap y$. □

It follows from the previous proposition (and, actually, also directly from Proposition 4) that if $\mathbf{A}$ is a complemented directoid and e is a central element, then e' is also central. Notice, also, that in case either x, y fail to be central elements or else they are (possibly) central but $\mathbf{A}$ is a bounded involutive directoid that is not complemented, $x \wedge y$ need not be equal to $x \sqcap y$; however, we still have that

$$x \wedge y = x \sqcap \left(x' \sqcup y\right).$$

Now, if $\mathbf{A}$ is a bounded involutive directoid and e is a central element of $\mathbf{A}$, let

$$\begin{aligned} &[0, e] = \left\langle \{a : a \le e\}, \sqcup, {}^e, 0, e\right\rangle \\ &a^e = e \sqcap a' \end{aligned}$$

In the following theorem, we freely avail ourselves of the characterisation of central elements in bounded involutive directoids given at the beginning of the section.

Theorem 8 *Let* $\mathbf{A}$ *be a bounded involutive directoid and* e *a central element of* $\mathbf{A}$. *Then* $\mathbf{A} \cong [0, e] \times \left[0, e'\right]$.

Proof By Theorem 7.2 and Proposition 2, upon observing that for all $a \le e$ we have that $e \wedge a = e \sqcap a$, all we have to prove is the following:

(1) $A_e = \{a : a \le e\}$
(2) for $a, b \le e$, $a \sqcup b = e \wedge (a \sqcup b)$
(3) for $a \le e$, $a^e = e \wedge a'$

(1) Let $a \sqcup e = e$. Then $a = a \sqcap (a \sqcup e) = a \sqcap e = e \wedge a$, whence $a \in A_e$. Conversely, if $a \in A_e$, then for some b we have that $a = e \sqcap \left(e' \sqcup b\right)$, and so

$$e \sqcup a = e \sqcup \left(e \sqcap \left(e' \sqcup b\right)\right) = e$$

(2)

$$\begin{aligned} e \wedge (a \sqcup b) &= e \sqcap \left(e' \sqcup (a \sqcup b)\right) \\ &= \left(e \sqcap e'\right) \sqcup (e \sqcap (a \sqcup b)) \\ &= e \sqcap (a \sqcup b) \\ &= (e \sqcap a) \sqcup (e \sqcap b) \\ &= a \sqcup b \end{aligned}$$

(3)

$$\begin{aligned} e \wedge a' &= e \sqcap \left(e' \sqcup a'\right) \\ &= e \sqcap a' \end{aligned}$$

□

Proposition 6 *Let* $\mathbf{A}$ *be a bounded involutive directoid,* $e \in \mathrm{Ce}(\mathbf{A})$ *and* $c \in A_e$. *Then,*

$$c \in \mathrm{Ce}(\mathbf{A}) \quad \Leftrightarrow \quad c \in \mathrm{Ce}(\mathbf{A}_e).$$

Moreover, if $c \in \mathrm{Ce}(\mathbf{A}_e)$*, then* $(\mathbf{A}_e)_c = \mathbf{A}_c$.

Proof ($\Rightarrow$) It is an immediate consequence of the fact that $h : \mathbf{A} \to \mathbf{A}_e$ in Theorem 7 is an onto homomorphism such that for every $a \in A_e$, $h(a) = a$, and central elements are characterised by equations, in virtue of Proposition 4. ($\Leftarrow$) Since central elements are characterised by equations, if c_i is a central element of a bounded involutive directoid $\mathbf{A}_i$, for $i = 1, 2$, then $(c_1, c_2) \in \mathrm{Ce}(\mathbf{A}_1 \times \mathbf{A}_2)$. Therefore, if $c \in A_e$, the image of c by the isomorphism of Theorem 7 is $(c, 0)$. The element 0 is always central, and c is central by hypothesis. Hence $(c, 0)$ is central in $\mathbf{A}_e \times \mathbf{A}_{e'}$. But, this implies that $c \in \mathrm{Ce}(\mathbf{A})$, because $\mathbf{A} \cong \mathbf{A}_e \times \mathbf{A}_{e'}$. Finally, if we have $c \in \mathrm{Ce}(\mathbf{A}_e)$, we have already proved that $c \in \mathrm{Ce}(\mathbf{A})$, and the only thing we have to check is the fact that the definition of the involution ^{c} does not depend on whether we are defining it in terms of $'$ or of ^{e}. That is to say, we have to prove that for every $a \leq c$, $a' \sqcap c = a^e \sqcap c$. Indeed, $a^e \sqcap c = (a' \sqcap e) \sqcap c = (a' \sqcap c) \sqcap (e \sqcap c) = (a' \sqcap c) \sqcap c = a' \sqcap c$, where we have used Lemma 2, the fact that $c \leq e$, and the dual of Eq. (1) of Lemma 1. □

As we have seen, $\mathrm{Ce}(\mathbf{A})$ is a Boolean algebra, and we can consider the set of its atoms, which we denote by $At(\mathbf{A})$. Note that an atom of $\mathrm{Ce}(\mathbf{A})$ needs not be an atom of $\mathbf{A}$.

Lemma 3 *If* $\mathbf{A}$ *is a complemented directoid and* e *is an atomic central element of* $\mathbf{A}$*, then* $At(\mathbf{A}_{e'}) = At(\mathbf{A}) - \{e\}$.

Proof ($\supseteq$) Since e is an atom in the Boolean algebra $\mathrm{Ce}(\mathbf{A})$, for any other atomic central element c of $\mathbf{A}$, $e \sqcap c = 0$, and therefore $e' \sqcup c' = 1$. Hence, $c = c \sqcap 1 = c \sqcap (e' \sqcup c') = (c \sqcap e') \sqcup (c \sqcap c') = (c \sqcap e') \sqcup 0 = c \sqcap e'$, which shows that $c \leq e'$. Thus, by Proposition 6, $c \in \mathrm{Ce}(\mathbf{A}_{e'})$. Moreover, if d is a central element of $\mathbf{A}_{e'}$ such that $d < c$, then d is a central element of $\mathbf{A}$, and since we are assuming that c is atomic central of $\mathbf{A}$, then $d = 0$. Which shows that c is also an atom in $\mathbf{A}_{e'}$. ($\subseteq$) If $c \in At(\mathbf{A}_{e'})$, then in particular, by Proposition 6, $c \in \mathrm{Ce}(\mathbf{A})$. If d is a central element of $\mathbf{A}$ such that $d < c$, then we have $d \leq e'$, because $c \in A_{e'}$, and therefore $d \in \mathrm{Ce}(\mathbf{A}_{e'})$, again by Proposition 6. Since by hypothesis c is atomic central in $\mathbf{A}_{e'}$, then $d = 0$. Which shows that c is atomic central in $\mathbf{A}$. Finally, $c \leq e'$, and therefore $c \neq e$. Otherwise, we would have $e \leq e'$, and hence $e = e \sqcap e' = 0$, which is impossible because e is atomic central. □

Theorem 9 *If* $\mathbf{A}$ *is a complemented directoid such that* $\mathrm{Ce}(\mathbf{A})$ *is an atomic Boolean algebra with finitely many atoms, then*

$$\mathbf{A} = \prod_{e \in At(\mathbf{A})} \mathbf{A}_e$$

is a decomposition of $\mathbf{A}$ *as a product of directly indecomposable algebras.*

Proof In order to proceed with the proof of the theorem, we will use induction on the number of elements of $At(\mathbf{A})$. If 1 is the only atomic central element of $\mathbf{A}$, then $\mathbf{A}$ is directly indecomposable, and the result follows because $\mathbf{A}_1 = \mathbf{A}$. If there is an atomic central element $e \neq 1$, then $\mathbf{A} = \mathbf{A}_e \times \mathbf{A}_{e'}$ in virtue of Theorem 8. Since e is an atom, $\mathrm{Ce}(\mathbf{A}_e) = \{0, e\}$, because if $\mathbf{A}_e$ had another central element, say c, then c would be a central element of $\mathbf{A}$ in virtue of Proposition 6, and such that $0 < c < e$, contradicting the fact that e is an atom. Therefore, $\mathbf{A}_e$ is directly indecomposable. Now, $At(\mathbf{A}_{e'}) = At(\mathbf{A}) - \{e\}$, by Lemma 3, and by the induction hypothesis, $\mathbf{A}_{e'} = \prod_{c \in At((\mathbf{A}_{e'}))} \mathbf{A}_c$, whence the result readily follows. □

5 Strong amalgamation property

A *V-formation* (Fig. 1) is a tuple $(\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, i, j)$ such that $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2$ are similar algebras, and $i : \mathbf{A} \to \mathbf{B}_1$, $j : \mathbf{A} \to \mathbf{B}_2$

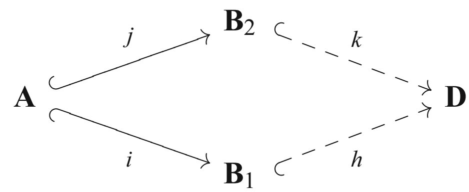


Fig. 1 A generic amalgamation schema

are embeddings. A class $\mathcal{K}$ of similar algebras is said to have the *amalgamation property* if for every V-formation with $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{K}$ and $A \neq \emptyset$ there exists an algebra $\mathbf{D} \in \mathcal{K}$ and embeddings $h : \mathbf{B}_1 \to \mathbf{D}$, $k : \mathbf{B}_2 \to \mathbf{D}$ such that $k \circ j = h \circ i$. In such an event, we also say that k and h *amalgamate* the V-formation $(\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, i, j)$. $\mathcal{K}$ is said to have the *strong amalgamation property* if, in addition, such embeddings can be taken s.t. $k \circ j\,(\mathbf{A}) = h\,(\mathbf{B}_1) \cap k\,(\mathbf{B}_2)$.

Amalgamations were first considered for groups by Schreier (1927) in the form of amalgamated free products. The general form of the AP was first formulated by Fraïsse (1954), and the significance of this property to the study of algebraic systems was further demonstrated in Jónsson's pioneering work on the topic Jónsson (1956, 1960, 1961, 1962). The added interest in the AP for algebras of logic is due to its relationship with various syntactic interpolation properties. We refer the reader to Metcalfe et al. (2014) for relevant references and an extensive discussion of these relationships.

In this section, we show that the variety of directoids has the strong amalgamation property.

Theorem 10 *The variety of directoids has the strong amalgamation property.*

Proof Let us suppose that we have a V-formation like the solid part of Fig. 1, and without loss of generality, let us assume that $B_1 \cap B_2 = A$. We are going to give an explicit construction of the amalgam of this V-formation. Let us consider $D = B_1 \cup B_2 \cup \{1\}$, where 1 is a new element. We proceed to define a partial order in D as follows: $x \leq 1$, for all $x \in D$, and if $x \in B_i$ and $y \in B_j$:

$$x \leq y \Leftrightarrow \begin{cases} i = j \text{ and } x \leq^{\mathbf{B}_i} y, \\ \text{or} \\ i \neq j,\ x, y \notin A, \text{ and there is } b \in A, \\ \quad \text{such that } x \leq^{\mathbf{B}_i} b \leq^{\mathbf{B}_j} y. \end{cases}$$

We show in what follows that $\leq$ is a partial ordering on D. First, note that if $x \leq y$ and $x, y \in B_i$, then $x \leq^{\mathbf{B}_i} y$. Now, $\leq$ is obviously reflexive. In order to see that it is antisymmetric, let us suppose that $x \in B_i$, $y \in B_j$, and $x \leq y$ and $y \leq x$. We will distinguish three different cases:

(1) $i = j$. Then, $x \leq^{\mathbf{B}_i} y$ and $y \leq^{\mathbf{B}_i} x$, and by the antisymmetry of $\leq^{\mathbf{B}_i}$, $x = y$.

(2) $i \neq j$, $x, y \notin A$. Then there are $b_1, b_2 \in A$ such that $x \leq^{\mathbf{B}_i} b_1 \leq^{\mathbf{B}_j} y$ and $y \leq^{\mathbf{B}_j} b_2 \leq^{\mathbf{B}_i} x$. In that case, $b_2 \leq^{\mathbf{B}_i} x \leq^{\mathbf{B}_i} b_1$ and $b_1 \leq^{\mathbf{B}_j} y \leq^{\mathbf{B}_j} b_2$, whence $b_2 \leq^{\mathbf{A}} b_1$ and $b_1 \leq^{\mathbf{A}} b_2$, and therefore $b_1 = b_2$. This would imply that $x = b_1 = b_2 = y$, and actually that $x, y \in A$, which is a contradiction. So this case is impossible.

(3) The only remaining case is $x \leq 1$ and $1 \leq x$. However, $1 \leq x$ implies by definition that $x = 1$.

In order to prove the transitivity of $\leq$, let us suppose that $x, y, z \in D$ and $x \leq y$ and $y \leq z$. Obviously, if $z = 1$, then $x \leq z$ and there is nothing to prove. We assume then that $z \neq 1$, which implies that $x, y \neq 1$ as well, and distinguish three cases:

- If $x, y, z \in B_i$ for some $i = 1, 2$, then $x \leq^{\mathbf{B}_i} y \leq^{\mathbf{B}_i} z$, and then obviously $x \leq^{\mathbf{B}_i} z$, which implies $x \leq z$.
- $x \in B_i$, $z \in B_j$, $x, z \notin A$. We have different subcases depending on the position of y. If $y \in B_i$ and $y \notin A$, then there exists $b \in A$ such that $y \leq^{\mathbf{B}_i} b \leq^{\mathbf{B}_j} z$, and therefore $x \leq^{\mathbf{B}_i} b \leq^{\mathbf{B}_j} z$, which by definition implies $x \leq z$. If $y \in A$, then we have $x \leq^{\mathbf{B}_i} y \leq^{\mathbf{B}_j} z$, which again by definition implies $x \leq z$. If $y \in B_j$ and $y \notin A$, then there exists $b \in A$ such that $x \leq^{\mathbf{B}_i} b \leq^{\mathbf{B}_j} y$ and therefore $x \leq^{\mathbf{B}_i} b \leq^{\mathbf{B}_j} z$, whence $x \leq z$.
- $x, z \in B_i$, $y \in B_j$, and $y \notin A$. If $x \in A$ and $z \notin A$, then there is $b \in A$ such that $y \leq^{\mathbf{B}_i} b \leq^{\mathbf{B}_j} z$. But then, $x \leq^{\mathbf{B}_i} y \leq^{\mathbf{B}_i} b$, which implies $x \leq^{\mathbf{A}} b$, and therefore $x \leq^{\mathbf{B}_j} b \leq^{\mathbf{B}_j} z$, whence $x \leq z$. If $x \notin A$ and $z \in A$, then we argue analogously. If $x, z \notin A$, then there are $b_1, b_2 \in A$ such that $x \leq^{\mathbf{B}_i} b_1 \leq^{\mathbf{B}_j} y \leq^{\mathbf{B}_j} b_2 \leq^{\mathbf{B}_i} z$. Hence, $b_1 \leq^{\mathbf{B}_j} b_2$, which is the same as $b_1 \leq^{\mathbf{A}} b_2$, and then $b_1 \leq^{\mathbf{B}_i} b_2$. Thus, $x \leq^{\mathbf{B}_i} b_1 \leq^{\mathbf{B}_i} b_2 \leq^{\mathbf{B}_i} z$, and by the transitivity of $\leq^{\mathbf{B}_i}$, we obtain $x \leq z$.

Thus, we have turned D into a poset. We can readily see that it is directed, because it is bounded above. Then we take the directoid $\mathbf{D} = \mathcal{D}(D, \leq)$, where $\sqcup$ is defined as follows:

$$x \sqcup y = y \sqcup x = \begin{cases} y & \text{if } x \leqslant y, \\ x \sqcup^{\mathbf{B}_i} y & \text{if } x, y \in B_i \text{ and } x \parallel y, \\ 1 & \text{otherwise.} \end{cases}$$

This operation is well-defined, because if $x, y \in B_1 \cap B_2 = A$, then $x \sqcup^{\mathbf{B}_1} y = x \sqcup^{\mathbf{A}} y = x \sqcup^{\mathbf{B}_2} y$. Now, it is not difficult to prove that $\mathbf{B}_i$ is a subalgebra of $\mathbf{D}$. Indeed, if $x, y \in B_i$, then it could be that $x \leq^{\mathbf{B}_i} y$, $y \leq^{\mathbf{B}_i} x$, or $x \parallel y$. In any of those three cases $x \sqcup y = x \sqcup^{\mathbf{B}_i} y$. And as we saw in the first section, $\mathbf{D}$ retains the information relative to the ordering of $(D, \leq)$, that can be recovered by stipulating that $x \leq y$ if and only if $x \sqcup y = y$. By construction, the intersection of $\mathbf{B}_1$ and $\mathbf{B}_2$ as subalgebras of $\mathbf{D}$ is the algebra $\mathbf{A}$. Therefore, we have proven that $\mathbf{D}$ is a strong amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$. □

Note that we needed to add a new element 1 to $B_1 \cup B_2$ just to ensure that $U(x, y)$ is nonempty, for every $x, y \in D$, in particular when $x \in B_i$, $y \in B_j$ and $x, y \notin A$. If $\mathbf{B}_1$ and $\mathbf{B}_2$ are algebras with a common subalgebra $\mathbf{A}$, in a language with the constant 1, which is interpreted as the top element on each of these algebras, there is no need to add any new element to $B_1 \cup B_2$. The construction of the amalgam is otherwise entirely analogous.

Theorem 11 *The varieties of bounded directoids, involutive directoids, bounded involutive directoids, and complemented directoids have the strong amalgamation property.*

Proof Essentially, the amalgam of a V-formation in each one of those varieties is the amalgam of the $\sqcup$-reducts, although some technical but innocuous modifications are needed in some cases. If we are in the variety of bounded directoids, we have to identify the new top element 1 in the amalgam with the top element of $\mathbf{A}$ (which would also be the top element of $\mathbf{B}_1$ and $\mathbf{B}_2$). Otherwise said, we do not need to add a new element 1. If we are in the variety of involutive directoids, then we not only have to add the new element 1 to $B_1 \cup B_2$, but also another new element 0, which should be defined to be the infimum. And the involution * of $\mathbf{D}$ should be defined to be $1^* = 0$, $0^* = 1$, and $x^* = x'^{\mathbf{B}_i}$ if $x \in B_i$.

As established in Kiss et al. (1983), the epimorphisms of the varieties enjoying the strong amalgamation property can be characterised as the onto homomophisms. Therefore, we have the following result:

Corollary 2 *In each one of the varieties of directoids, bounded directoids, involutive directoids, bounded involutive, and complemented directoids, the epimorphisms are onto.*

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