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SPECIAL FILTERS IN BOUNDED LATTICES

Ivan Chajda* — Miroslav Kolařík** — Helmut Länger***, [c]

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ABSTRACT. M. Sambasiva Rao recently investigated some sorts of special filters in distributive pseudocomplemented lattices. In our paper, we extend this study to lattices which need neither be distributive nor pseudocomplemented. For this sake, we define a certain modification of the notion of a pseudocomplement as the set of all maximal elements belonging to the annihilator of the corresponding element. We prove several basic properties of this notion and then define coherent, closed and median filters as well as D-filters. In order to be able to obtain valuable results, we often must add some additional assumptions on the underlying lattice, e.g. that this lattice is Stonean or D-Stonean. Our results relate properties of lattices and to those of corresponding filters. We show how the structure of a lattice influences the form of its filters and vice versa.

1. Introduction

M. Sambasiva Rao recently studied various types of filters and ideals in distributive lattices with pseudocomplements, see [4] and [5]. Some of these findings require the lattice under consideration to be either Stone or weakly Stone, as indicated in the referenced papers. The question is whether the distributivity assumption can be relaxed and, more specifically, if the pseudocomplementation assumption can be eliminated. This approach, in which we consider instead of the pseudocomplement a certain subset (in fact an antichain) of elements behaving similar to the pseudocomplement, has already been introduced by I. Chajda and H. Länger in [2]. In this paper the authors show that this construction can be successfully used in order to introduce the implication and negation connectives in an intuitionistic-like logic based on any bounded lattice that satisfies the Ascending Chain Condition.

It is well known that some of non-classical logics are formalized by means of lattice structures, i.e., by lattices endowed with an additional unary operation. Concerning classical propositional logic, it is formalized by a Boolean algebra, and intuitionistic logic is usually formalized by relatively pseudocomplemented semilattices or lattices, see, e.g., [2] and [3]. However, other non-classical logics use logical connectives that do not necessarily have a sharp meaning, i.e., the result of negation or implication for given entries need not be an element of the lattice in question, but may be a subset of mutually incomparable elements, see [2] for details. In order to axiomatize such logics bounded lattices are used where pseudocomplementation is replaced by the so-called annihilator,

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[c] Corresponding author.

i.e., the subset of maximal elements being disjoint to a given one. Using such an operator, we can introduce sharp and dense elements and a certain generalization of a Stone lattice.

It is generally accepted that the structure of a lattice can be revealed by investigating its filters. This motivated us to study some special filters on bounded lattices where the annihilator serves as an operator replacing pseudocomplementation. For distributive and pseudocomplemented lattices such a study was performed by M. Sambasiva Rao, see [4] and [5]. We obtain similar results for lattices that are neither distributive nor pseudocomplemented. The study of these special filters deepens our understanding of how lattice structure influences logical connectives.

2. Preliminaries

In this paper, we will deal with bounded lattices $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ satisfying the *Ascending Chain Condition* (ACC). A poset is said to satisfy the ACC if it has no infinite ascending chains. We identify singletons with their unique element, i.e., we will write a instead of $\{a\}$. For $a \in L$ and for subsets A, B of L, we define

$$\begin{aligned} A \vee B &:= \{x \vee y \mid x \in A \text{ and } y \in B\}, \\ A \wedge B &:= \{x \wedge y \mid x \in A \text{ and } y \in B\}, \\ A^0 &:= \operatorname{Max}\{x \in L \mid x \wedge y = 0 \text{ for all } y \in A\}, \\ a^0 &:= \operatorname{Max}\{x \in L \mid x \wedge a = 0\}. \end{aligned}$$

Here and in the following, $\operatorname{Max} A$ denotes the set of all maximal elements of A which obviously is an antichain. Observe that $0^0 = 1$ and $1^0 = 0$ and that $A^0 \neq \emptyset$. The set a^0 is in fact a generalization of the *pseudocomplement* of a introduced by O. Frink [3] or a modification of the annihilator since the set

$$\{x \in L \mid x \wedge y = 0 \text{ for all } y \in A\}$$

is in fact the *annihilator* of the set A as known in lattice theory. Recall that the *pseudocomplement* a^* of a is the greatest element of $\{x \in L \mid x \wedge a = 0\}$. The advantage of our approach is that a^0 can be defined in any lattice with 0. However, a^0 need not be an element of L, but may be a subset of L, namely an antichain of $\mathbf{L}$.

It is easy to see that in a bounded completely distributive lattice every element has a pseudocomplement. We will indicate the fact that a bounded lattice is pseudocomplemented by denoting it in the form $\mathbf{L} = (L, \vee, \wedge, {}^*, 0, 1)$.

The element a is called *dense* if $a^0 = 0$ and *sharp* if $a^{00} = a$. Let D and S denote the set of all dense and sharp elements of L, respectively. Clearly the following holds:

$$\begin{aligned} &a \in D \text{ if and only if } a^{00} = 1, \\ &1 \in D, \\ &0, 1 \in S, \\ &D \cap S = 1. \end{aligned}$$

Moreover, we define the following binary relations and unary operators on 2^L:

$$\begin{aligned} &A \leq B \text{ if } x \leq y \text{ for all } x \in A \text{ and all } y \in B, \\ &A \leq_1 B \text{ if for every } x \in A \text{ there exists some } y \in B \text{ with } x \leq y, \\ &A \leq_2 B \text{ if for every } y \in B \text{ there exists some } x \in A \text{ with } x \leq y, \\ &A =_1 B \text{ if both } A \leq_1 B \text{ and } B \leq_1 A, \end{aligned}$$

$$\overline{A} := \{x \in L \mid 1 \in x^{00} \vee y^{00} \text{ for each } y \in A\},$$
$$A^D := \{x \in L \mid x \vee y \in D \text{ for all } y \in A\}.$$

The set A is called *closed* if $\overline{\overline{A}} = A$. Observe that $\overline{0} = D$ and $\overline{1} = L$.

Lemma 2.1. *Let* $(L, \vee, \wedge, 0, 1)$ *be a bounded lattice satisfying the* ACC. *Then* $\overline{D} = L$ *and* $\overline{L} = D$.

Proof. If $a \in L$, then $1 \in a^{00} \vee x^{00}$ for all $x \in D$ and hence $a \in \overline{D}$. This shows $L \subseteq \overline{D}$, i.e., $\overline{D} = L$. Now let $a \in \overline{L}$. Then $1 \in a^{00} \vee x^{00}$ for all $x \in L$, especially $1 \in a^{00} \vee 0^{00} = a^{00}$. Since a^{00} is an antichain, we have $a^{00} = 1$ whence $a \in D$. This shows $\overline{L} \subseteq D$. Conversely, if $a \in D$, then $a^{00} = 1$ and hence $1 \in a^{00} \vee x^{00}$ for all $x \in L$, i.e., $a \in \overline{L}$. This shows $D \subseteq \overline{L}$. Altogether, we obtain $\overline{L} = D$. □

Remark 1. The pair $(A \mapsto \overline{A}, A \mapsto \overline{A})$ is the Galois-correspondence between $(2^L, \subseteq)$ and $(2^L, \subseteq)$ induced by the binary relation $\{(x, y) \in L^2 \mid 1 \in x^{00} \vee y^{00}\}$ on L. Hence $A \mapsto \overline{\overline{A}}$ is a closure operator on $(2^L, \subseteq)$, and therefore the set of all closed subsets of L forms a closure system of L, i.e., it is closed under arbitrary intersections. This means that the closed subsets of L form a complete lattice with respect to inclusion with smallest element $\overline{L} = D$ and greatest element $\overline{\emptyset} = L$, and we have the following properties:

$$A \subseteq B \text{ implies } \overline{B} \subseteq \overline{A},$$
$$A \subseteq \overline{\overline{A}},$$
$$\overline{\overline{\overline{A}}} = \overline{A},$$
$$A \subseteq \overline{B} \text{ if and only if } B \subseteq \overline{A}.$$

Analogously, the pair $(A \mapsto A^D, A \mapsto A^D)$ is the Galois-correspondence between $(2^L, \subseteq)$ and $(2^L, \subseteq)$ induced by the binary relation $\{(x, y) \in L^2 \mid x \vee y \in D\}$ on L.

Recall that a non-empty subset F of L is called a *filter* of $\mathbf{L}$ if $x \wedge y, x \vee z \in F$ for all $x, y \in F$ and all $z \in L$.

Observe that for every $x \in L$ the set $F_x := \{y \in L \mid x \leq y\}$ is a filter of $\mathbf{L}$, the so-called *principal filter* generated by x. A filter F of $\mathbf{L}$ is called a D*-filter* if $D \subseteq F$. It is obvious that every filter of a finite lattice is principal.

3. Filters and D-filters

It is evident that if a given bounded lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is pseudocomplemented, then for each element $x \in L$, we have $x^0 = x^*$. We start this section with emphasizing certain differences between the properties of our concept x^0 and pseudocomplements.

It is well known that for distributive pseudocomplemented lattices $(L, \vee, \wedge, {}^*, 0)$ with bottom element 0 and $a, b \in L$ the following holds:

(i) $a \leq b$ implies $b^* \leq a^*$,

(ii) $a \leq a^{**}$ and $a^{***} = a^*$,

(iii) $(a \vee b)^* = a^* \wedge b^*$,

(iv) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

The following example shows that not all statements valid for pseudocomplements hold for x^0.

***Example* 1.** Consider the non-distributive lattice $\mathbf{L}$ depicted in Figure 1.

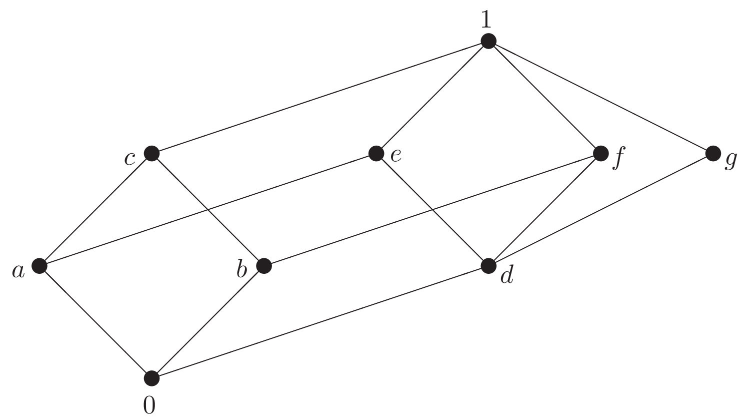

FIGURE 1. Non-distributive non-pseudocomplemented lattice

We have

x	0	a	b	c	d	e	f	g	1
x^0	1	fg	eg	g	c	c	a	c	0
x^{00}	0	a	b	c	g	g	fg	g	1
$\overline{x}$	F_1	F_d	F_d	F_d	$abcf1$	$abcf1$	$abcdefg1$	$abcf1$	L

$D = F_1$ and $S := \{0, a, b, c, g, 1\}$. (Here and in the following we often write fg instead of $\{f, g\}$, and so on.) Hence $\mathbf{L}$ is not pseudocomplemented.

In $\mathbf{L}$ we have:

(i) $a \leq c$ and $c^0 = g \not\leq \{f, g\} = a^0$, but $c^0 \leq_1 a^0$.

(ii) $f \not\leq \{f, g\} = f^{00}$, but $f \leq_1 f^{00}$.

(iii) $(a \vee b)^0 = c^0 = g \neq \{d, g\} = \{f, g\} \wedge \{e, g\} = a^0 \wedge b^0$, but $(a \vee b)^0 \leq_1 a^0 \wedge b^0$.

(iv) $(d \wedge f)^{00} = d^{00} = g \neq \{d, g\} = g \wedge \{f, g\} = d^{00} \wedge f^{00}$, but $(d \wedge f)^{00} =_1 d^{00} \wedge f^{00}$.

On the other hand, some of the properties of pseudocomplements are also preserved here, see the following result.

THEOREM 3.1. *Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the* ACC, *$x, y \in L$ and $A, B \subseteq L$. Then the following holds:*

(i) *$a \wedge b = 0$ for all $a \in A$ and all $b \in A^0$, especially, $x \wedge x^0 = 0$;*

(ii) *If $a \wedge b = 0$ for all $a \in A$ and all $b \in B$, then $A \leq_1 B^0$, especially, $x \wedge y = 0$ implies $x \leq_1 y^0$;*

(iii) *$A \leq_1 A^{00}$, especially, $x \leq_1 x^{00}$;*

(iv) *$A \leq_1 B$ implies $B^0 \leq_1 A^0$, especially, $x \leq y$ implies $y^0 \leq_1 x^0$;*

(v) *$A \leq_1 B^0$ if and only if $B \leq_1 A^0$, especially, $x \leq_1 y^0$ if and only if $y \leq_1 x^0$;*

(vi) *$a =_1 B^0$ implies $a = B^0$;*

(vii) *$A^0 =_1 B^0$ implies $A^0 = B^0$;*

(viii) *$A^{000} = A^0$, especially, $x^{000} = x^0$;*

(ix) *$(A \vee B)^0 \leq_1 A^0 \wedge B^0$, especially, $(x \vee y)^0 \leq_1 x^0 \wedge y^0$;*

(x) *$(A \wedge B)^{00} =_1 A^{00} \wedge B^{00}$, especially, $(x \wedge y)^{00} =_1 x^{00} \wedge y^{00}$;*

(xi) $A^{00} \vee B^{00} \leq_1 (A^0 \wedge B^0)^0$ *implies* $(A^{00} \vee B^{00})^0 =_1 A^0 \wedge B^0$, *especially,* $x^{00} \vee y^{00} \leq_1 (x^0 \wedge y^0)^0$ *implies* $(x^{00} \vee y^{00})^0 =_1 x^0 \wedge y^0$.

Proof. (i) If $b \in A \wedge A^0$, then there exists some $c \in A$ and some $d \in A^0$ with $c \wedge d = b$. Since $x \wedge d = 0$ for all $x \in A$, we have $b = c \wedge d = 0$.

(ii) If $A \wedge B = 0$ and $b \in A$, then $b \wedge x = 0$ for all $x \in B$ and hence there exists some $c \in B^0$ with $b \leq c$.

(iii) This follows from (i) and (ii).

(iv) If $A \leq_1 B$, then

$$\{x \in L \mid x \wedge B = 0\} \subseteq \{x \in L \mid x \wedge A = 0\}$$

and hence $B^0 \leq_1 A^0$.

(v) If $A \leq_1 B^0$, then $B \leq_1 B^{00} \leq_1 A^0$ by (iii) and (iv). Analogously, $B \leq_1 A^0$ implies $A \leq_1 B^0$.

(vi) Let $b \in B^0$. Because of $B^0 \leq_1 a$, we have $b \leq a$, and because of $a \leq_1 B^0$ there exists some $c \in B^0$ with $a \leq c$. Together we obtain $b \leq a \leq c$ and hence $b \leq c$. Since both b and c belong to the antichain B^0, we conclude $b = c$ and hence $b = a$ showing $B^0 = a$.

(vii) Assume $b \in A^0$. Because of $A^0 \leq_1 B^0$ there exists some $c \in B^0$ with $b \leq c$, and because of $B^0 \leq_1 A^0$ there exists some $d \in A^0$ with $c \leq d$. Together we obtain $b \leq c \leq d$ and hence $b \leq d$. Since both b and d belong to the antichain A^0, we conclude $b = d$ and hence $b = c \in B^0$. This shows $A^0 \subseteq B^0$. Interchanging the roles of A^0 and B^0 gives $B^0 \subseteq A^0$. Together we obtain $A^0 = B^0$.

(viii) Because of (iii), we have $A \leq_1 A^{00}$ from which we conclude $A^{000} = (A^{00})^0 \leq_1 A^0$ by (iv). Again by (iii) we have $A^0 \leq_1 (A^0)^{00} = A^{000}$. Altogether, $A^{000} =_1 A^0$. Applying (vii) yields $A^{000} = A^0$.

(ix) We have $A, B \leq_1 A \vee B$ and hence $(A \vee B)^0 \leq_1 A^0, B^0$ by (iv) which implies $(A \vee B)^0 \leq_1 A^0 \wedge B^0$.

(x) We have $A \wedge B \leq_1 A^{00} \wedge B^{00}$ according to (iii) and hence $(A \wedge B)^{00} \leq_1 (A^{00} \wedge B^{00})^{00}$ according to (iv). Using (ii), (i) and (iv), we see that any of the following statements implies the next one:

$$\begin{aligned}
(A \wedge B)^0 \wedge A^{00} \wedge B^{00} \wedge A \wedge B &\subseteq \{0\},\\
(A \wedge B)^0 \wedge A^{00} \wedge B^{00} \wedge A &\leq_1 B^0,\\
(A \wedge B)^0 \wedge A^{00} \wedge B^{00} \wedge A &\leq_1 B^0 \wedge B^{00},\\
(A \wedge B)^0 \wedge A^{00} \wedge B^{00} \wedge A &\subseteq \{0\},\\
(A \wedge B)^0 \wedge A^{00} \wedge B^{00} &\leq_1 A^0,\\
(A \wedge B)^0 \wedge A^{00} \wedge B^{00} &\leq_1 A^0 \wedge A^{00},\\
(A \wedge B)^0 \wedge A^{00} \wedge B^{00} &= 0,\\
(A \wedge B)^0 &\leq_1 (A^{00} \wedge B^{00})^0,\\
(A^{00} \wedge B^{00})^{00} &\leq_1 (A \wedge B)^{00}.
\end{aligned}$$

Together we obtain $(A \wedge B)^{00} =_1 (A^{00} \wedge B^{00})^{00}$ and hence $(A \wedge B)^{00} = (A^{00} \wedge B^{00})^{00}$ according to (vii). Now we have $A^{00} \wedge B^{00} \leq_1 A^{00}, B^{00}$ and hence $(A^{00} \wedge B^{00})^{00} \leq_1 A^{00}, B^{00}$ by (iv) and (viii) which implies $(A^{00} \wedge B^{00})^{00} \leq_1 A^{00} \wedge B^{00}$. Together with $A^{00} \wedge B^{00} \leq_1 (A^{00} \wedge B^{00})^{00}$ that follows from (iii) this yields $(A \wedge B)^{00} =_1 (A^{00} \wedge B^{00})^{00} =_1 A^{00} \wedge B^{00}$.

(xi) According to (iii), (iv), (ix), (viii) and the assumption, we have

$$A^0 \wedge B^0 \leq_1 (A^0 \wedge B^0)^{00} \leq_1 (A^{00} \vee B^{00})^0 \leq_1 A^{000} \wedge B^{000} = A^0 \wedge B^0. \qquad \square$$

Theorem 3.1 is very important because we will use it many times in the proofs of most of the following statements.

Remark 2. Assume $a, b \in L$ and $a^{00} \vee b^{00} \leq_1 (a^0 \wedge b^0)^0$. Then because of (xi) of Theorem 3.1, we have $(a^{00} \vee b^{00})^0 =_1 a^0 \wedge b^0$. Hence any of the following statements implies the next one:

$$\begin{aligned} 1 &\in a^{00} \vee b^{00}, \\ 1 &\leq_1 a^{00} \vee b^{00}, \\ (a^{00} \vee b^{00})^0 &\leq_1 0, \\ (a^{00} \vee b^{00})^0 &= 0, \\ a^0 \wedge b^0 &= 0. \end{aligned}$$

Applying the previous results, we can show that the set of sharp elements of $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ forms a subsemilattice of $(L, \wedge)$.

Proposition 3.2. *Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the* ACC *and $a, b \in S$. Then $a \wedge b \in S$.*

Proof. According to (x) of Theorem 3.1, we have $(a \wedge b)^{00} =_1 a^{00} \wedge b^{00} = a \wedge b$ and hence $(a \wedge b)^{00} = a \wedge b$ by (vi) of Theorem 3.1, i.e., $a \wedge b \in S$. □

Remark 3. Unfortunately, S need not be a sublattice of $\mathbf{L}$. Namely, consider the lattice $\mathbf{L}$ visualized in Figure 2. Here, obviously, $a, b \in S$ and also $a \wedge b = 0 \in S$ in accordance with Proposition 3.2, but $a \vee b = e \notin S$ since $e^{00} = 1 \neq e$.

Now we prove an expected statement about the set of dense elements.

Proposition 3.3. *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the* ACC. *Then D forms a D-filter of $\mathbf{L}$.*

Proof. Let $a, b \in D$ and $c \in L$. Then according to (x) of Theorem 3.1, we conclude $(a \wedge b)^{00} =_1 a^{00} \wedge b^{00} = 1 \wedge 1 = 1$ and hence $(a \wedge b)^{00} = 1$ according to (vi) of Theorem 3.1 proving $a \wedge b \in D$. If $a \leq c$, then $c^0 \leq_1 a^0 = 0$ according to (iv) of Theorem 3.1 and hence $c^0 = 0$, i.e., $c \in D$. □

It is well known that the set $\mathcal{F}$ of all filters of $\mathbf{L}$ forms a complete lattice with respect to inclusion. Using Proposition 3.3 one can recognize that the set of all D-filters of $\mathbf{L}$ forms a complete sublattice of $\mathcal{F}$ with bottom element D.

D-filters were described for distributive pseudocomplemented lattices by M. Sambasiva Rao [5]. However, we are going to show that similar results can be stated also for lattices which are neither pseudocomplemented nor distributive, but satisfy a weaker condition.

Theorem 3.4. *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the* ACC *and A a non-empty subset of L such that for each $x, y \in \overline{A}$ the following condition holds:*

$$\textit{if } 1 \in (x^{00} \vee z^{00}) \wedge (y^{00} \vee z^{00}) \textit{ for all } z \in A \textit{ then } 1 \in (x^{00} \wedge y^{00}) \vee z^{00} \textit{ for all } z \in A.$$

Then $\overline{A}$ is a D-filter of $\mathbf{L}$.

Proof. Obviously, $D \subseteq \overline{A}$. Let $a, b \in \overline{A}$. Then $1 \in (a^{00} \vee z^{00}) \wedge (b^{00} \vee z^{00})$ for all $z \in A$ and hence $1 \in (a^{00} \wedge b^{00}) \vee z^{00} =_1 (a \wedge b)^{00} \vee z^{00}$ for all $z \in A$ according to (x) of Theorem 3.1, i.e., $a \wedge b \in \overline{A}$. Further, if $c \in L$ and $a \leq c$, then $a^{00} \leq_1 c^{00}$ according (iv) of Theorem 3.1 and hence $1 \in a^{00} \vee z^{00} \leq_1 c^{00} \vee z^{00}$ for all $z \in A$ showing $c \in \overline{A}$. Altogether, $\overline{A}$ is a D-filter of $\mathbf{L}$. □

In the next example, we show a lattice being neither distributive nor pseudocomplemented, but satisfying the condition from Theorem 3.4.

***Example* 2.** Consider the non-distributive lattice $\mathbf{L}$ depicted in Figure 2:

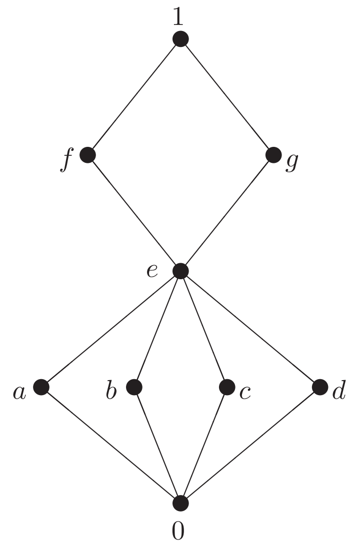

FIGURE 2. Non-distributive non-pseudocomplemented lattice

We have

x	0	a	b	c	d	e	f	g	1
x^0	1	bcd	acd	abd	abc	0	0	0	0
x^{00}	0	a	b	c	d	1	1	1	1
$\overline{x}$	F_e	F_e	F_e	F_e	F_e	F_0	F_0	F_0	F_0

$D = F_e$ and $S = \{0, a, b, c, d, 1\}$. Hence $\mathbf{L}$ is not pseudocomplemented. Further, F_a is a D-filter of $\mathbf{L}$, $\overline{\{a, b\}} = F_e$ is a D-filter of $\mathbf{L}$ in accordance with Theorem 3.4 and $\{a, b\}^D = \{c, d, e, f, g, 1\}$.

4. Stonean and D-Stonean lattices

The concept of a Stone lattice was introduced by R. Balbes and A. Horn [1], see also the paper [6] by T. P. Speed. Recall from [1] that a bounded pseudocomplemented lattice $(L, \vee, \wedge, {}^*, 0, 1)$ is called *Stone* if

$$x^* \vee x^{**} = 1 \text{ and } x^* \vee y^* = (x \wedge y)^* \text{ for all } x, y \in L.$$

The theory of Stone lattices is well developed, see e.g. [1] and [6] and for filters [5]. This motivated us to introduce and study an analogous concept for bounded lattices that are not necessarily pseudocomplemented.

In analogy to the above definition, we define the following two concepts.

DEFINITION 1. Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC. Then $\mathbf{L}$ is called *Stonean* if

$$1 \in x^{00} \vee y^{00} \text{ for every } x \in L \text{ and every } y \in x^0 \tag{4.1}$$

and *D-Stonean* if it is both Stonean and if

$$\text{for all } x, y \in L, x \vee y \in D \text{ is equivalent to } 1 \in x^{00} \vee y^{00}. \tag{4.2}$$

The Stonean property ensures that certain relations between elements hold which simplify the investigation of filters. The stronger D-Stonean property, moreover, establishes a connection to dense elements. These properties help to bridge the gap between classical results for distributive lattices and more general results concerning not necessarily distributive lattices.

Observe that (4.1) is equivalent to $x^0 \subseteq \overline{x}$ for all $x \in L$, and (4.2) is equivalent to $\overline{x} = x^D$ for all $x \in L$. Hence $\mathbf{L}$ is Stonean if and only if $x^0 \subseteq \overline{x}$ for all $x \in L$, and $\mathbf{L}$ is D-Stonean if and only if $x^0 \subseteq \overline{x} = x^D$ for all $x \in L$. The lattice visualized in Figure 2 is not Stonean since $b \in a^0$, but $1 \notin e = a \vee b = a^{00} \vee b^{00}$.

The following result relates the concept of a Stonean lattice to concepts mentioned before.

Proposition 4.1. *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the* ACC *and $a \in L$. Then the following holds:*

(i) *If $\mathbf{L}$ is Stonean and $a^0 \leq_2 \overline{a}$, then $\overline{\overline{a}} = \overline{a^0}$.*

(ii) *$\mathbf{L}$ is Stonean if and only if $\overline{\overline{x}} \subseteq \overline{x^0}$ for all $x \in L$.*

Proof. (i) Assume that $\mathbf{L}$ is Stonean and $a^0 \leq_2 \overline{a}$. Then $a^0 \subseteq \overline{a}$ and hence $\overline{\overline{a}} \subseteq \overline{a^0}$. Now let $b \in \overline{a^0}$ and $c \in \overline{a}$. Since $a^0 \leq_2 \overline{a}$ there exists some $d \in a^0$ with $d \leq c$. We conclude $1 \in b^{00} \vee d^{00} \leq_1 b^{00} \vee c^{00}$ and hence $1 \in b^{00} \vee c^{00}$. This shows $b \in \overline{\overline{a}}$ and hence $\overline{a^0} \subseteq \overline{\overline{a}}$. Together we obtain $\overline{\overline{a}} = \overline{a^0}$.

(ii) According to the observation after Definition 1, $\mathbf{L}$ is Stonean if and only if $x^0 \subseteq \overline{x}$ for all $x \in L$. Now for every $x \in L$ the inclusion $x^0 \subseteq \overline{x}$ is equivalent to $\overline{\overline{x}} \subseteq \overline{x^0}$. □

The next result is elementary, but we will use it in the sequel.

Lemma 4.1. *Let $(L, \vee, \wedge, 0, 1)$ be a D-Stonean lattice and $a \in L$. Then $a \vee a^0 \subseteq D$.*

Proof. If $b \in a^0$, then $1 \in a^{00} \vee b^{00}$ according to (4.1) which is equivalent to $a \vee b \in D$ because of (4.2). □

***Example* 3.**

(i) The lattice visualized in Figure 2 satisfies neither (4.1) nor (4.2) since $b \in a^0$ and $a \vee b = e \in D$, but $1 \notin e = a \vee b = a^{00} \vee b^{00}$.

(ii) Consider the non-distributive lattice $\mathbf{L}$ depicted in Figure 3:

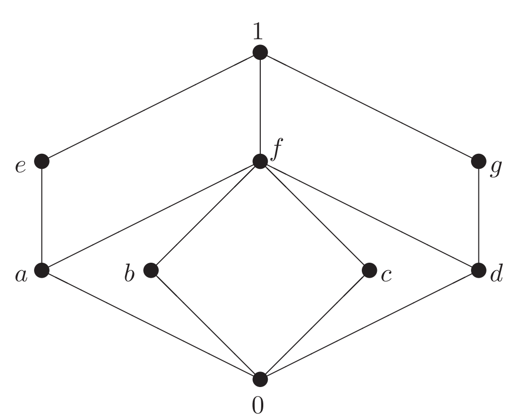

Figure 3. Non-distributive non-pseudocomplemented non-Stonean lattice

We have

x	0	a	b	c	d	e	f	g	1
x^0	1	bcg	ceg	beg	bce	bcg	0	bce	0
x^{00}	0	e	b	c	g	e	1	g	1
$\overline{x}$	F_f	$bcdfg1$	$adefg1$	$adefg1$	$abcef1$	$bcdfg1$	L	$abcef1$	F_0

$D = F_f$ and $S = \{0, b, c, e, g, 1\}$. Hence **L** is not pseudocomplemented. Moreover, **L** satisfies neither (4.1) nor (4.2) since $c \in b^0$ and $b \vee c = f \in D$, but $1 \notin f = b \vee c = b^{00} \vee c^{00}$. Therefore **L** is not Stonean.

Theorem 4.2. *Conditions* (4.1) *and* (4.2) *of* Definition 1 *are independent.*

P r o o f. The lattice visualized in Figure 1 satisfies (4.1), but not (4.2) since $e \vee g = 1 \in D$, but $1 \notin g = g \vee g = e^{00} \vee g^{00}$. Hence it is Stonean, but not D-Stonean. Now consider the non-distributive lattice **L** depicted in Figure 4:

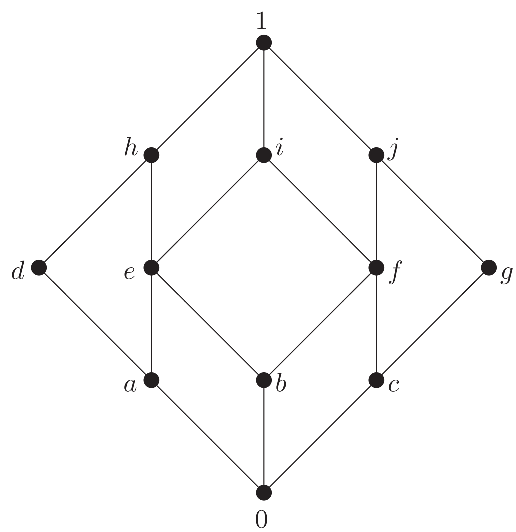

Figure 4. Non-distributive non-pseudocomplemented non-Stonean lattice

We have

x	0	a	b	c	d	e	f	g	h	i	j	1
x^0	1	j	dg	h	j	g	d	h	g	0	d	0
x^{00}	0	d	b	g	d	h	j	g	h	1	j	1
$\overline{x}$	F_i	F_c	F_i	F_a	F_c	F_c	F_a	F_a	F_c	F_0	F_a	F_0

$D = F_i$ and $S = \{0, b, d, g, h, j, 1\}$. Hence **L** is not pseudocomplemented. The lattice **L** satisfies (4.2), but not (4.1) since $d \in b^0$, but $1 \notin h = b \vee d = b^{00} \vee d^{00}$. Therefore it is not Stonean. □

The following theorem shows how the concept of a Stonean lattice is related with its filters.

Theorem 4.3. *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a Stonean lattice satisfying the* ACC. *Then* (4.2) *is equivalent to any single of the following statements:*

(i) $\overline{F} = F^D$ *for all filters* F *of* **L**,

(ii) $\overline{x} = x^D$ *for all* $x \in L$,

(iii) *for every two filters* F, G *of* **L**, $F \cap G \subseteq D$ *is equivalent to* $F \subseteq \overline{G}$.

P r o o f. (4.2) $\Rightarrow$ (i): This follows from

$$\overline{F} = \{x \in L \mid 1 \in x^{00} \vee y^{00} \text{ for all } y \in F\},$$
$$F^D = \{x \in L \mid x \vee y \in D \text{ for all } y \in F\}$$

for all filters F of $\mathbf{L}$.

(i) $\Rightarrow$ (ii): For all $x \in L$, we have $\overline{x} = \overline{F_x} = F_x^D = x^D$.

(ii) $\Rightarrow$ (iii): Let F, G be filters of $\mathbf{L}$. If $a \in F \cap G$, then $a = a \vee a \in F \vee G$. If, conversely, $a \in F \vee G$, then there exists some $b \in F$ and some $c \in G$ with $b \vee c = a$ and hence $a \in F \cap G$. This shows $F \cap G = F \vee G$. Now the following are equivalent:

$$F \cap G \subseteq D,$$
$$F \vee G \subseteq D,$$
$$x \vee y \in D \text{ for all } x \in F \text{ and all } y \in G,$$
$$y \in x^D \text{ for all } x \in F \text{ and all } y \in G,$$
$$y \in \overline{x} \text{ for all } x \in F \text{ and all } y \in G,$$
$$1 \in x^{00} \vee y^{00} \text{ for all } x \in F \text{ and all } y \in G,$$
$$F \subseteq \overline{G}.$$

(iii) $\Rightarrow$ (4.2): For all $x, y \in L$ the following are equivalent: $x \vee y \in D$; $F_{x \vee y} \subseteq D$; $F_x \cap F_y \subseteq D$; $F_x \subseteq \overline{F_y}$; $x \in \overline{y}$; $1 \in x^{00} \vee y^{00}$. □

The results of the previous theorem can be checked in the following example.

***Example* 4.** Consider the non-distributive lattice $\mathbf{L}$ depicted in Figure 5.

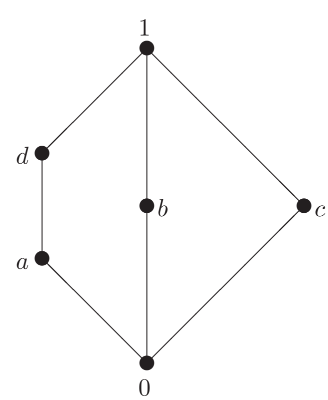

FIGURE 5. Non-distributive non-pseudocomplemented D-Stonean lattice

We have

x	0	a	b	c	d	1
x^0	1	bc	cd	bd	bc	0
x^{00}	0	d	b	c	d	1
$\overline{F_x} = F_x^D = \overline{x} = x^D$	F_1	$bc1$	$acd1$	$abd1$	$bc1$	F_0

$D = F_1$ and $S = \{0, b, c, d, 1\}$. Hence $\mathbf{L}$ is not pseudocomplemented, but it is D-Stonean. Moreover, for $x, y \in L$ both $F_x \cap F_y \subseteq D$ and $F_x \subseteq \overline{F_y}$ are equivalent to $1 \in \{x, y\}$ or ($x, y \in \{a, b, c, d\}$ and $x \neq y$ and $\{x, y\} \neq \{a, d\}$) in accordance with Theorem 4.3.

5. Coherent and closed filters

In the following we define a certain operator c on the lattice of filters of a bounded lattice which shares some properties with a closure operator. It is a natural task to investigate when a filter F coincides with its closure $c(F)$. Such filters will be called *coherent*. In particular, we describe coherent filters in D-Stonean lattices.

Now let us define the operator c on filters of $\mathbf{L}$ as follows:

$$c(F) := \{x \in L \mid \overline{x} \wedge F = L\}.$$

Definition 2. A *filter* F of a bounded lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ satisfying the ACC is called *coherent* if $c(F) = F$.

One can easily show that if F and G are filters of $\mathbf{L}$ with $F \subseteq G$, then $c(F) \subseteq c(G)$. Hence $c(F \cap G) \subseteq c(F) \cap c(G)$ for all filters F, G of $\mathbf{L}$.

***Example* 5.** Consider the lattice from Figure 2. Then we have

x	0	a	b	c	d	e	f	g	1
$c(F_x)$	F_0	F_e	F_e	F_e	F_e	F_e	F_e	F_e	F_e

and hence F_0 and F_e are the only coherent filters. The filters F_a and F_f of the lattice depicted in Figure 3 are coherent, but the filter F_b is not since $b \in F_b \setminus c(F_b)$.

Lemma 5.1. *If $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a bounded lattice satisfying the* ACC *and F a filter of $\mathbf{L}$ satisfying $\overline{x} \cup F = L$ for all $x \in F$, then $F \subseteq c(F)$.*

Proof. Let $a \in F$ and $b \in L$. Then $1 \in F \cap \overline{a}$ and either $b \in \overline{a}$ and hence $b = b \wedge 1 \in \overline{a} \wedge F$ or $b \in F$ and hence $b = 1 \wedge b \in \overline{a} \wedge F$. This shows $a \in c(F)$ and therefore $F \subseteq c(F)$. □

Since $\overline{x} = L$ for all $x \in D$, we have $F \subseteq c(F)$ for all filters F of $\mathbf{L}$ being contained in D.

We can show that $c(F)$ is closed with respect to $\wedge$ under a weak condition.

Proposition 5.1. *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the* ACC *and F a proper filter of $\mathbf{L}$ and $a, b \in L$. Then the following holds:*

(i) *If $c(F)$ is closed with respect to $\wedge$, then $c(F)$ is a D-filter of F,*

(ii) *the inclusion $\overline{a} \wedge \overline{b} \subseteq \overline{a \wedge b}$ holds if and only if $x, y \in L$ and $1 \in (x^{00} \vee a^{00}) \wedge (y^{00} \vee b^{00})$ together imply $1 \in (x \wedge y)^{00} \vee (a \wedge b)^{00}$,*

(iii) *if $\overline{x} \wedge \overline{y} \subseteq \overline{x \wedge y}$ for all $x, y \in c(F)$, then $c(F)$ is closed with respect to $\wedge$ and hence a D-filter of $\mathbf{L}$.*

Proof. (i) If $a \in D$, then $a^{00} = 1$ and hence $\overline{a} = L$ whence $L = L \wedge 1 \subseteq L \wedge F = \overline{a} \wedge F \subseteq L$, i.e., $\overline{a} \wedge F = L$ showing $a \in c(F)$. Therefore $D \subseteq c(F)$. If $b \in c(F)$, $c \in L$ and $b \leq c$, then $b^{00} \leq_1 c^{00}$ and hence $L = \overline{b} \wedge F \subseteq \overline{c} \wedge F \subseteq L$ which implies $\overline{c} \wedge F = L$, i.e., $c \in c(F)$.

(ii) We have

$$\overline{a} \wedge \overline{b} = \{x \wedge y \mid x, y \in L \text{ and } 1 \in (x^{00} \vee a^{00}) \wedge (y^{00} \vee b^{00})\},$$
$$\overline{a \wedge b} = \{x \in L \mid 1 \in x^{00} \vee (a \wedge b)^{00}\}.$$

(iii) If $\overline{x} \wedge \overline{y} \subseteq \overline{x \wedge y}$ for all $x, y \in c(F)$, then for all $x, y \in c(F)$, we have

$$L = L \wedge L = (\overline{x} \wedge F) \wedge (\overline{y} \wedge F) = (\overline{x} \wedge \overline{y}) \wedge (F \wedge F) = (\overline{x} \wedge \overline{y}) \wedge F \subseteq \overline{x \wedge y} \wedge F \subseteq L$$

and hence $\overline{x \wedge y} \wedge F = L$, which means nothing else than $x \wedge y \in c(F)$. □

The condition in (ii) of Proposition 5.1 holds for all proper filters of the lattice from Figure 2. For D-Stonean lattices, we can prove the following result.

Theorem 5.2. *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a* D*-Stonean lattice and* F *a* D*-filter of* $\mathbf{L}$*. Then* $c(F) \subseteq F$*.*

Proof. Let $a \in c(F)$. Then $\overline{a} \wedge F = L$ and hence there exists some $b \in \overline{a}$ and some $c \in F$ with $b \wedge c = a$. Now

$$\overline{a} = \{x \in L \mid 1 \in x^{00} \vee a^{00}\} = \{x \in L \mid x \vee a \in D\}$$

and hence $b = b \vee (b \wedge c) = b \vee a \in D \subseteq F$ which implies $a = b \wedge c \in F$. □

Note that the condition of $\mathbf{L}$ being D-Stonean is only sufficient but not necessary.

***Example* 6.** The filter F_a of the non-Stonean lattice visualized in Figure 2 is not coherent since $a \in F_a \setminus c(F_a)$, but $c(F_a) = F_e \subseteq F_a$.

Combining Lemma 5.1 and Theorem 5.2, we obtain the following corollary.

Corollary 5.2.1. *If* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *is a* D*-Stonean lattice and* F *a* D*-filter of* $\mathbf{L}$ *satisfying* $\overline{x} \cup F = L$ *for all* $x \in F$*, then* F *is coherent.*

Now we turn our attention to the so-called closed filters. A *filter* F of $\mathbf{L}$ is called *closed* if it is a closed subset of L as defined in the introduction, i.e., if $\overline{\overline{F}} = F$. Of course, $F \subseteq \overline{\overline{F}}$ holds for every filter F of $\mathbf{L}$.

***Example* 7.** The filter F_a of the D-Stonean lattice visualized in Figure 5 is closed and coherent since $\overline{\overline{F_a}} = \overline{\{b, c, 1\}} = F_a$ and $c(F_a) = F_a$.

Corollary 5.2.2. *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a bounded lattice satisfying the* ACC *and* $A \subseteq L$*. Then the following holds:*

(i) $D \subseteq \overline{A}$,

(ii) $\overline{A} = L$ *if and only if* $A \subseteq D$,

(iii) *every closed filter of* $\mathbf{L}$ *is a* D*-filter.*

Proof. (i) and (ii) We use Lemma 2.1.
(iii) This follows from (i). □

***Example* 8.** The filter F_e of the lattice depicted in Figure 2 is both coherent and closed since $c(F_e) = F_e$ and $\overline{\overline{F_e}} = \overline{F_0} = F_e$, but its subfilter F_f is neither coherent nor closed because $e \in c(F_f) \setminus F_f$ and $\overline{\overline{F_f}} = \overline{F_0} = F_e \neq F_f$.

Remark 4. Since the set of closed subsets of L is closed under arbitrary intersections, the same is true for the set of closed filters, and the latter forms a complete lattice with respect to inclusion. Moreover, for every filter F of $\mathbf{L}$ there is a smallest closed filter of $\mathbf{L}$ including it.

6. Maximal, prime and median filters

Prime filters and maximal filters play an important role in the theory of rings, but also for semirings, especially for bounded distributive lattices. The question is if similar notions can be developed for lattices which need not be distributive. The aim of this section is to answer this question positively. We establish connections between maximal, prime, median and coherent filters as well as D-filters.

First let us recall some well-known classes of lattice filters. We call a *filter* F of a lattice $(L, \vee, \wedge)$

- *proper* if $F \neq L$,
- *maximal* if it is a maximal proper filter,
- *prime* if $x, y \in L$ and $x \vee y \in F$ imply $x \in F$ or $y \in F$.

It is well known (see, e.g., [7]) that every maximal filter of a distributive lattice is prime. Unfortunately, this does not hold for non-distributive lattices. For example, the filter F_a of the lattice in Figure 5 is maximal, but it is not prime since $b \vee c = 1 \in F_a$, but $b, c \notin F_a$. However, we can prove the following result.

Theorem 6.1. *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a bounded lattice satisfying the* ACC *and* F *a proper filter of* $\mathbf{L}$. *Then the following holds:*

(i) *F is maximal if and only if $x^0 \cap F \neq \emptyset$ for all $x \in L \setminus F$,*

(ii) *if F is maximal, then F is a D-filter,*

(iii) *if F is maximal and $(x \vee y) \wedge z = 0$ for all $x, y \in L \setminus F$ and all $z \in F$ with $x \wedge z = y \wedge z = 0$, then F is prime.*

Proof. (i) Assume F to be maximal and $a \in L \setminus F$. Then $G := \{x \in L \mid \text{there exists some } f \in F \text{ with } f \wedge a \leq x\}$ is a filter of $\mathbf{L}$ strictly including F. Since F is maximal, we conclude $G = L$. Hence $0 \in G$, i.e., there exists some $g \in F$ with $g \wedge a \leq 0$, it means $g \wedge a = 0$. Therefore there exists some $b \in a^0$ with $g \leq b$. Since F is a filter, we conclude $b \in F$ and hence $b \in a^0 \cap F$ whence $a^0 \cap F \neq \emptyset$. Conversely, assume $x^0 \cap F \neq \emptyset$ for all $x \in L \setminus F$. Suppose F not to be maximal. Then there exists a proper filter H of $\mathbf{L}$ strictly including F. Let $c \in H \setminus F$. Then $c^0 \cap F \neq \emptyset$. Let $d \in c^0 \cap F$. Then $c, d \in H$. Since H is a filter of $\mathbf{L}$, we have $0 = c \wedge d \in H$ and hence $H = L$, a contradiction. This shows that F is maximal.

(ii) Assume F to be maximal, but not being a D-filter. Then there exists some $a \in D \setminus F$. According to (i) $0 \cap F = a^0 \cap F \neq \emptyset$ and hence $0 \in F$ which implies $F = L$, a contradiction. Therefore F is a D-filter.

(iii) Assume F to be maximal and $(x \vee y) \wedge z = 0$ for all $x, y \in L \setminus F$ and all $z \in F$ with $x \wedge z = y \wedge z = 0$. Suppose F to be not prime. Then there exist some $a, b \in L \setminus F$ with $a \vee b \in F$. According to (i), we have $a^0 \cap F, b^0 \cap F \neq \emptyset$. Let $f \in a^0 \cap F$ and $g \in b^0 \cap F$ and put $h := f \wedge g$. Then $h \in F$ and $a \wedge f = b \wedge g = 0$ and hence $a \wedge h = b \wedge h = 0$. By the above assumption, we conclude $0 = (a \vee b) \wedge h \in F$ which implies that F is not proper which is a contradiction. Hence F is prime. $\square$

Of course, condition (iii) of Theorem 6.1 holds for any distributive lattice. However, the following example shows that it may be satisfied also in a non-distributive lattice.

***Example* 9.** The non-distributive lattice visualized in Figure 6 satisfies the condition in (iii) of Theorem 6.1 for the maximal proper filter F_a. In accordance with this theorem, this filter is prime.

Let us now adopt a certain modification of the concept of a median filter defined in [5].

Definition 3. Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC. A filter F of $\mathbf{L}$ is called *median* if it is maximal and if for each $x \in F$ there exists some $y \in L \setminus F$ with $1 \in x^{00} \vee y^{00}$.

***Example* 10.** The filter F_a of the non-Stonean lattice $(L, \vee, \wedge, 0, 1)$ visualized in Figure 3 is median, closed and coherent since $b \in L \setminus F_a$, $1 \in x^{00} \vee b^{00}$ for every $x \in F_a$ and $\overline{\overline{F_a}} = \overline{\{b, c, d, f, g, 1\}} = F_a$. The filter F_a of the non-Stonean lattice $(L, \vee, \wedge, 0, 1)$ depicted in Figure 4 is prime, median,

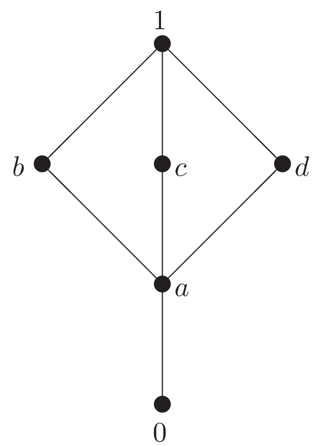

FIGURE 6. Non-distributive lattice

closed and coherent since $c \in L \setminus F_a$, $1 \in x^{00} \vee c^{00}$ for all $x \in F_a$ and $\overline{\overline{F_a}} = \overline{F_c} = F_a$. We list all closed filters of the lattices from Figure 1 to Figure 5:

Fig.	closed filters
1	F_0, F_d, F_f, F_1
2	F_0, F_e
3	F_0, F_a, F_d, F_f
4	F_0, F_a, F_c, F_i
5	F_0, F_b, F_c, F_d, F_1

Next, we present several basic properties of proper, prime and median filters.

THEOREM 6.2. *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a bounded lattice satisfying the* ACC*,* F *a proper filter of* $\mathbf{L}$ *and* $a \in L$*. Then the following holds:*

(i) *If* $x^{00} \vee y^{00} \leq_1 (x \vee y)^{00}$ *for all* $x, y \in L$*,* F *is a prime D-filter and* $a \in L \setminus F$*, then* $\overline{a} \subseteq F$*;*

(ii) *if* $x^{00} \vee y^{00} \leq_1 (x \vee y)^{00}$ *for all* $x, y \in L$*,* F *is a median prime D-filter and* $a \in F$*, then* $\overline{\overline{a}} \subseteq F$*;*

(iii) *if* $a \in F$*, then* $a^0 \nsubseteq F$*;*

(iv) *if* $\mathbf{L}$ *is D-Stonean and* F *a prime D-filter of* $\mathbf{L}$*, then* $a \in F$ *if and only if* $a^0 \nsubseteq F$*.*

Proof. (i) Assume $x^{00} \vee y^{00} \leq_1 (x \vee y)^{00}$ for all $x, y \in L$ and let F be a prime D-filter of $\mathbf{L}$, $a \in L \setminus F$ and $b \in \overline{a}$. Then $1 \in a^{00} \vee b^{00} \leq_1 (a \vee b)^{00}$. Since $(a \vee b)^{00}$ is an antichain, we have $(a \vee b)^{00} = 1$ which implies $(a \vee b)^0 = (a \vee b)^{000} = 0$, i.e., $a \vee b \in D$. Since F is a D-filter, we have $D \subseteq F$ and therefore $a \vee b \in F$. Because $a \notin F$ and F is prime, we conclude $b \in F$ proving $\overline{a} \subseteq F$.

(ii) Assume $x^{00} \vee y^{00} \leq_1 (x \vee y)^{00}$ for all $x, y \in L$ and let F be a median D-filter of $\mathbf{L}$, $a \in F$ and $b \in \overline{\overline{a}}$. Since F is median there exists some $c \in L \setminus F$ with $1 \in a^{00} \vee c^{00}$. Hence $c \in \overline{a} = \overline{\overline{\overline{a}}} \subseteq \overline{b}$. Since $c \notin F$, we have $\overline{c} \subseteq F$ according to (i) and we obtain $b \in \overline{\overline{b}} \subseteq \overline{c} \subseteq F$ proving $\overline{\overline{a}} \subseteq F$.

(iii) If we would have $a^0 \subseteq F$ for some $a \in F$, then $0 = a \wedge a^0 \subseteq F$ and hence $F = L$ contradicting the assumption that F is proper.

(iv) Suppose $\mathbf{L}$ to be D-Stonean, F to be a prime D-filter of $\mathbf{L}$ and $a^0 \nsubseteq F$. Then there exists some $b \in a^0 \setminus F$. Since $\mathbf{L}$ is D-Stonean, we have $1 \in a^{00} \vee b^{00}$ and hence $a \vee b \in D \subseteq F$ according to the assumption that F is a D-filter. Because F is prime and $b \notin F$, we obtain $a \in F$. The converse implication follows from (iii). ☐

The next result illuminates the relationship between coherent and median filters.

Theorem 6.3. *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a bounded lattice satisfying the* ACC *and* F *a maximal filter of* $\mathbf{L}$. *Then the following holds:*

(i) *If* F *is coherent, then it is median,*

(ii) *if* $\overline{F} \not\subseteq F$, *then* F *is median,*

(iii) *if* $F = \overline{L \setminus F}$, *then* F *is median.*

P r o o f. Let $a \in F$.

(i) Assume F to be coherent. Then $a \in c(F)$, i.e., $\overline{a} \wedge F = L$ and hence there exists some $b \in \overline{a}$ and some $c \in F$ with $b \wedge c = 0$. Because of $b \in \overline{a}$, we have $1 \in a^{00} \vee b^{00}$. Now $b \in F$ would imply $0 = b \wedge c \in F$ and hence $F = L$ contradicting the maximality of F. Hence $b \notin F$ showing F to be median.

(ii) Assume $\overline{F} \not\subseteq F$. Then there exists some $b \in \overline{F} \setminus F$. Hence $1 \in a^{00} \vee b^{00}$ and $b \in L \setminus F$ proving F to be median.

(iii) Since $a \in \overline{L \setminus F}$, we have $1 \in a^{00} \vee x^{00}$ for all $x \in L \setminus F$. Because F is proper there exists such an x. □

M. Sambasiva Rao [5] has found that median prime filters of a distributive pseudocomplemented lattice have an interesting property. The same applies to more general lattices.

Proposition 6.4. *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a* D*-Stonean lattice,* F *a median prime filter of* $\mathbf{L}$ *and* $a, b \in L$. *Then the following holds:*

(i) *If* F *is a* D*-filter of* $\mathbf{L}$, $a \in F$ *and* $\overline{a} = \overline{b}$, *then* $b \in F$,

(ii) *if* $a \vee b \in F$, *then there exists some* $c \in L \setminus F$ *such that* $\{a \vee c, b \vee c\} \cap D \neq \emptyset$.

P r o o f. (i) Assume F to be a D-filter of $\mathbf{L}$, $a \in F$ and $\overline{a} = \overline{b}$. Since F is median there exists some $c \in L \setminus F$ with $1 \in a^{00} \vee c^{00}$. Hence $c \in \overline{a} = \overline{b}$, i.e., $1 \in b^{00} \vee c^{00}$. Since $\mathbf{L}$ is D-Stonean and F is a D-filter, we have $b \vee c \in D \subseteq F$. Because F is prime, we conclude $b \in F$.

(ii) Suppose $a \vee b \in F$. Since F is prime, we have $a \in F$ or $b \in F$. Because F is median, in the first case we see that there exists some $c \in L \setminus F$ with $1 \in a^{00} \vee c^{00}$. Since $\mathbf{L}$ is D-Stonean, we obtain $a \vee c \in D$. The case $b \in F$ can be treated analogously. □

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REFERENCES

[1] BALBES, R.—HORN, A.: *Stone lattices*, Duke Math. J. **37** (1970), 537–545.

[2] CHAJDA, I.—LÄNGER, H.: *Algebraic structures formalizing the logic with unsharp implication and negation*, Logic J. IGPL **33** (2025), 36–48.

[3] FRINK, O.: *Pseudo-complements in semi-lattices*, Duke Math. J. **29** (1962), 505–514.

[4] SAMBASIVA RAO, M.: *Median prime ideals of pseudo-complemented distributive lattices*, Arch. Math. (Brno) **58** (2022), 213–226.

[5] SAMBASIVA RAO, M.: *Median filters of pseudo-complemented distributive lattices*, Discuss. Math. General Algebra Appl. **44** (2024), 147–161.

[6] SPEED, T. P.: *On Stone lattices*, J. Austral. Math. Soc. **9** (1969), 297–307.

[7] SZÁSZ, G.: *Introduction to Lattice Theory*, Akadémiai Kiadó, Budapest, 1963.

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* *Department of Algebra and Geometry*
Palacký University Olomouc
Faculty of Science
17. listopadu 12
CZ–771 46 Olomouc
CZECH REPUBLIC
E-mail: ivan.chajda@upol.cz

** *Department of Computer Science*
Palacký University Olomouc
Faculty of Science
17. listopadu 12
CZ–771 46 Olomouc
CZECH REPUBLIC
E-mail: miroslav.kolarik@upol.cz

*** *Institute of Discrete Mathematics and Geometry*
Faculty of Mathematics and Geoinformation
TU Wien
Wiedner Hauptstraße 8-10
A–1040 Vienna
AUSTRIA

Department of Algebra and Geometry
Palacký University Olomouc
Faculty of Science
17. listopadu 12
CZ–771 46 Olomouc
CZECH REPUBLIC
E-mail: helmut.laenger@tuwien.ac.at