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Pseudo Basic Algebras

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Since basic algebras are equivalent to bounded lattices with sectional antitone involutions, it motivated us to study an algebraic counterpart of semilattices with sectional switching involutions. These algebras are called pseudo basic algebras. They are determined by four independent identities and hence the class of these algebras forms a variety. Several basic properties of these algebras are presented and a particular interest is devoted to pseudo basic algebras whose main involution is even antitone (so-called strict pseudo basic algebras) and to those whose binary operation is commutative. Congruence properties of the varieties of these algebras are investigated.

Keywords: Sectional switching involution, join-semilattice, basic algebra, pseudo basic algebra, congruence regularity, arithmeticity.

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1 MOTIVATION

The concept of basic algebra was introduced by the first author, R. Halaš and J. Kühr as a common generalization of an MV-algebra and an orthomodular lattice, see [5], [6] and [7]. Remember that MV-algebras serve as an algebraic axiomatization of the so-called Łukasiewicz many-valued logics and orthomodular lattices form an algebraic counterpart of the logic of quantum

mechanics. Hence, basic algebras form a common algebraic axiomatization of both logics mentioned above.

Recall that a **basic algebra** (see e.g. [6, 7, 9]) is an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type (2, 1, 0) satisfying the following identities

(B1) $x \oplus 0 = x$

(B2) $\neg\neg x = x$

(B3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

(B4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$, where $1 = \neg 0$.

As it was shown in [5], every basic algebra has its second face, namely $\mathcal{A} = (A; \oplus, \neg, 0)$ can be organized into a bounded lattice $(A; \vee, \wedge)$, where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y),$$

whose order is given by

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

Of course, $0 \leq x \leq 1$ for each $x \in A$. Moreover, this lattice $(A; \vee, \wedge)$ is endowed by a set $({}^a)_{a \in A}$ of so-called **sectional antitone involutions**, i.e. for each $a \in A$ there exists a mapping $x \mapsto x^a$ of the interval $[a, 1]$ (called **section**) into itself such that

$$x^{aa} = x \quad \text{and} \quad x \leq y \;\Rightarrow\; y^a \leq x^a \quad \text{for all } x, y \in [a, 1].$$

This system $\mathcal{L}(\mathcal{A}) = (L; \vee, \wedge, ({}^a)_{a \in L}, 0, 1)$ is called a **lattice with sectional antitone involutions** assigned to $\mathcal{A} = (A; \oplus, \neg, 0)$. Also conversely, having a bounded lattice with sectional antitone involutions $\mathcal{L} = (L; \vee, \wedge, ({}^a)_{a \in L}, 0, 1)$, one can convert it into a basic algebra $\mathcal{A}(\mathcal{L}) = (L; \oplus, \neg, 0)$, where

$$\neg x = x^0 \quad \text{and} \quad x \oplus y = (\neg x \vee y)^y,$$

see e.g. [6] for details. Moreover, the assignments $\mathcal{A} \to \mathcal{L}(\mathcal{A})$ and $\mathcal{L} \to \mathcal{A}(\mathcal{L})$ are one-to-one correspondences, i.e. $\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}$ and $\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}$.

For MV-algebras, the underlying lattice can be induced in the same way as for basic algebras and for each element a, a mapping $x \mapsto \neg x \oplus a = x^a$ is a sectional antitone involution again.

For an orthomodular lattice $\mathcal{L} = (L; \vee, \wedge, {}^\perp, 0, 1)$ and an arbitrary element $a \in L$, the mapping $x \mapsto x^a = x^\perp \vee a$ is an antitone involution in the section $[a, 1]$ which is, moreover, a sectional complementation.

This motivated us to investigate a more general case. Consider e.g. a bounded modular lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ with a complementation $x \mapsto x'$. Let $a \in L$ and $x \in [a, 1]$. It is well-known [1] that $x' \vee a$ is a complement of x in section $[a, 1]$. Hence, complemented modular lattices can be also considered as lattices with sectional involutions and we can consider a logic induced by them. However, there is an essential difference from the above mentioned cases, see the following.

Example 1. *Let $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ be the modular lattice depicted in Figure 1.*

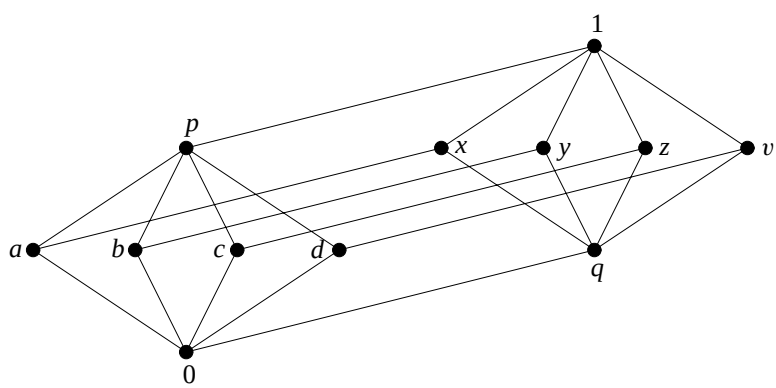

FIGURE 1

It is evident that $\mathcal{L}$ is isomorphic to the direct product of M_4 and a two-element chain C_2. Although every possible complementation in M_4 is antitone as well as the unique complementation in C_2, there exists a complementation in $\mathcal{L}$ which is not antitone (and hence it is not a direct product of any complementations in M_4 and C_2). We can get e.g. the following:

t	0	a	b	c	d	x	y	z	v	p	q	1
t'	1	y	z	v	x	d	a	b	c	q	p	0

One can easily check that the complementation $t \to t'$ is really an involution in $\mathcal{L}$ but it is not antitone:

$$a \leq x \quad \textit{but} \quad a' = y \parallel d = x'.$$

Hence, although $\mathcal{L}$ is a complemented modular lattice, it is not orthocomplemented (with respect to the complementation by our choice).

It means that for complemented modular lattices we can study sectional involutions similarly as for basic algebras, MV-algebras and orthomodular lattices but we cannot assume that these involutions are antitone. Hence, for a conversion of complemented modular lattices into algebras similar to basic algebras, we have to consider lattices (or semilattices) whose sectional involutions need not be antitone but only switching the endpoints of the section.

Concerning the motivation, there is a close connection of the mentioned algebras with various propositional logics. As listed above, MV-algebras serve as an algebraic axiomatization of many-valued Łukasiewicz logics and orthomodular lattices are used for an axiomatization of the logic of quantum mechanics. An interesting connection of commutative basic algebras and certain non-associative fuzzy logics is pointed out by M. Botur and R. Halaš [2]. These logics play an important role in so-called expert systems, see [2] for details. From the algebraic point of view, the logical connective implication is expressed as a term operation $x \to y = \neg x \oplus y$. Applying the alter ego of basic algebras, i.e. lattices with sectional antitone involutions, implication can be equivalently defined by $x \to y = (x \vee y)^y$. Since this expression does not contain the lattice operation $\wedge$, it can be used also for semilattices with sectional involutions which is just our case. At the first glance, one can see that antitonicity does not play a role in this definition. Hence, we can introduce a logic (based on implication and, possibly, negation and constants) which can be algebraically axiomatized via this algebra. In what follows, we will call this algebra as pseudo basic algebra and we will search for its axiomatization and important algebraic properties. A description of the induced logic is not the aim of this paper and will be done in a separate paper in the sequel.

2 BASIC CONCEPTS

Consider a section $[a, 1]$ of an ordered set with greatest element 1. A mapping $x \mapsto x^a$ of $[a, 1]$ into itself is called a **sectional switching involution** if $x^{aa} = x$ for each $x \in [a, 1]$ and $a^a = 1$, $1^a = a$. In general, we do not ask that this involution should be antitone; it only switches the endpoints of the section.

Now, we can consider a bounded lattice with sectional switching involutions $\mathcal{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ and study what an algebra can be obtained by using a similar construction as that for basic algebras. For our purposes, we will consider only a semilattice since the operation meet is not applied in the construction of the operations of the new algebra.

Theorem 1. *Let $\mathcal{S} = (S; \vee, (^a)_{a\in S}, 0, 1)$ be a bounded semilattice with sectional switching involutions. Define $\neg x = x^0$ and $x \oplus y = (x^0 \vee y)^y$. Then*

the algebra $\mathcal{A}(\mathcal{S}) = (S; \oplus, \neg, 0)$ *assigned to* $\mathcal{S}$ *satisfies the following identities:*

(P1) $\neg x \oplus x = 1$
(P2) $x \oplus 0 = x$
(P3) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$
(P4) $\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z = \neg(\neg(\neg(\neg y \oplus z) \oplus z) \oplus x) \oplus x.$

If, moreover, the involution $x \mapsto x^0$ *is antitone, then* $\mathcal{A}(\mathcal{S})$ *satisfies the identity*

(A) $(\neg(x \oplus y) \oplus y) \oplus x = 1.$

Proof. It is easy to see that $\neg x \oplus x = (x \vee x)^x = x^x = 1$ and $x \oplus 0 = (x^0 \vee 0)^0 = x^{00} = x$ proving (P1) and (P2).

For (P3), we compute

$$\begin{aligned} \neg(\neg(x \oplus y) \oplus y) \oplus y &= (((x^0 \vee y)^y \vee y)^y \vee y)^y = (x^0 \vee y)^{yyy} \\ &= (x^0 \vee y)^y = x \oplus y \end{aligned}$$

due to the fact that $(x^0 \vee y)^y \in [y, 1]$, i.e. $y \leq (x^0 \vee y)^y$.

To prove (P4), we derive

$$\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z = (((x \vee y)^y \vee y)^y \vee z)^z \vee z)^z$$
$$= (((x \vee y)^{yy} \vee z)^z \vee z)^z = ((x \vee y) \vee z)^{zz} = (x \vee y) \vee z = x \vee y \vee z,$$

similarly it can be shown $\neg(\neg(\neg(\neg y \oplus z) \oplus z) \oplus x) \oplus x = x \vee y \vee z$.

Assume that the involution $x \mapsto x^0$ is antitone. Then if $a \leq b$ in $\mathcal{S}$, we have $\neg a \oplus b = (a \vee b)^b = b^b = 1$. Using this and the fact that $y \vee x^0 \geq x^0$, we have $(y \vee x^0)^0 \leq x^{00} = x$ and $(y \vee x^0) \oplus x = 1$, hence

$$(\neg(x \oplus y) \oplus y) \oplus x = 1$$

proving (A).

Remark 1. *(a) Every bounded semilattice* $\mathcal{S} = (S; \vee, 0, 1)$ *can be considered as a semilattice with sectional switching involutions. Namely, for each* $a \in S$ *one can define a switching involution on* $[a, 1]$ *as follows:* $a^a = 1, 1^a = a$ *and* $x^a = x$ *for each* $x \in [a, 1]$, $a \neq x \neq 1$.

(b) If the involution $x \mapsto x^0$ is antitone, then $\mathcal{S} = (S; \vee, 0, 1)$ is in fact a lattice due to the DeMorgan laws because

$$x \wedge y = (x^0 \vee y^0)^0.$$

(c) There exist bounded semilattices with sectional switching involutions which are not lattices. Such a semilattice $\mathcal{K}$, call a "kite" is visualized in Figure 2.

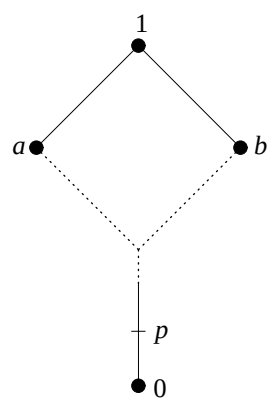

FIGURE 2

It is an ordinal sum of an infinite chain C with the least element 0 and without a greatest element and a three element semilattice $\{a, b, 1\}$, i.e. $p \leq a, b$ for each $p \in C$. Then $\mathcal{K}$ is a $\vee$-semilattice which is not a lattice since $\inf\{a, b\}$ *does not exists. Moreover, for each $c \in \mathcal{K}$ we define $c^c = 1$, $1^c = c$ and $x^c = x$ for $x \in [c, 1]$, $c \neq x \neq 1$.*

3 ASSIGNED ALGEBRAS

In the previous paragraph, we have shown that to every bounded semilattice $\mathcal{S}$ with sectional switching involutions we can assign an algebra $\mathcal{A}(\mathcal{S}) = (S; \oplus, \neg, 0)$ satisfying (P1)–(P4). We are going to show that algebras satisfying (P1)–(P4) are interesting for their own sake.

Definition 1. *An algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities (P1)–(P4) will be called a* ***pseudo basic algebra****. If, moreover, $\mathcal{A}$ satisfies also the identity (A), it will be called a* ***strict pseudo basic algebra****.*

Since both pseudo basic algebras and strict pseudo basic algebras are determined by identities, their classes are in fact varieties. Denote by $\mathcal{P}$ (or $\mathcal{PS}$) the variety of pseudo basic algebras (or strict pseudo basic algebras, respectively).

Lemma 1. *Every pseudo basic algebra satisfies the following identities:*

(i) $0 \oplus x = x$
(ii) $\neg\neg x = x$
(iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
(iv) $\neg x \oplus (y \oplus x) = 1$, *where* $1 = \neg 0$
(v) $x \oplus \neg x = 1$
(vi) $1 \oplus x = 1 = x \oplus 1$.

Proof. By putting $y = 0$ in (P3), we get by (P2)

$$\neg\neg x = \neg(\neg(x \oplus 0) \oplus 0) \oplus 0 = x \oplus 0 = x$$

which is (ii).

Putting $z = 0$ in (P4) we get (iii) immediately. If we substitute x by $\neg x$ in (P1) and use (ii), we infer (v).

Using (P3) (with $\neg x$ instead of x and $y \oplus x$ instead of y), (iii) (also with $y \oplus x$ instead of y), (P3) again and (P1), we have

$$\begin{aligned}\neg x \oplus (y \oplus x) &= \neg(\neg(\neg x \oplus (y \oplus x)) \oplus (y \oplus x)) \oplus (y \oplus x)\\ &= \neg(\neg(\neg(y \oplus x) \oplus x) \oplus x) \oplus (y \oplus x) = \neg(y \oplus x) \oplus (y \oplus x) = 1\end{aligned}$$

which is (iv).

From (iv) we obtain $\neg\neg x \oplus (x \oplus \neg x) = 1$. Using (ii) and (v) we infer $x \oplus 1 = 1$. Further, from (iv) we have $1 \oplus (x \oplus 0) = 1$, which together with (P2) yields $1 \oplus x = 1$. Thus we have proved (vi).

Finally, putting $y = 0$ in (iii) and applying (P2) and (ii) we obtain $x = \neg(\neg 0 \oplus x) \oplus x$ which yields (i) by (vi).

Theorem 2. *The axioms (P1)–(P4) are independent.*

Proof. Let $A = \{0, 1\}$ for $0 \neq 1$.

(a) Define $\neg 0 = 0$, $\neg 1 = 1$ and $0 \oplus 0 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1 \oplus 1 = 1$. Then $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfies (P2)–(P4) but not (P1).
(b) Define $\neg 0 = 1$, $\neg 1 = 0$ and $x \oplus y = 1$ for all $x, y \in \{0, 1\}$. Then $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfies (P1), (P3) and (P4) but not (P2).

(c) Define $\neg 0 = \neg 1 = 1$ and $0 \oplus 0 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1 \oplus 1 = 1$. Then $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfies (P1), (P2) and (P4) but not (P3).

(d) Define $\neg 0 = 1$, $\neg 1 = 0$ and $0 \oplus 0 = 1 \oplus 1 = 0$, $1 \oplus 0 = 0 \oplus 1 = 1$. Then $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfies (P1)–(P3) but not (P4) since for $x = 0 = z$, $y = 1$ we would obtain $1 = 0$.

Our next task is to show that also conversely, every pseudo basic algebra can be organized into a bounded semilattice with sectional switching involutions.

Theorem 3. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra. Define $1 = \neg 0$, $x \vee y = \neg(\neg x \oplus y) \oplus y$ and for any $a \in A$, let $x^a = \neg x \oplus a$. Then*

(a) $(A; \vee)$ *is a join-semilattice with least element* 0 *and greatest element* 1

(b) $x \leq y$ *if and only if* $\neg x \oplus y = 1$ *is the induced order of the semilattice* $(A; \vee)$

(c) *for each* $a \in A$ *and* $x \in [a, 1]$, *the mapping* $x \mapsto x^a = \neg x \oplus a$ *is a sectional switching involution on the section* $[a, 1]$.

If, moreover, $\mathcal{A}$ is a strict pseudo basic algebra, then $(A; \vee)$ *is a lattice where* $x \wedge y = \neg(\neg x \vee \neg y)$.

Proof. Define $1 = \neg 0$ and $x \vee y = \neg(\neg x \oplus y) \oplus y$. By (iii) of Lemma 1, we have $x \vee y = y \vee x$. By (P1) and (i) we have $x \vee x = \neg(\neg x \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x = x$. By (P4) we conclude $(x \vee y) \vee z = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z = \neg(\neg(\neg(\neg y \oplus z) \oplus z) \oplus x) \oplus x = (y \vee z) \vee x = x \vee (y \vee z)$ thus $\vee$ is a semilattice operation on A.

Let now $\leq$ be the induced order of $(A; \vee)$, i.e. $a \leq b$ if and only if $a \vee b = b$. Assume $x \leq y$. Then, by (P3) and (P1), $\neg x \oplus y = \neg(\neg(\neg x \oplus y) \oplus y) \oplus y = \neg(x \vee y) \oplus y = \neg y \oplus y = 1$. Conversely, let $\neg x \oplus y = 1$. Then $x \vee y = \neg(\neg x \oplus y) \oplus y = \neg 1 \oplus y = 0 \oplus y = y$ by (i) of Lemma 1 and hence $x \leq y$.

Now, let $a \in A$ and $x \in [a, 1]$. Define $x^a = \neg x \oplus a$. Then $x^{aa} = \neg(\neg x \oplus a) \oplus a = x \vee a = x$ thus $x \mapsto x^a$ is an involution of the section $[a, 1]$. Further, $a^a = \neg a \oplus a = 1$ and $1^a = \neg 1 \oplus a = 0 \oplus a = a$ thus it is also switching.

Finally, assume that $\mathcal{A}$ is a strict pseudo basic algebra, i.e. it satisfies the identity (A). Let $x \leq y$. Then $\neg x \oplus y = 1$ and, using (A), we get $y \oplus \neg x = (\neg(\neg x \oplus y) \oplus y) \oplus \neg x = 1$ proving $\neg y \leq \neg x$. Hence, the mapping $x \mapsto \neg x$ is antitone. By (ii) of Lemma 1 it is an involution and thus we can use the DeMorgan laws to show that $(A; \wedge)$ for $x \wedge y = \neg(\neg x \vee \neg y)$ is a meet-semilattice and hence $(A; \vee, \wedge)$ is a lattice.

We can show that the assignment between pseudo basic algebras and bounded semilattices with sectional switching involutions is a one-to-one correspondence.

Theorem 4. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra, $\mathcal{S}(\mathcal{A})$ its assigned semilattice with sectional switching involutions. Then $\mathcal{A}(\mathcal{S}(\mathcal{A})) = \mathcal{A}$.*

Let $\mathcal{S} = (S; \vee, ({}^a)_{a\in S}, 0, 1)$ be a bounded semilattice with sectional switching involutions, $\mathcal{A}(\mathcal{S})$ its assigned pseudo basic algebra. Then $\mathcal{S}(\mathcal{A}(\mathcal{S})) = \mathcal{S}$.

Proof. Denote by $+$ and $'$ the binary and the unary operation in $\mathcal{A}(\mathcal{S}(\mathcal{A}))$. Then $x' = x^0 = \neg x$ and

$$x + y = (x^0 \vee y)^y = \neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$$

whence $\mathcal{A}(\mathcal{S}(\mathcal{A})) = \mathcal{A}$.

In the second case, denote by $\cup$ the binary operation in $\mathcal{S}(\mathcal{A}(\mathcal{S}))$. Then

$$x \cup y = \neg(\neg x \oplus y) \oplus y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y.$$

Denote by x_a the involution in the section $[a, 1]$ of $\mathcal{S}(\mathcal{A}(\mathcal{S}))$. Then $x_a = \neg x \oplus a = (x \vee a)^a = x^a$. Since both 0 and 1 are the same in $\mathcal{S}$ and $\mathcal{S}(\mathcal{A}(\mathcal{S}))$, we conclude $\mathcal{S}(\mathcal{A}(\mathcal{S})) = \mathcal{S}$.

4 STRICT AND COMMUTATIVE PSEUDO BASIC ALGEBRAS

In this paragraph, we reveal several interesting properties of strict and/or commutative pseudo basic algebras.

Lemma 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra and $\mathcal{S}(\mathcal{A})$ its assigned semilattice. Then $\neg a \vee a = 1$ in $\mathcal{S}(\mathcal{A})$ if and only if $a \oplus a = a$ in $\mathcal{A}$.*

Proof. We have $x \oplus x = x$ if and only if $(\neg x \vee x)^x = x$. Since the sectional involution in $[x, 1]$ is switching, it means that $\neg x \vee x = 1$.

As mentioned in (b) of Remark 1, if a pseudo basic algebra $\mathcal{A}$ is strict, then the assigned semilattice $\mathcal{S}(\mathcal{A})$ is a lattice $(A; \vee, \wedge)$ where $x \wedge y = \neg(\neg x \vee \neg y)$. In what follows, we will use this fact and $\mathcal{S}(\mathcal{A})$ will be called an **assigned lattice**.

Theorem 5. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a strict pseudo basic algebra. Then $\neg$ is a complementation in the induced lattice $(A; \vee, \wedge, 0, 1)$ if and only if $\mathcal{A}$ satisfies the identity $x \oplus x = x$.*

Proof. By Lemma 2, $\neg x \vee x = 1$ if and only if $x \oplus x = x$. Due to the DeMorgan laws, $\neg x \wedge x = 0$ if and only if $\neg x \vee x = 1$.

A pseudo basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called **commutative** if it satisfies the identity $x \oplus y = y \oplus x$ and $\mathcal{A}$ is called **associative** if it satisfies the identity $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

An interesting connection is given by the following.

Theorem 6. *(a) Every commutative pseudo basic algebra is strict.*
(b) A pseudo basic algebra is an MV-algebra if and only if it is associative.

Proof. (a) Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a commutative pseudo basic algebra and $x, y \in A$ with $x \leq y$. Then $\neg x \oplus y = 1$, i.e.

$$\neg\neg y \oplus \neg x = y \oplus \neg x = \neg x \oplus y = 1$$

whence $\neg y \leq \neg x$ proving that the involution $\neg$ is antitone.
(b) It is clear that every MV-algebra is a pseudo basic algebra which is commutative and associative.

Conversely, assume that a pseudo basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is associative. Then by (P1) and Lemma 1 we have

$$1 = \neg(\neg y \oplus \neg x) \oplus 1 = \neg(\neg y \oplus \neg x) \oplus (\neg x \oplus x)$$

$$= (\neg(\neg y \oplus \neg x) \oplus \neg x) \oplus x = (\neg(\neg\neg x \oplus y) \oplus y) \oplus x$$

$$= (\neg(x \oplus y) \oplus y) \oplus x = \neg(x \oplus y) \oplus (y \oplus x)$$

whence $x \oplus y \leq y \oplus x$. Analogously we infer $y \oplus x \leq x \oplus y$ and hence $\mathcal{A}$ is also commutative.

Now, let $a \in A$ and $x, y \in [a, 1]$ with $x \leq y$. Then $\neg x \oplus y = 1$ which yields

$$\neg x \oplus (\neg(\neg y \oplus a) \oplus a) = \neg x \oplus (y \vee a) = \neg x \oplus y = 1.$$

Due to associativity, this is equivalent to

$$(\neg x \oplus \neg(\neg y \oplus a)) \oplus a = 1$$

and, by commutativity, also $(\neg(\neg y \oplus a) \oplus \neg x) \oplus a = 1$ and hence $\neg(\neg y \oplus a) \oplus (\neg x \oplus a) = 1$ which means $y^a \leq x^a$. This proves that every sectional switching involution is in fact antitone and hence $\mathcal{A} = (A; \oplus, \neg, 0)$ is an associative basic algebra. By [11], $\mathcal{A}$ is an MV-algebra.

It was proved by M. Botur and R. Halaš [3] that every finite commutative basic algebra is in fact an MV-algebra. Hence, it is a natural question if there really exist commutative pseudo basic algebras which are not basic algebras. The answer is positive also for a finite pseudo basic algebra.

Example 2. *Consider the commutative pseudo basic algebra* $\mathcal{A} = (\{0, a, b, c, d, e, 1\}; \oplus, \neg, 0)$*, where the operations* $\neg$ *and* $\oplus$ *are given by the tables*

x	0	a	b	c	d	e	1
$\neg x$	1	c	b	a	e	d	0

$\oplus$	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	c	d	1	d	b	1
b	b	d	1	1	1	c	1
c	c	1	1	1	1	c	1
d	d	d	1	1	1	1	1
e	e	b	c	c	1	d	1
1	1	1	1	1	1	1	1

By Theorem 6 (a), $\mathcal{A}$ *is strict and its induced lattice is visualized in Figure 3.*

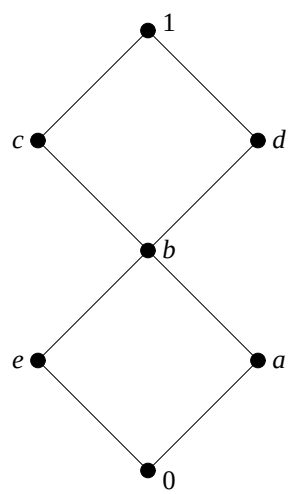

FIGURE 3

The sectional switching involutions are as follows:

In $[0, 1]$, $x^0 = \neg x$.

In $[e, 1]$ *it is:* $e^e = 1$, $b^e = c$, $c^e = b$, $d^e = d$, $1^e = e$.

In $[a, 1]$ *it is:* $a^a = 1$, $b^a = d$, $d^a = b$, $c^a = c$, $1^a = a$.

In $[b, 1]$ *it is:* $b^b = 1$, $c^b = d$, $d^b = c$ *and* $1^b = b$.

In $[c, 1]$, $[d, 1]$ *and* $[1, 1]$ *it is determined uniquely. One can easily check that* $x \mapsto x^e$ *and* $x \mapsto x^a$ *are not antitone since e.g.*

$$b \leq d \quad \textit{but} \quad b^e = c \parallel d = d^e.$$

Hence, $\mathcal{A}$ *cannot be a basic algebra.*

Call a lattice $\mathcal{L}$ with greatest element 1 **1-distributive** if it does not contain a sublattice isomorphic to N_5 or M_3 whose greatest element coincides with 1, see Figure 5.

Example 3. *Consider the pseudo basic algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$ *where* $A = \{0, a, b, c, p, x, y, z, 1\}$ *and the operations are determined by the tables*

t	0	a	b	c	p	x	y	z	1
$\neg t$	1	x	y	z	p	a	b	c	0

$\oplus$	0	a	b	c	p	x	y	z	1
0	0	a	b	c	p	x	y	z	1
a	a	y	y	y	z	1	1	1	1
b	b	x	x	x	y	z	1	1	1
c	c	p	p	p	x	y	z	1	1
p	p	z	z	z	1	1	1	1	1
x	x	1	z	z	1	1	1	1	1
y	y	z	1	z	1	1	1	1	1
z	z	z	z	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1

Its assigned semilattice $\mathcal{S}(\mathcal{A})$ *is in fact a lattice and it is depicted in Figure 4.*
It is evident that the lattice $\mathcal{S}(\mathcal{A})$ *is not distributive since it contains a sublattice* $\{0, a, b, c, p\}$ *isomorphic to* M_3 *but it is 1-distributive.*

We can show that 1-distributivity of $\mathcal{S}(\mathcal{A})$ is a property of assigned lattices of commutative pseudo basic algebras.

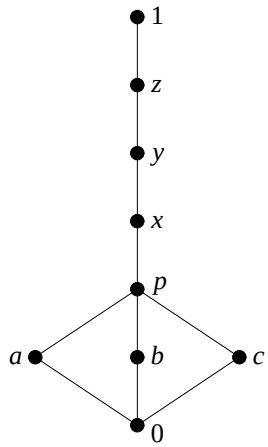

FIGURE 4

Theorem 7. *If a pseudo basic algebra $\mathcal{A}$ is commutative then the assigned lattice $\mathcal{S}(\mathcal{A})$ is* 1*-distributive.*

Proof. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a commutative pseudo basic algebra. By Theorem 6 (a) it is strict and hence its assigned semilattice $\mathcal{S}(\mathcal{A})$ is a lattice.

Assume that $\mathcal{S}(\mathcal{A})$ contains a sublattice N_5 or M_3 depicted in Figure 5.

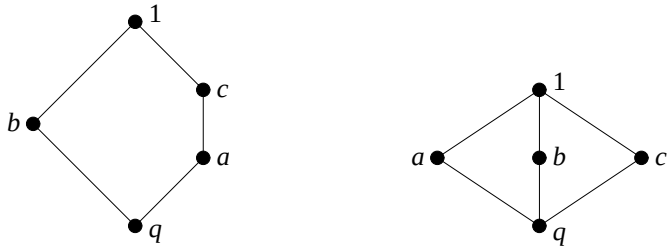

FIGURE 5

Since the involution $x \mapsto \neg x$ is antitone, it contains also a sublattice N_5 or M_3 equal to $\{0 = \neg 1, \neg a, \neg b, \neg c, \neg q\}$. In both possible cases we have

$$a = 1^a = (b \vee a)^a = \neg b \oplus a = a \oplus \neg b = (\neg a \vee \neg b)^{(\neg b)} = (\neg q)^{(\neg b)}$$

$$c = 1^c = (b \vee c)^c = \neg b \oplus c = c \oplus \neg b = (\neg c \vee \neg b)^{(\neg b)} = (\neg q)^{(\neg b)}$$

whence $a = c$, a contradiction. Thus the lattice $\mathcal{S}(\mathcal{A})$ is 1-distributive.

5 CONGRUENCE PROPERTIES AND SECTION ALGEBRAS

Now we turn our attention to congruence properties of pseudo basic algebras. Recall that an algebra $\mathcal{A}$ with a constant 1 is **weakly regular** (see [8], [10]) if for any two congruences $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ we have

$$[1]_\Theta = [1]_\Phi \quad \text{implies} \quad \Theta = \Phi,$$

i.e. each congruence on $\mathcal{A}$ is uniquely determined by its 1-class. An algebra $\mathcal{A}$ is called **congruence regular** if for any two congruence $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ and any element $a \in A$ we have

$$[a]_\Theta = [a]_\Phi \quad \text{implies} \quad \Theta = \Phi,$$

i.e. every congruence $\mathcal{A}$ is uniquely determined by each class.

A variety $\mathcal{V}$ is called **weakly regular** or **congruence regular** if every $\mathcal{A} \in \mathcal{V}$ has this property.

Theorem 8. *The variety $\mathcal{P}$ of pseudo basic algebras is weakly regular. The variety $\mathcal{PS}$ of strict pseudo basic algebras is congruence regular.*

Proof. By a theorem of Csákány [10] (see also Theorem 6.4.3 in [8]), a variety is weakly regular if and only if there exist an integer $n \geq 1$ and binary terms $t_1(x, y), \ldots, t_n(x, y)$ such that

$$t_1(x, y) = \cdots = t_n(x, y) = 1 \quad \text{if and only if} \quad x = y.$$

We can take $n = 2$ and $t_1(x, y) = \neg x \oplus y$, $t_2(x, y) = \neg y \oplus x$. Of course, $t_1(x, x) = t_2(x, x) = \neg x \oplus x = 1$. Conversely, if $t_1(x, y) = t_2(x, y) = 1$ then $\neg x \oplus y = 1 = \neg y \oplus x$, i.e. $x \leq y$ and $y \leq x$ whence $x = y$. Thus the variety $\mathcal{P}$ is weakly regular.

By another theorem of Csákány [10] (see also Theorem 6.1.3 in [8]), a variety is congruence regular if and only if there exist an integer n and ternary terms $t_1(x, y, z), \ldots, t_n(x, y, z)$ such that

$$t_1(x, y, z) = \cdots = t_n(x, y, z) = z \quad \text{if and only if} \quad x = y.$$

We can take $n = 2$ and $t_1(x, y, z) = (\neg x \oplus y) \wedge (\neg y \oplus x) \wedge z$, $t_2(x, y, z) = \neg((\neg x \oplus y) \wedge (\neg y \oplus x)) \oplus z$, where the derived operation $\wedge$ is defined in the variety $\mathcal{PS}$ via the DeMorgan laws by means of $a \vee b = \neg(\neg a \oplus b) \oplus b$.

Then

$$t_1(x, x, z) = (\neg x \oplus x) \wedge z = 1 \wedge z = z$$

$$t_2(x, x, z) = \neg 1 \oplus z = 0 \oplus z = z.$$

If, conversely, $t_1(x, y, z) = t_2(x, y, z) = z$ then

$$z \leq (\neg x \oplus y) \wedge (\neg y \oplus x). \qquad (*)$$

As it follows from $t_2(x, y, z) = z$ and $(*)$,

$$\begin{aligned} z &= \neg((\neg x \oplus y) \wedge (\neg y \oplus x)) \oplus z = (((\neg x \oplus y) \wedge (\neg y \oplus x)) \vee z)^z \\ &= ((\neg x \oplus y) \wedge (\neg y \oplus x))^z. \end{aligned}$$

Since the involution in $[z, 1]$ is switching, it yields

$$(\neg x \oplus y) \wedge (\neg y \oplus x) = 1$$

thus also $\neg x \oplus y = 1$ and $\neg y \oplus x = 1$. Hence $x \leq y$ and $y \leq x$ giving $x = y$. We have shown that the variety $\mathcal{PS}$ is congruence regular.

An algebra $\mathcal{A}$ is **arithmetical** if its congruence lattice $\mathrm{Con}\mathcal{A}$ is distributive and $\Theta \circ \Phi = \Phi \circ \Theta$ for each two congruences $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ (i.e. congruences are permutable). A variety $\mathcal{V}$ is arithmetical if each $\mathcal{A} \in \mathcal{V}$ has this property.

Theorem 9. *The variety $\mathcal{PS}$ of strict pseudo basic algebras is arithmetical.*

Proof. A variety $\mathcal{V}$ is arithmetical if it has a so-called Pixley term, i.e. a ternary term $m(x, y, z)$ satisfying the identities

$$m(x, x, z) = z, \quad m(x, z, z) = x \quad \text{and} \quad m(x, y, x) = x,$$

see e.g. Theorem 3.2.11 in [8]. For the variety $\mathcal{PS}$, we can take

$$m(x, y, z) = (\neg(\neg x \oplus y) \oplus z) \wedge (\neg(\neg z \oplus y) \oplus x) \wedge (x \vee z).$$

Then

$$m(x, x, z) = (\neg(\neg x \oplus x) \oplus z) \wedge (\neg(\neg z \oplus x) \oplus x) \wedge (x \vee z)$$

$$= z \wedge (z \vee x) \wedge (x \vee z) = z,$$

$$m(x, z, z) = (\neg(\neg x \oplus z) \oplus z) \wedge (\neg(\neg z \oplus z) \oplus x) \wedge (x \vee z)$$

$$= (x \vee z) \wedge x \wedge (x \vee z) = x$$

and

$$m(x, y, x) = (\neg(\neg x \oplus y) \oplus x) \wedge (\neg(\neg x \oplus y) \oplus x) \wedge x.$$

By (iv) of Lemma 1 we have $x \leq a \oplus x$ for each $x, a \in A$ thus also $x \leq \neg(\neg x \oplus y) \oplus x$. Hence $m(x, y, x) = x$. Thus $m(x, y, z)$ is a Pixley term proving that the variety $\mathcal{PS}$ is arithmetical.

As shown above, pseudo basic algebras are equivalent to bounded join-semilattices with sectional switching involutions. However, every section is a semilattice, i.e. it is again a bounded semilattice with sectional switching involutions. Hence, it can be converted into a pseudo basic algebra. How to organize its operations is shown in the following.

Theorem 10. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra, let $\leq$ be its induced order and $p \in A$. The section $[p, 1]$ can be organized into a pseudo basic algebra $([p, 1]; \oplus_p, \neg_p, p)$ as follows:*

$$\neg_p x = \neg x \oplus p \quad \textit{and} \quad x \oplus_p y = \neg(\neg x \oplus p) \oplus y$$

for $x, y \in [p, 1]$.

The proof is straightforward and hence omitted.

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