c-IDEALS IN COMPLEMENTED POSETS

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Abstract. In their recent paper on posets with a pseudocomplementation denoted by $*$ the first and the third author introduced the concept of a $*$-ideal. This concept is in fact an extension of a similar concept introduced in distributive pseudocomplemented lattices and semilattices by several authors, see References. Now we apply this concept of a c-ideal (dually, c-filter) to complemented posets where the complementation need neither be antitone nor an involution, but still satisfies some weak conditions. We show when an ideal or filter in such a poset is a c-ideal or c-filter, and we prove basic properties of them. Finally, we prove the so-called separation theorems for c-ideals. The text is illustrated by several examples.

Keywords: complemented poset; antitone involution; ideal; filter; ultrafilter; c-ideal; c-filter; c-condition; separation theorem

MSC 2020: 06A11, 06C15

1. Introduction

The concept of a δ-ideal was introduced recently in pseudocomplemented distributive lattices and semilattices in [6], [7] and [8]. Later on, it was extended to pseudocomplemented posets under the name $*$-ideal in [3] where $*$ means the pseudocomplementation on the poset in question. The authors used also some results taken from their previous paper (see [2]).

It turns out that in complemented posets the aforementioned concepts of ideals and filters play also an important role and this fact motivated us to extend our study to complemented posets where the complementation need not be an antitone involution in all cases. Several such examples are included in the paper. Hence, we

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obtain the concept of a c-ideal. Our goals are to present several basic properties of c-ideals and prove the so-called separation theorems showing that under certain assumptions for every ideal I and certain filters F of a complemented poset such that $I \cap F = \emptyset$ there exists a c-ideal J including I with $J \cap F = \emptyset$.

In what follows we collect the concepts used throughout the paper. Some of them are familiarly known and can be found e.g. in [1].

Let $(P, \leqslant)$ be a poset and $a, b \in P$ and $A, B \subseteq P$. We define

$$L(A) := \{x \in P\colon\ x \leqslant y \text{ for all } y \in A\}$$

and

$$U(A) := \{x \in P\colon\ y \leqslant x \text{ for all } y \in A\},$$

the so-called *lower cone* and *upper cone* of A, respectively. Instead of $L(\{a\})$, $L(\{a,b\})$, $L(A \cup \{a\})$, $L(A \cup B)$ and $L(U(A))$ we simply write $L(a)$, $L(a,b)$, $L(A,a)$, $L(A,B)$ and $LU(A)$, respectively. Analogously we proceed in similar cases.

Consider a *bounded* poset $(P, \leqslant, 0, 1)$, i.e., a poset with the bottom element 0 and top element 1. A unary operation $'$ on P is called a *complementation* if for every $x \in P$ there exist $x \vee x'$ and $x \wedge x'$ and, moreover, $x \vee x' = 1$ and $x \wedge x' = 0$. If $'$ is a complementation on $(P, \leqslant, 0, 1)$ then $(P, \leqslant, {}', 0, 1)$ is called a *complemented poset*. Clearly, $0' = 0 \vee 0' = 1$ and $1' = 1 \wedge 1' = 0$.

A unary operation $'$ on a poset $(P, \leqslant)$ is called an *involution* if it satisfies the identity $x'' \approx x$ and it is called *antitone* if $x, y \in P$ and $x \leqslant y$ together imply $y' \leqslant x'$.

Let us note that if $'$ is an antitone involution on a given poset $(P, \leqslant)$ then $'$ satisfies the identity $x''' \approx x'$ and we can use *De Morgan's laws*, i.e., $(L(x,y))' = U(x',y')$ and $(U(x,y))' = L(x',y')$ for all $x, y \in P$.

Let $\mathbf{P} = (P, \leqslant, {}', 0, 1)$ be a complemented poset. For a subset A of P we put

$$A' := \{x'\colon\ x \in A\}, \quad A_0 := \{x \in P\colon\ x' \in A\}.$$

It is an easy observation that $A \subseteq B$ implies $A_0 \subseteq B_0$. An element a of P is called *Boolean* if $a'' = a$. For nonempty subsets I and F of P we define:

- ▷ I is called an *ideal* of $\mathbf{P}$ if $L(x) \subseteq I$ and $U(x,y) \cap I \neq \emptyset$ for all $x, y \in I$,
- ▷ F is called a *filter* of $\mathbf{P}$ if $U(x) \subseteq F$ and $L(x,y) \cap F \neq \emptyset$ for all $x, y \in F$.

Lemma 1. *Let $\mathbf{P} = (P, \leqslant, {}', 0, 1)$ be a complemented poset with antitone complementation satisfying $x \leqslant x''$ for all $x \in P$, let $a \in P$, and assume that every ideal of $\mathbf{P}$ containing a contains a''. Then a is a Boolean element of $\mathbf{P}$.*

P r o o f. Every of the following statements implies the next one: $a \in L(a)$ and $L(a)$ is an ideal of $\mathbf{P}$, $a'' \in L(a)$, $a \leqslant a'' \leqslant a$, $a'' = a$, a is a Boolean element of $\mathbf{P}$. □

Let I be an ideal of $\mathbf{P}$. Then

- ▷ I is called *proper* if $I \neq P$,
- ▷ I is called *maximal* if I is a maximal proper ideal of $\mathbf{P}$,
- ▷ I is called a *prime ideal* if $I \neq P$ and $\{x, y\} \cap I \neq \emptyset$ for all $x, y \in P$ with $L(x, y) \subseteq I$.

For a filter F of $\mathbf{P}$,

- ▷ F is called *proper* if $F \neq P$,
- ▷ F is called an *ultrafilter* if F is a maximal proper filter of $\mathbf{P}$,
- ▷ F is called a *prime filter* if $F \neq P$ and $\{x, y\} \cap F \neq \emptyset$ for all $x, y \in P$ with $U(x, y) \subseteq F$.

Now we define our main concepts. An ideal I of $\mathbf{P}$ is called a *c-ideal* if there exists some filter F of $\mathbf{P}$ with $F_0 = I$. A filter F of $\mathbf{P}$ is called a *c-filter* if there exists some ideal I of $\mathbf{P}$ with $I_0 = F$. Namely, these concepts enable to separate ideals from filters in complemented posets as expressed in the *separation theorems* (Theorems 16 and 20).

It is evident that the concept of an ideal and a filter are dual to each other. Clearly, 0, 1 are Boolean elements of a complemented poset $\mathbf{P}$. Further, $\{0\}$ and P are ideals of $\mathbf{P}$, and $0 \in I$ for all ideals I of $\mathbf{P}$. Moreover, an ideal I of $\mathbf{P}$ is proper if and only if $1 \notin I$. Dual statements hold for filters.

The ideals of the form $L(a)$ with $a \in P$ are called *principal ideals*, and the filters of the form $U(a)$ with $a \in P$ *principal filters*.

Let us repeat the following useful results from [3].

Lemma 2 ([3])**.** *Let $\mathbf{P} = (P, \leqslant)$ be a poset and I an ideal of $\mathbf{P}$. Then the following are equivalent:*

(i) *I is a prime ideal of $\mathbf{P}$.*

(ii) *$P \setminus I$ is a prime filter of $\mathbf{P}$.*

(iii) *$P \setminus I$ is a filter of $\mathbf{P}$.*

Moreover, the mapping $I \mapsto P \setminus I$ is a bijection from the set of all prime ideals of $\mathbf{P}$ to the set of all prime filters of $\mathbf{P}$.

Recall that a poset $(P, \leqslant)$ satisfies the *Ascending Chain Condition* if it has no infinite ascending chains. The *Descending Chain Condition* is defined dually.

Lemma 3 ([3])**.** *Let $\mathbf{P} = (P, \leqslant)$ be a poset. Then the following statements hold.*

(i) *Every ideal of $\mathbf{P}$ is principal if and only if $\mathbf{P}$ satisfies the Ascending Chain Condition.*

(ii) *Every filter of* $\mathbf{P}$ *is principal if and only if* $\mathbf{P}$ *satisfies the Descending Chain Condition.*

Note that if P is finite then $(P, \leqslant)$ satisfies the Ascending Chain Condition as well as the Descending Chain Condition.

Lemma 4. *Let* $\mathbf{P} = (P, \leqslant, ', 0, 1)$ *be a complemented poset,* $a \in P$ *and* I *a proper ideal of* $\mathbf{P}$. *Then either* $a \notin I$ *or* $a' \notin I$.

Proof. If $a, a' \in I$ then $\{1\} \cap I = U(a, a') \cap I \neq \emptyset$, i.e., $1 \in I$ which implies $I = P$ contradicting the assumption of I being a proper ideal of $\mathbf{P}$. □

In the following we demonstrate the role of I_0.

Proposition 5. *Let* $\mathbf{P} = (P, \leqslant, ', 0, 1)$ *be a complemented poset,* I *an ideal, and* F *a filter of* $\mathbf{P}$. *Then the following are equivalent:*

(i) I *is proper,*
(ii) $I_0 \neq P$,
(iii) $I \cap I_0 = \emptyset$.

Moreover, also the following are equivalent:

(iv) F *is proper,*
(v) $F_0 \neq P$,
(vi) $F \cap F_0 = \emptyset$.

Proof. (i) ⇒ (ii): $I_0 = P$ would imply $0 \in I_0$ and hence $1 = 0' \in I$, i.e., $I = P$, a contradiction.

(ii) ⇒ (iii): Suppose $I \cap I_0 \neq \emptyset$. Then there exists some $a \in I \cap I_0$. Hence $a, a' \in I$. Since I is an ideal of $\mathbf{P}$ we conclude $\{1\} \cap I = U(a, a') \cap I \neq \emptyset$, i.e., $1 \in I$ and therefore $I = P$ whence $I_0 = P$, a contradiction.

(iii) ⇒ (i): $I = P$ would imply $I \cap I_0 = P \cap P = P \neq \emptyset$, a contradiction.

The proof for filters follows by duality. □

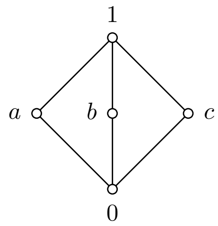
Lemma 6. *Let* $\mathbf{P} = (P, \leqslant, ', 0, 1)$ *be a complemented poset,* I *a c-ideal, and* F *a c-filter of* $\mathbf{P}$. *Then the following hold:*

(i) *Assume* $x' \leqslant x'''$ *for all* $x \in P$. *Then* $I'' \subseteq I$.
(ii) *Assume* $x''' \leqslant x'$ *for all* $x \in P$. *Then* $F'' \subseteq F$.

Proof. (i) Since I is a c-ideal of $\mathbf{P}$ there exists some filter F of $\mathbf{P}$ with $F_0 = I$ and every of the following statements implies the next one: $x \in I$, $x \in F_0$, $x' \in F$, $x''' \in F$, $x'' \in F_0$, $x'' \in I$.

(ii) It follows by duality. □

E x a m p l e 7. Consider the complemented poset $\mathbf{P} = (P, \leqslant, ', 0, 1)$ shown in Figure 1 and the table for its complementation:



x	0	a	b	c	1
x'	1	b	c	b	0
x''	0	c	b	c	1
x'''	1	b	c	b	0

Figure 1. A bounded poset.

We have:

▷ Boolean elements: 0, b, c, 1,

▷ maximal ideals: $L(a)$, $L(b)$, $L(c)$,

▷ ultrafilters: $U(a)$, $U(b)$, $U(c)$,

▷ **P** has neither prime ideals nor prime filters,

▷ c-ideals: $L(0)$, $L(b)$, $L(1)$,

▷ c-filters: $U(0)$, $U(b)$, $U(1)$.

The complementation defined by the above table is antitone and satisfies the identity $x''' \approx x'$, but it is not an involution. We have

$$L(0)'' = L(0),$$
$$L(a)'' = L(c) \nsubseteq L(a),$$
$$L(b)'' = L(b),$$
$$L(c)'' = L(c),$$
$$L(1)'' = \{0, b, c, 1\} \subseteq L(1).$$

Lemma 8. *Let $(P, \leqslant, ', 0, 1)$ be a complemented poset satisfying $x''' \approx x'$ and let $a \in P$ and $A \subseteq P$. Then $a \in A_0$ if and only if $a'' \in A_0$.*

P r o o f. The following are equivalent: $a \in A_0$, $a' \in A$, $a''' \in A$, $a'' \in A_0$. □

If we suppose that the complementation is antitone then we can formulate easy assumptions ensuring that every ideal is a c-ideal and every filter is a c-filter.

Theorem 9. *Let $\mathbf{P} = (P, \leqslant, ', 0, 1)$ be a complemented poset with antitone complementation, F a filter, and I an ideal of $\mathbf{P}$. Then the following hold:*

(i) *Assume $x \leqslant x''$ for all $x \in P$. Then F_0 is a c-ideal of $\mathbf{P}$.*

(ii) *Assume $x'' \leqslant x$ for all $x \in P$. Then I_0 is a c-filter of $\mathbf{P}$.*

P r o o f. (i) Let $a, b \in F_0$. Since $0' = 1 \in F$, we have $0 \in F_0$ and hence $F_0 \neq \emptyset$. Because $a', b' \in F$ and F is a filter of $\mathbf{P}$ there exists some $c \in L(a', b') \cap F$. Since $c \leqslant c''$ we have $c'' \in F$, i.e., $c' \in F_0$. Together $c' \in U(a'', b'') \cap F_0 \subseteq U(a, b) \cap F_0$, which proves $U(a, b) \cap F_0 \neq \emptyset$. If $d \in P$, $e \in F_0$ and $d \leqslant e$ then $e' \in F$ and $e' \leqslant d'$ and hence $d' \in F$, i.e., $d \in F_0$. Altogether, F_0 is an ideal and F a filter of $\mathbf{P}$ and hence F_0 is a c-ideal of $\mathbf{P}$.

(ii) It follows by duality. □

If we assume that the complementation is an antitone involution, which is a rather strong assumption, we can state the following result.

Corollary 10. *Let $\mathbf{P} = (P, \leqslant, ', 0, 1)$ be a complemented poset with an antitone involution, I an ideal, and F a filter of $\mathbf{P}$. Then*

(i) *I_0 is a filter of $\mathbf{P}$, $I = (I_0)_0$ and hence I is a c-ideal of $\mathbf{P}$.*
(ii) *F_0 is an ideal of $\mathbf{P}$, $F = (F_0)_0$ and hence F is a c-filter of $\mathbf{P}$.*

P r o o f. (i) From (ii) of Theorem 9 we obtain that I_0 is a filter of $\mathbf{P}$. Moreover, for $x \in P$ the following are equivalent: $x \in (I_0)_0$, $x' \in I_0$, $x'' \in I$, $x \in I$. This shows $(I_0)_0 = I$. By (i) of Theorem 9 we conclude that I is a c-ideal of $\mathbf{P}$.

(ii) It follows by duality. □

R e m a r k 11. If $(P, \leqslant, ', 0, 1)$ is a complemented poset whose complementation is an antitone involution and $a \in P$ then $L(a)_0 = U(a')$ since the following are equivalent: $x \in L(a)_0$, $x' \in L(a)$, $x' \leqslant a$, $a' \leqslant x$, $x \in U(a')$. According to (ii) of Corollary 10 we have $L(a) = U(a')_0$.

Now we formulate a condition which will be helpful for the separation theorems.

Definition 12. A subset A of a complemented poset $(P, \leqslant, ', 0, 1)$ satisfies the *c-condition* if for every $x \in P$ the set A contains exactly one of x and x'.

Recall from Lemma 4 that for a proper ideal I of $\mathbf{P}$ and for arbitrary $x \in P$ the situation $x, x' \in I$ is impossible.

The following lemma shows that ideals satisfying the c-condition can be found among prime ideals.

Lemma 13. *Let $\mathbf{P} = (P, \leqslant, ', 0, 1)$ be a complemented poset, I a prime ideal, and F a prime filter of $\mathbf{P}$. Then I and F satisfy the c-condition.*

P r o o f. If $a \in P$ then $L(a, a') = \{0\} \subseteq I$ and hence $\{a, a'\} \cap I \neq \emptyset$. The rest follows by duality. □

Recall from [5] that a *poset* $(P, \leqslant)$ is called *distributive* if it satisfies one of the following equivalent conditions:

$$L(U(x,y),z) = LU(L(x,z),L(y,z)) \quad \forall\, x,y,z \in P,$$
$$U(L(x,y),z) = UL(U(x,z),U(y,z)) \quad \forall\, x,y,z \in P.$$

The next result illuminates the role of distributivity of the poset $\mathbf{P}$ for the c-condition both for ideals and filters of $\mathbf{P}$.

Theorem 14. *Let* $\mathbf{P} = (P, \leqslant, ', 0, 1)$ *be a complemented poset,* I *an ideal, and* F *a filter of* $\mathbf{P}$*. Consider the following statements:*

(i) I *satisfies the c-condition,*
(ii) I *is a maximal ideal of* $\mathbf{P}$*,*
(iii) $(P, \leqslant)$ *is distributive,*
(iv) $\bigcup\{LU(a,i)\colon\ i \in I\}$ *is an ideal of* $\mathbf{P}$ *for all* $a \in P \setminus I$*,*
(v) F *satisfies the c-condition,*
(vi) F *is an ultrafilter of* $\mathbf{P}$*,*
(vii) $\bigcup\{UL(a,f)\colon\ f \in F\}$ *is a filter of* $\mathbf{P}$ *for all* $a \in P \setminus F$*.*

Then
▷ (i) $\Rightarrow$ (ii),
▷ ((ii), (iii) *and* (iv)) $\Rightarrow$ (i),
▷ (v) $\Rightarrow$ (vi),
▷ ((iii), (vi) *and* (vii)) $\Rightarrow$ (v).

Proof. (i) $\Rightarrow$ (ii): Since I satisfies the c-condition, $I \neq P$. Let J be an ideal of $\mathbf{P}$ strictly including I. Then there exists some $a \in J \setminus I$. Because of (i) we conclude $a' \in I$, which implies $a' \in J$. Since J is an ideal of $\mathbf{P}$ we have $\{1\} \cap J = U(a,a') \cap J \neq \emptyset$ and hence $1 \in J$ which implies $J = P$.

((ii), (iii) and (iv)) $\Rightarrow$ (i): Let $a \in P \setminus I$. Because of (iv), $K := \bigcup\{LU(a,i)\colon\ i \in I\}$ is an ideal of $\mathbf{P}$ including $I \cup \{a\}$ and hence strictly including I. Since I is a maximal ideal of $\mathbf{P}$ we conclude $K = P$. Hence $1 \in K$ and therefore there exists some $i \in I$ with $1 \in LU(a,i)$. This means $U(a,i) = \{1\}$. Using (iii) we have

$$\begin{aligned} i \in U(i) &= U(0,i) = U(L(a,a'),i) = UL(U(a,i),U(a',i)) \\ &= UL(1,U(a',i)) = ULU(a',i) = U(a',i) \subseteq U(a'), \end{aligned}$$

i.e., $a' \leqslant i$. Since $i \in I$ and I is an ideal of $\mathbf{P}$ we conclude $a' \in I$. The rest follows by duality. □

The following lemma shows that the condition (iv) of Theorem 14 is satisfied automatically if the poset $\mathbf{P}$ in question is a join-semilattice. Dually, the condition (vii) of Theorem 14 holds if $\mathbf{P}$ is a meet-semilattice.

Lemma 15. *Let* $\mathbf{P} = (P, \leqslant)$ *be a join-semilattice,* I *an ideal of* $\mathbf{P}$*, and* $a \in P$*. Then* $\bigcup\{LU(a,i)\colon\ i \in I\}$ *is an ideal of* $\mathbf{P}$*.*

P r o o f. If $b, c \in I$ then there exists some $d \in U(b,c) \cap I$ whence $b \vee c \leqslant d \in I$ and hence $b \vee c \in I$. Put $J := \bigcup\{LU(a,i)\colon\ i \in I\}$. Then

$$J = \bigcup\{LU(a \vee i)\colon\ i \in I\} = \bigcup\{L(a \vee i)\colon\ i \in I\}.$$

If $b, c \in J$ then there exist $j, k \in I$ with $b \leqslant a \vee j$ and $c \leqslant a \vee k$ and hence

$$b \vee c \in U(b,c) \cap L(a \vee (j \vee k)) \subseteq U(b,c) \cap J.$$

Since J is downward closed it is an ideal of $\mathbf{P}$. □

Now we are ready to prove our first separation theorem.

Theorem 16 (First Separation Theorem)**.** *Let* $\mathbf{P} = (P, \leqslant, ', 0, 1)$ *be a complemented poset with antitone complementation satisfying* $x \leqslant x''$ *for every* $x \in P$*, let* I *be an ideal, and* F *a filter of* $\mathbf{P}$ *satisfying the c-condition and* $I \cap F = \emptyset$*. Then there exists some c-ideal* J *of* $\mathbf{P}$ *with* $I \subseteq J$ *and* $J \cap F = \emptyset$*.*

P r o o f. By Theorem 9, F_0 is a c-ideal of $\mathbf{P}$. Since $I \neq \emptyset$ and $I \cap F = \emptyset$ we have $F \neq P$ which implies $F \cap F_0 = \emptyset$ according to Proposition 5. Since F satisfies the c-condition, we have $F \cup F_0 = P$. Thus $I \subseteq P \setminus F = F_0$. This shows that one can take $J := F_0$. □

E x a m p l e 17. Consider the following bounded posets $\mathbf{P}_1$ and $\mathbf{P}_2$ which are not lattices:

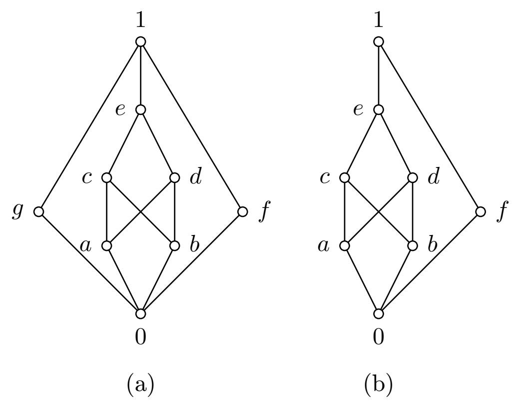


Figure 2. Bounded posets.

If the unary operation $'$ is defined by

x	0	a	b	c	d	e	f	g	1
x'	1	f	f	f	f	f	e	c	0
x''	0	e	e	e	e	e	f	f	1

then $\mathbf{P}_1 = (P_1, \leqslant, ', 0, 1)$ and $\mathbf{P}_2 = (P_2, \leqslant, ', 0, 1)$ are complemented posets and the complementation is antitone, but not an involution. Moreover, it can be seen from the table that all elements x except the element g satisfy the inequality $x \leqslant x''$. We have

$\mathbf{P}_1$: Boolean elements: 0, e, f, 1,
maximal ideals: $L(e)$, $L(f)$, $L(g)$,
ultrafilters: $U(a)$, $U(b)$, $U(f)$, $U(g)$,
$\mathbf{P}_1$ has neither prime ideals nor prime filters,
c-ideals: $L(0)$, $L(e)$, $L(f)$, $L(1)$,
c-filters: $U(0)$, $U(g)$, $U(1)$,
$\mathbf{P}_1$ has no filter satisfying the c-condition.

$\mathbf{P}_2$: Boolean elements: 0, e, f, 1,
maximal ideals: $L(e)$, $L(f)$,
ultrafilters: $U(a)$, $U(b)$, $U(f)$,
prime ideals: $L(e)$,
prime filters: $U(f)$,
c-ideals: $L(0)$, $L(e)$, $L(f)$, $L(1)$,
c-filters: $U(0)$, $U(f)$, $U(1)$,
filters satisfying the c-condition: $U(f)$.
$\mathbf{P}_1$ has no filter satisfying the c-condition.

The situation for $\mathbf{P}_2$ is different. Here $U(f)$ is the unique filter satisfying the c-condition. For every ideal $I \subseteq L(e)$ we have $I \cap U(f) = \emptyset$ and taking $J := L(e)$, J is a c-ideal of $\mathbf{P}_2$ with $I \subseteq J$ and $J \cap U(f) = \emptyset$.

Example 17 shows that the implication (ii) $\Rightarrow$ (i) in Theorem 14 does not hold in general.

The next result is in fact another version of the First Separation Theorem where we use the result from Lemma 13.

Corollary 18. *Let $\mathbf{P} = (P, \leqslant, ', 0, 1)$ be a complemented poset with antitone complementation satisfying $x \leqslant x''$ for every $x \in P$, let I be an ideal, and F a prime filter of $\mathbf{P}$ and assume $I \cap F = \emptyset$. Then there exists some c-ideal J of $\mathbf{P}$ with $I \subseteq J$ and $J \cap F = \emptyset$.*

P r o o f. From Lemma 13 we conclude that F satisfies the c-condition. Now apply Theorem 16. □

E x a m p l e 19. Consider the complemented poset $\mathbf{P}$ visualized in Figure 3:

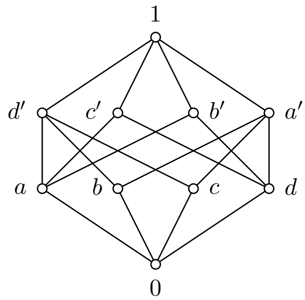


Figure 3. A complemented poset with an antitone involution.

Evidently, $\mathbf{P}$ is neither a lattice nor distributive since

$$L(U(a,b),c) = L(d',c) = L(c) \neq L(0) = LU(0) = LU(L(a,c),L(b,c)).$$

We have

Boolean elements: 0, a, b, c, d, a', b', c', d', 1,
maximal ideals: $L(a')$, $L(b')$, $L(c')$, $L(d')$,
ultrafilters: $U(a)$, $U(b)$, $U(c)$, $U(d)$,
prime ideals: $L(a')$, $L(d')$,
prime filters: $U(a)$, $U(d)$,
c-ideals: $L(0)$, $L(a)$, $L(b)$, $L(c)$, $L(d)$, $L(a')$, $L(b')$, $L(c')$, $L(d')$, $L(1)$,
c-filters: $U(0)$, $U(a)$, $U(b)$, $U(c)$, $U(d)$, $U(a')$, $U(b')$, $U(c')$, $U(d')$, $U(1)$.

The complementation is an antitone involution. Thus the assumption $x \leqslant x''$ from Corollary 18 is satisfied. If we consider the prime filter $F = U(d)$ and the ideal $I = L(a)$ then $I \cap F = \emptyset$ and there exists a c-ideal $J = L(d')$ with $I \subseteq J$ and $J \cap F = \emptyset$.

Now we can formulate our second separation theorem for distributive complemented posets satisfying the Descending Chain Condition, in particular for finite distributive complemented posets. Here we need not assume that the complementation satisfies $x \leqslant x''$ nor that the filter in question satisfies the c-condition.

Theorem 20 (Second Separation Theorem)**.** *Let $\mathbf{P} = (P, \leqslant, ', 0, 1)$ be a distributive complemented poset satisfying the Descending Chain Condition, let I be an ideal, and F an ultrafilter of $\mathbf{P}$. Then there exists some $g \in F$ with $U(g) = F$. Now assume that $x \wedge g$ exists for every $x \in P \setminus F$ and that $I \cap F = \emptyset$. Then there exists some c-ideal J of $\mathbf{P}$ with $I \subseteq J$ and $J \cap F = \emptyset$.*

P r o o f. Let $a \in P \setminus F$ and $b \in P$. Since $\mathbf{P}$ satisfies the Descending Chain Condition we have that F is principal according to Lemma 3, i.e., there exists some $g \in F$ with $U(g) = F$. Because of $L(a, g) \subseteq L(a, f)$ for all $f \in F$ we have $UL(a, f) \subseteq UL(a, g)$ for all $f \in F$ and hence $\bigcup\{UL(a, f)\colon f \in F\} = UL(a, g) = UL(a \wedge g) = U(a \wedge g)$ which shows that $\bigcup\{UL(a, f)\colon f \in F\}$ is a filter of $\mathbf{P}$. According to Theorem 14 we conclude that F satisfies the c-condition. Moreover,

$$\begin{aligned} L(b'') &= L(1, b'') = L(U(b', b), b'') \\ &= LU(L(b', b''), L(b, b'')) = LU(0, L(b, b'')) \\ &= LU(0, L(b'', b)) = LU(L(b', b), L(b'', b)) \\ &= L(U(b', b''), b) = L(1, b) = L(b) \end{aligned}$$

and hence $b'' = b$, i.e., the complementation is an involution. Now apply Theorem 16. □

E x a m p l e 21. The complemented poset $\mathbf{P}$ depicted in Figure 4 is distributive, but not a semilattice.

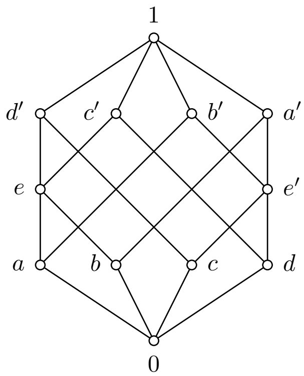


Figure 4. A distributive complemented poset.

We have

Boolean elements: $0, a, b, c, d, e, a', b', c', d', e', 1$,

maximal ideals: $L(a'), L(b'), L(c'), L(d')$,

ultrafilters: $U(a), U(b), U(c), U(d)$,

c-ideals: $L(0), L(a), L(b), L(c), L(d), L(e), L(a'), L(b'), L(c'), L(d'), L(e'), L(1)$.

One can easily check that for the ultrafilter $F = U(b)$ we have $x \wedge b = 0$ for all $x \in P \setminus F$. Thus the assumptions of Theorem 20 are satisfied. Now if I denotes the ideal $L(e')$ then $I \cap F = \emptyset$ and there exists a c-ideal $J = L(b')$ with $I \subseteq J$ and $J \cap F = \emptyset$.

C l o s i n g R e m a r k s . It is well-known (see e.g. [4]) that for a distributive lattice $\mathbf{L} = (L, \vee, \wedge)$ there holds the separation theorem saying that if F is a filter of $\mathbf{L}$ then every ideal I of $\mathbf{L}$ being disjoint to F can be extended to some prime ideal P of $\mathbf{L}$ being disjoint to F. Using this fact, a Stone space can be introduced on $\mathbf{L}$ whose subbase for open sets is given by the sets $r(a) := \{P\colon\ P$ is a prime ideal of $\mathbf{L}$ and $a \notin P\}$, $a \in L$. One cannot assume that such a strong result would be valid in our case where only posets with involution are considered. However, it can encourage the readers to try to define a certain structure on a poset which could be close to the Stone space where the "open" subsets can be generated by means of c-ideals.

Let us note that the assumption of the Descending Chain Condition in Theorem 20 can be weakened by assuming that every nonempty subset of P contains at least one minimal element, but we do not see an advantage of this approach.

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