Fuzzy rule based systems

Universal approximation property

Approximation of functions $f : \mathbb{R}^n \to \mathbb{R}$

1. metric $d : X \times X \to \mathbb{R}$

   satisfies:

   a) $d(x,x) = 0$ (pseudometric), $d(x,y) = 0$ iff $x = y$
   b) $d(x,y) = d(y,x)$
   c) $d(x,z) \leq d(x,y) + d(y,z)$

   examples:

   $X = \mathbb{R}$, $d(x,y) = |x - y|$
   $X = \mathbb{R}^n$, $d((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sqrt{(x_1-y_1)^2 + \ldots + (x_n-y_n)^2}$ (Euclidean)
   $d((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = sup_{i=1}^n |x_i - y_i|$
   $d((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{i=1}^n |x_i - y_i|$

2. distance of functions $f, g : K \subseteq \mathbb{R}^n \to \mathbb{R}$
   $d(f,g) = sup_{x \in X} |f(x) - g(x)|$ - sup-distance, a metric of $\{f : K \to \mathbb{R}\}$

X, d - metric on X, $A \subseteq B \subseteq X$

A is **dense** in B if $\forall \varepsilon > 0\ \forall x \in B\ \exists y \in A : d(x,y) \leq \varepsilon$

Ex.: $\mathbb{R} = X = B$, $A = Q \subseteq B$: Q is dense in $\mathbb{R}$

In the following we restrict our attention to $\mathcal{F}(M,L,D)$ - a class of fuzzy rule based systems given by the following restrictions:

M …is the class of all membership functions (fuzzy sets A_{ij}, B_i) that are allowed:

there is some fuzzy set μ_0 …continuous and positive on some subinterval of $\mathbb{R}$, a fuzzy set $\mu \in M$ iff $\mu(x) = \mu_0(ax + b), a, b \in \mathbb{R}$

Ex.: if $\mu_0 = \underline{\bigwedge}\mu_0 \rightsquigarrow M = \{$all triangular fuzzy sets$\}$
$\mu_0 = \underline{\bigcap}$ (bell-shaped) $\rightsquigarrow M = \{$all bell-shaped fuzzy sets$\}$

L …consists of all continuous t-norms and t-conorms

$B'(y) = \vee_j\ A_{j1}(x_1) \otimes \ldots \otimes A_{jn}(x_n) \otimes B_j(y)$, $A'_1, \ldots, A'_n$ are singletons $\{1/x_1\}, \ldots, \{1/x_n\}$

t-norm $\triangle : [0,1]^2 \to [0,1]$, commutative, associative, monotonous, $x \triangle 1 = x$

t-conorm $\bigtriangledown : [0,1]^2 \to [0,1]$, commutative, associative, monotonous, $x \bigtriangledown 0 = x$

t-norms $\sim$ conjunctions, t-conorms $\sim$ disjunctions

$x \bigtriangledown y = (xy, 1)$, $\bigtriangledown$ - Lukasiewicz t-conorm

$B'(y) = \bigtriangledown_j \; A_{j1}(x_1) \bigtriangleup \ldots \bigtriangleup A_{jn}(x_n) \bigtriangleup B_j(y)$

D ... defuzzyfication procedure, which transforms a given fuzzy set μ into a real number in such a way, that if $\mu(x) = 0$ outside some interval $(\alpha, \beta) \subseteq \mathbb{R}$, then $D(\mu) \in [\alpha, \beta]$

Ex.: COG is one possible choice of D

$K \subseteq \mathbb{R}^n$ ... a compact subset

Ex.: hyperrectangles $K = [a_1, b_1] \times \ldots \times [a_n, b_n], a_i \leq b_i$

K is **compact** with respect to some metric d (for ex. sup-metric) if for each covering $C = \{A_i \mid A_i \subseteq \mathbb{R}^n, A_i \; is \; open, i \in I\}$ of K ($\bigcup_i C \supseteq K$) there is a finite $I' \subseteq I$ such that $C' = \{A_i \mid i \in I'\}$ is a covering of K as well.

$C(K)$ ... class of all continuous functions $f : K \to \mathbb{R}$

$\mathcal{F}(M, L, D)$ ... for each particular choice of $A_{ij} \in M, B_i \in M$, and each choice of $\bigtriangleup, \bigtriangledown \in L$, and for D this represents a function $F : K \to \mathbb{R}$.

$F(x_1, \ldots, x_n) = D(B'), B'(y) = \bigtriangledown_j \; A_{j1}(x_1) \bigtriangleup \ldots \bigtriangleup B_i(y)$

$\mathcal{F}(M, L, D) = \{F \mid F \; can \; be \; obtained \; like \; F(x_1, \ldots, x_n)\}$

Theorem: For each compact $K \subseteq \mathbb{R}^n$, $\mathcal{F}(M, L, D)$ is a dense subset of $C(K)$.

$$\forall f \in C(K) \; \forall \varepsilon > 0 \; \exists F \in \mathcal{F}(M, L, D) : d(f, F) \leq \varepsilon.$$