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Invited lecture

SSIE Dept., T. J. Watson School, Binghamton University—SUNY
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Outline

1. introduction and motivation
2. existing approaches and methods
   - drawing by hand
   - layer approach and force directed approach
   - new approaches for lattices
   - level method (my own)
   - nested diagrams
3. geometric heuristic
   - geometric method: intro
   - rules of parallelograms and lines
   - evaluation & comparison
   - open questions and problems
   - forthcoming research
4. software for drawing lattices
   - existing software
   - our software: LatVis and EllenaArt
Introduction and motivation

- **important role of lattices** in computer science and applied math. (data analysis, information retrieval, machine learning, intelligent systems, industrial engineering, ...)
- information usually represented by hierarchical structures, often described by graphs or lattices
- need to visualize (draw) lattices – (commonly) by drawings of **Hasse (upward, linear) diagram**

= **oriented graph** \( \langle V, E \rangle \), where nodes \( V = \) lattice elements and edges \( E = \) lattice cover relation \( \prec \)

+ **drawing conventions:**
  1. node for \( x \) is drawn (as a dot or a circle) below node for \( y \) \( \iff x < y \)
  2. nodes for \( x \) and \( y \) are connected by a straight line \( \iff x \prec y \) (i.e. no lines for transitive edges and no cycles)
Introduction and motivation (con’t)

Problem:
We can draw many different Hasse diagram drawings of (the Hasse diagram of) a given lattice.
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Task:
Arrange the nodes and lines of the lattice diagram drawing in order to achieve the **best visual quality, readability**, etc. . . . and do it fast and automatically.
Lattice drawing

- evolved from **graph drawing** (well-elaborated)
- several **subjective human aesthetics criteria**: minimizing line crossings, eliminating line breaks (produced by e.g. layer approach), maximizing conflict distance, angle between incident lines, symmetries, compactness etc.

= optimization criteria used when drawing by hand
- however, **what makes the best readable diagram?**
- criteria are often contradictory and lead to computationaly difficult (NP-complete) problems

→ **heuristic approaches** to drawing, but the task remains difficult (how to precisely mathematize the criteria?)

= several automated drawing methods, but none universal, the best
- drawing by hand is traditionally better, but slow and tedious
- automated drawing by computer is at least a good starting point

Note: We also have criteria for labelling diagram nodes (e.g. in concept lattices, depends on application area).
How large lattices one can draw by a computer?

Up to about a hundred of elements.

There is no point in drawing whole larger lattices.
Lattice drawing (con’t)

*How large lattices one can draw by a computer?*

Up to about a hundred of elements.

There is no point in drawing whole larger lattices.
Lattice drawing (con’t)

→ divide and draw substructures (only)
Existing approaches and methods

Presumptions:
- drawing a lattice top-down, i.e. downwards from the top element
- (usually) no need of initial drawing, the input is the underlying order relation only

Drawing by hand ("intuitively")

= arranging nodes of lower neighbors of actual node followed by placing nodes of infima of the neighbors or further neighbors and so on

problem: concrete placement of nodes \(\rightarrow\) "intuitively" in iterations

EXERCISE:

Draw the Hasse diagram of the following lattice:

<table>
<thead>
<tr>
<th>elements:</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower neighbors:</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>b</td>
<td>d</td>
<td>c</td>
</tr>
</tbody>
</table>
Existing approaches and methods (con’t)

Drawing graphs

G. Di Battista, P. Eades, R. Tamassia, I.–G. Tollis: 

- many elaborate methods
- some can be used and adapted to draw Hasse diagrams

Layer approach

= nodes in **layers** based on their distance from the top node, **sorted** to achieve minimal line crossings

- steps:
  1. layer assignment (determining y-coordinates): longest path layering, Coffman-Graham layering, usage of broken lines
  2. crossing reduction: layer-by-layer node sweep by solving two-layer crossing problem (using sorting, averaging, linear programming methods etc.)
  3. x-coordinate assignment: e.g. straightening broken lines
Existing approaches and methods (con’t)

Layer approach – example
Existing approaches and methods (con’t)

Force directed approach

- balancing imaginary repulsive and attractive forces between nodes and lines, based on spring models
- resulting drawing depends on initial arrangement of nodes
- works in iterations, results are “unpredictable” (methods are quite difficult to adapt to Hasse diagrams)
- several variants: force placement, edge-edge repulsion and others
Existing approaches and methods (con’t)

New approaches for lattices

- **attribute additivity** = node position $\vec{p}$ determined by the node positions $\vec{x}$ of greater *inf-irreducible elements* $x \in M(V)$

\[
\vec{p} := \vec{a} + \sum_{x \in M(V) \mid p \leq x} \vec{x}
\]

- combinations of methods, e.g. **hybrid method** = layer + attribute additivity
- special methods, due to visualizing **concept lattices** in FCA, e.g. **grid method** = projection of the lattice placed into a multidimensional grid onto a suitable plane
Existing approaches and methods (con’t)

Level method (my own)

- similar to drawing by hand and layer approach
- solves concrete placement of nodes minimizing line crossings and maximizing compactness of the diagram
- arranging nodes of lower neighbors of nodes from previous level/layer in a new level/layer, **evenly** below the nodes and **ordered** by ordering of the nodes (based on non-decreasing numbering)
Existing approaches and methods (con’t)

Present methods:

- produce quite good, readable, diagrams, however, for smaller lattices only (around 30 elements at max) – because of global optimization?
- attribute additivity based win
- performance of drawing or interactive altering is not a problem (up to a hundred of elements)

Nested diagrams

- separated parts of the whole diagram drawn as nested diagrams, bunches of parallel lines replaced by a single (or double) line
- used by Wille et al. in FCA
- interesting idea, well-readable drawings (even for tens of elements), but unusual and not very used
- problem: identifying parts – needs structural analysis of the lattice (hard!)
Existing approaches and methods (con’t)

Nested diagrams – example
Geometric method: intro

- proposed by Wille et al. in 1989, re-introduced in 1993


- originally developed for drawing **concept lattices** in FCA

- (originally) based on a **geometric interpretation** of the lattice:
  - finding a best possible diagram layout with the help of a **geometrical diagram** (auxiliary picture when drawing by hand)
Geometric method: intro (con’t)

Geometrical diagram

- a look at the 3D visualization of the lattice from its top element:

  - lower neighbor of top element → circled label
  - element with one upper neighbor → circled label partly covered by the neighbor’s label
  - elements with two upper neighbors → circled label partly covered by a line connecting the neighbors’ labels
  - elements with three upper neighbors → label inside a (sloped filled) triangle connecting neighbors’ labels

  … (the top and the least element are omitted)

**EXERCISE:** Draw the geometrical diagram for the following lattice:

<table>
<thead>
<tr>
<th>elements</th>
<th>upper neighbors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6 3 5 11 3 9</td>
</tr>
<tr>
<td>2 1</td>
<td>7 1 12 4 7 16</td>
</tr>
<tr>
<td>3 2</td>
<td>8 7 13 8 12 17</td>
</tr>
<tr>
<td>4 1</td>
<td>9 2 7 14 5 9 12</td>
</tr>
<tr>
<td>5 2 4</td>
<td>10 9 15 10 14</td>
</tr>
</tbody>
</table>
Geometric method: intro (con’t)

Geometrical diagram – example
Geometric method: intro (con’t)

→ drawing of the Hasse diagram by recognizing and realizing geometrical patterns

- using mainly two geometric rules:
  1. rule of parallelograms
  2. rule of lines
Definition (Rule of parallelograms)

A new node should be placed in such a way that the node together with some already placed nodes and lines forms a **parallelogram** (the geometric shape with parallel lines, e.g. diamond or rhomboid).

- rather a general rule, looks simple, but
- immediate problem: vague formulation “some already placed nodes”

→ most commonly selected nodes = of **pair of upper neighbors** + their **common upper neighbor** (supremum)
Rule of lines

**Definition (Rule of lines)**

A new node should be placed on a **prolonged line** connecting some already placed nodes, preferably at the same distance as the distance between the nodes.

- again, rather a general simple looking rule, but
- again, immediate problem: vague formulation “some already placed nodes”

→ most commonly selected nodes = of a **single upper neighbor + its upper neighbor**
Geometric method (con’t)

Application of rules

- results in many parallel lines and regular geometric shapes in the diagram (fulfilling aesthetics criteria) → good level of readability even for medium sized lattices (30–50 elements)

= essential part of discovering regular geometrical shapes, structures and patterns

- aim: best possible overall geometric regularity of the diagram
Comparison of methods

hand drawn  level  layer  geometric
Open questions and problems (1)

However, application of rules is not a straightforward action at all, there are many decision points in a new node placement:

Rule of parallelograms

- there can be more than one pair of upper neighbors → the supremum of a pair should be as “close” as possible (ideally an upper neighbor) = make the parallelogram as small as possible
- if the supremum is not an upper neighbor, should we place the new node in the middle bellow its upper neighbors? (violating the rule, e.g. the least element)
- ...
Open questions and problems (2)

Rule of lines
- there can be more than one upper neighbor of the single upper neighbor → ?
- *should we place the new node straight bellow its upper neighbor?* (violating the rule)
- ...

The final choices in the the decisions often depends on suggested placements for *other elements*.

complex heuristic on (semi)local optimization problems
Open questions and problems (3)

Arrangement of co-atoms (inf-irreducibles)

- important **starting point** in drawing the diagram!
- or whenever the rule of lines suggests the same location
  → we can place them on an imaginary horizontal line, a parabola, using a force directed approach, . . .

- **which order of the elements?** → should place elements which have more lower neighbors aside (since we will need a space for the neighbors)
- the same applies for elements with equal suggested placement (by the rules or other way)
EXERCISE:
Draw the Hasse lattice diagram for the geometrical diagram from the previous exercise.
**Geometric method (con’t)**

**EXERCISE:**
Draw the Hasse lattice diagram for the geometrical diagram from the previous exercise.
Forthcoming research

- no further papers on the geometric method (since the initial two or three in 1989-93)!
- we find the method very promising
- we have explored (some of) the problems and proposed (some) ideas and solutions

→ we are developing a new method for automated lattice drawing inspired by and further refining the geometric method

Main idea: (intermediate) logical diagram description

- similar to geometrical diagram used in the original geometric method, but more general
- contains: constraints of Hasse diagram, space constraints, node placements suggested by the geometric rules and other principles, evaluations of the suggested placements, . . .
- obtaining final diagram = heuristic solutions to both local and global optimization problems aimed at producing the best possible diagram
Software for drawing lattices: requirements

...to test, evaluate, develop, fine-tune and ... of course use the drawing method (by end users)

Requirements

- producing the **best possible diagram drawing** of any given lattice (of course)
- **fine-tuning** the diagram drawing by hand (or additional methods): moving nodes or parts, grid aligning, hiding/folding parts, zooming, rotating, etc.
- **displaying parts** of the lattice: lower/upper neighbors/cones, infs/sups, paths, etc.
- **exporting** the diagram drawing (or part of it) to the picture in a paper, look customization
- editing the lattice (underlying order relation)
- ...
- usable by **end user** (i.e. graphical)
Software for drawing lattices: concept lattice

Software for FCA (Formal Concept Analysis)
- original purpose: a tool for FCA
- drawing the resulting concept lattice only

Toscana, Anaconda, Diagram
- FCA tools from the (former) FCA group of TU Darmstadt, Germany
- not available anymore

ToscanaJ
- Java reimplementation of Toscana, open source, part of Tockit FCA framework (http://tockit.sourceforge.net)
- lattice diagram viewer only, displaying lower/upper cones, look customization
- drawing methods?, nested diagrams
- http://toscanaj.sourceforge.net/
Software for drawing lattices: concept lattice (con’t)

Galicia

- (rich) FCA platform
- layer and force directed approaches (including 3D variants)
- node moving, rotating, zooming
- written in Java, open source
- http://www.iro.umontreal.ca/~galicia/

Concept Explorer

- FCA tool
- layer, force directed (including the one in LatDraw) and grid approaches
- node moving, displaying lower/upper cones, grid aligning, zooming
- written in Java, open source
- http://conexp.sourceforge.net/
Software for drawing lattices: concept lattice (con’t)

**GaloisExplorer**

- FCA tool, lattice diagram viewer only
- force directed approach? (3D variant)
- features?
- written in C++ (MS Windows, Apple Mac OS), free software
- [http://galoisexplorer.sourceforge.net](http://galoisexplorer.sourceforge.net)

**JaLaBa**

- online FCA web application
- uses LatDraw for drawing the concept lattice
- [http://maarten.janssenweb.net/jalaba/JaLaBA.pl](http://maarten.janssenweb.net/jalaba/JaLaBA.pl)
Software for drawing lattices: any lattice

Software for drawing (arbitrary) lattices

LatDraw

- online Java applet or Java application by Ralph Freese from the University of Hawaii
- used by several other tools (e.g. JaLaBa, JavaMath plugin to Maple), source upon request (API for free)
- combined layer and force directed approach
- lattice diagram viewer only, rotating

Software for drawing lattices: any lattice (con’t)

GAP – poset visualization part

- online (only) Java applet by Peter Jipsen from the Chapman University, CA, open source
- (simplified) combined layer and force directed approach
- limited node moving
- [http://www1.chapman.edu/~jipsen/gap/posets.html](http://www1.chapman.edu/~jipsen/gap/posets.html)

Conclusion

- some FCA tools, limited in lattice drawing features
- **JUST TWO** Java applets for drawing arbitrary lattices!, yet very limited
- there is quite a lot of graph drawing tools ([http://www.graphviz.org](http://www.graphviz.org) [http://graphdrawing.org](http://graphdrawing.org)), but none of them with lattice Hasse diagram drawing feature
Software for drawing lattices: any lattice (con’t)

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Our software for drawing lattices

**LatVis** (Jan Outrata, 2003)

- lattice and poset editor and visualizer
- developed with my MSc. thesis at Dept. **Computer Science**, **Palacký University, Czech Rep.** in 2003
- layer approach, (author’s own) **level** and (original) **geometric** methods
- fine-tuning: user selected node moving (with coordinates displayed), node hiding
- displaying and selecting parts: lower/upper neighbors/cones, infs/sups, min/max, paths
- editing: copy&paste, undo/redo
- export: Metapost, Encapsulated Postscript (and PDF), look customization, saving to a XML document
- written in C++ (MS Windows, GNU/Linux), free software (GNU GPL)

http://phoenix.inf.upol.cz/~outrata/latvis/
Our software for drawing lattices (con’t)

EllenaArt (Lukas Hostalek, 2007)

- lattice and poset drawing tool
- developed with MSc. thesis of Lukas Hostalek at Dept. Computer Science, Palacký University, Czech Rep. in 2007
- force directed approach (three variants), (author’s own) heuristic and (original) geometric methods
- fine-tuning: node moving (with coordinates displayed), grid aligning, zooming
- export: Encapsulated Postscript and PDF, look customization, saving to a XML document
- written in Java, open source
Thank you!

**LatVis**  
http://phoenix.inf.upol.cz/~outrata/latvis/

**EllenaArt**  