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Existence principle for higher-order nonlinear differential equations with state-dependent impulses via fixed point theorem

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Dedicated to Professor Ivan Kiguradze for his merits in mathematical sciences

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Abstract

The paper provides an existence principle for a general boundary value problem of the form $\sum_{j=0}^{n} a_j(t)u^{(j)}(t) = h(t, u(t), \dots, u^{(n-1)}(t))$, a.e. $t \in [a, b] \subset \mathbb{R}$, $\ell_k(u, u', \dots, u^{(n-1)}) = c_k$, $k = 1, \dots, n$, with the state-dependent impulses $u^{(j)}(t+) - u^{(j)}(t-) = J_{ij}(u(t-), u'(t-), \dots, u^{(n-1)}(t-))$, where the impulse points t are determined as solutions of the equations $t = \gamma_i(u(t-), u'(t-), \dots, u^{(n-2)}(t-))$, $i = 1, \dots, p$, $j = 0, \dots, n-1$. Here, $n, p \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{R}$, the functions a_j/a_n, $j = 0, \dots, n-1$, are Lebesgue integrable on $[a, b]$ and h/a_n satisfies the Carathéodory conditions on $[a, b] \times \mathbb{R}^n$. The impulse functions J_{ij}, $i = 1, \dots, p$, $j = 0, \dots, n-1$, and the barrier functions γ_i, $i = 1, \dots, p$, are continuous on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. The functionals ℓ_k, $k = 1, \dots, n$, are linear and bounded on the space of left-continuous regulated (*i.e.* having finite one-sided limits at each point) on $[a, b]$ vector functions. Provided the data functions h and J_{ij} are bounded, transversality conditions which guarantee that each possible solution of the problem in a given region crosses each barrier γ_i at the unique impulse point τ_i are presented, and consequently the existence of a solution to the problem is proved.

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1 Introduction

In this paper we are interested in the nonlinear ordinary differential equation of the nth-order ($n \geq 2$) with state-dependent impulses and general linear boundary conditions on the interval $[a, b] \subset \mathbb{R}$. Studies of real-life problems with state-dependent impulses can be found *e.g.* in [1–6]. Here we consider the equation

$$\sum_{j=0}^{n} a_j(t)u^{(j)}(t) = h\big(t, u(t), \dots, u^{(n-1)}(t)\big), \quad \text{a.e. } t \in [a, b], \tag{1}$$

subject to the impulse conditions

$$\left.\begin{aligned} &u^{(j)}(t+) - u^{(j)}(t-) = J_{ij}(u(t-), u'(t-), \dots, u^{(n-1)}(t-)), \\ &\text{where } t = \gamma_i(u(t-), u'(t-), \dots, u^{(n-2)}(t-)) \\ &\text{for } i = 1, \dots, p, j = 0, \dots, n-1, \end{aligned}\right\} \tag{2}$$

and the linear boundary conditions

$$\ell_k\big(u, u', \dots, u^{(n-1)}\big) = c_k, \quad k = 1, \dots, n. \tag{3}$$

In what follows we use this notation. Let $k, m, n \in \mathbb{N}$. By $\mathbb{R}^{m\times n}$ we denote the set of all matrices of the type $m \times n$ with real valued coefficients. Let A^T denote the transpose of $A \in \mathbb{R}^{m\times n}$. Let $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ be the set of all n-dimensional column vectors $c = (c_1, \dots, c_n)^T$, where $c_i \in \mathbb{R}$, $i = 1, \dots, n$, and $\mathbb{R} = \mathbb{R}^{1\times 1}$. By $\mathbb{C}(\mathbb{R}^n; \mathbb{R}^m)$ we denote the set of all mappings $x : \mathbb{R}^n \to \mathbb{R}^m$ with continuous components. By $\mathbb{L}^\infty([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{L}^1([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{G}_{\mathrm{L}}([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{AC}([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{BV}([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{C}^k([a,b]; \mathbb{R}^{m\times n})$, we denote the sets of all mappings $x : [a,b] \to \mathbb{R}^{m\times n}$ whose components are, respectively, essentially bounded functions, Lebesgue integrable functions, left-continuous regulated functions, absolutely continuous functions, functions with bounded variation and functions with continuous derivatives of the kth order on the interval $[a,b]$. By $\mathrm{Car}([a,b] \times \mathbb{R}^n; \mathbb{R})$ we denote the set of all functions $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfying the Carathéodory conditions on the set $[a,b] \times \mathbb{R}^n$. Finally, by χ_M we denote the characteristic function of the set $M \subset \mathbb{R}$.

Note that a mapping $u : [a,b] \to \mathbb{R}^n$ is left-continuous regulated on $[a,b]$ if for each $t \in (a,b]$ and each $s \in [a,b)$ there exist finite limits

$$u(t) = u(t-) = \lim_{\tau \to t-} u(\tau), \qquad u(s+) = \lim_{\tau \to s+} u(\tau).$$

$\mathbb{G}_{\mathrm{L}}([a,b]; \mathbb{R}^n)$ is a linear space, and equipped with the sup-norm $\|\cdot\|_\infty$ it is a Banach space (see [7, Theorem 3.6]). In particular, we set

$$\|u\|_\infty = \max_{i \in \{1,\dots,n\}} \Big(\sup_{t\in[a,b]} \big|u_i(t)\big| \Big) \quad \text{for } u = (u_1, \dots, u_n)^T \in \mathbb{G}_{\mathrm{L}}\big([a,b]; \mathbb{R}^n\big).$$

A function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the Carathéodory conditions on $[a,b] \times \mathbb{R}^n$ if

- $f(\cdot, x) : [a,b] \to \mathbb{R}$ is measurable for all $x \in \mathbb{R}^n$,
- $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,
- for each compact set $K \subset \mathbb{R}^n$ there exists a function $m_K \in \mathbb{L}^1([a,b]; \mathbb{R})$ such that $|f(t,x)| \le m_K(t)$ for a.e. $t \in [a,b]$ and each $x \in K$.

In this paper we provide sufficient conditions for the solvability of problem (1)-(3). This problem is a generalization of problems studied in the papers [8–10] which are devoted to the second-order differential equation. Other types of initial or boundary value problems for the first- or second-order differential equations with state-dependent impulses can be found in [11–19]. To get the existence results for problem (1)-(3), we exploit the paper [20] with fixed-time impulsive problems.

Here we assume that

$$\left.\begin{array}{l}
n \ge 2, \frac{a_j}{a_n} \in \mathbb{L}^1([a,b]; \mathbb{R}), j = 0, \dots, n-1, \frac{h(t,x)}{a_n(t)} \in \mathrm{Car}([a,b] \times \mathbb{R}^n; \mathbb{R}), \\
c_j \in \mathbb{R}, J_{ij} \in \mathbb{C}(\mathbb{R}^n; \mathbb{R}), \gamma_i \in \mathbb{C}(\mathbb{R}^{n-1}; \mathbb{R}), i = 1, \dots, p, j = 0, \dots, n-1, \\
\ell_k : \mathbb{G}_{\mathrm{L}}([a,b]; \mathbb{R}^n) \to \mathbb{R} \text{ is a linear bounded functional, } \textit{i.e.} \\
\ell_k(z) = K_k z(a) + \int_a^b V_k(t)\, \mathrm{d}[z(t)], z \in \mathbb{G}_{\mathrm{L}}([a,b]; \mathbb{R}^{n\times 1}), \\
\text{where } K_k \in \mathbb{R}^{1\times n}, V_k \in \mathbb{BV}([a,b]; \mathbb{R}^{1\times n}), k = 1, \dots, n, n, p \in \mathbb{N}.
\end{array}\right\} \tag{4}$$

Remark 1 The integral in formula (4) is the Kurzweil-Stieltjes integral, whose definition and properties can be found in [21]. The fact that each linear bounded functional on $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^{n\times 1})$ can be written uniquely in the form described in (4) is proved in [22]. See also [20].

Now let us define a solution of problem (1)-(3).

Definition 2 A function $u \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ is said to be a solution of problem (1)-(3) if u satisfies (1) for a.e. $t \in [a,b]$ and fulfils conditions (2) and (3).

2 Problem with impulses at fixed times

In the paper [20] we have found an operator representation to the special type of problem (1)-(3) having impulses at fixed times. This is the case that the barrier functions γ_i in (2) are constant functions, *i.e.* there exist $t_1,\dots,t_p \in \mathbb{R}$ satisfying $a < t_1 < \cdots < t_p < b$ such that

$$\gamma_i(x_0,x_1,\dots,x_{n-2}) = t_i \quad \text{for } i = 1,\dots,p, x_0,x_1,\dots,x_{n-2} \in \mathbb{R}. \tag{5}$$

In this case, each solution of the problem crosses ith barrier at same time instant $\tau_i = t_i$ for $i = 1,\dots,p$.

Note that boundary value problems for higher-order differential equations with impulses at fixed times have been studied for example in [23–31] and for delay higher-order impulsive equations in [32, 33].

Let us summarize the results of the paper [20] according to our needs. Assume that the linear homogeneous problem

$$\left.\begin{aligned} &\sum_{j=0}^{n} a_j(t)u^{(j)}(t) = 0, \quad \text{a.e. } t \in [a,b], \\ &\ell_k\big(u,u',\dots,u^{(n-1)}\big) = 0, \quad k = 1,\dots,n, \end{aligned}\right\} \tag{6}$$

has only the trivial solution. Let $\{\tilde{u}_1,\dots,\tilde{u}_n\}$ be a fundamental system of solutions of the differential equation from (6), W be their Wronski matrix and w its first row, *i.e.*

$$W(t) = \begin{pmatrix} \tilde{u}_1(t) & \cdots & \tilde{u}_n(t) \\ \tilde{u}_1'(t) & \cdots & \tilde{u}_n'(t) \\ \tilde{u}_1^{(n-1)}(t) & \cdots & \tilde{u}_n^{(n-1)}(t) \end{pmatrix}, \qquad w(t) = \big(\tilde{u}_1(t),\dots,\tilde{u}_n(t)\big), \quad t \in [a,b]. \tag{7}$$

Denote

$$\ell(W) = \Big(\ell_i\big(\tilde{u}_j,\tilde{u}_j',\dots,\tilde{u}_j^{(n-1)}\big)\Big)_{i,j=1}^{n}. \tag{8}$$

From [20, Lemma 8] (see also Chapter 3 in [34]) it follows that the unique solvability of (6) is equivalent to the condition

$$\det \ell(W) \neq 0. \tag{9}$$

Further assume (9), consider V_j, $j = 1, \dots, n$, from (4), and denote

$$V(t) = \begin{pmatrix} V_1(t) \\ V_2(t) \\ \cdots \\ V_n(t) \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_0(t)}{a_n(t)} & -\frac{a_1(t)}{a_n(t)} & -\frac{a_2(t)}{a_n(t)} & \cdots & -\frac{a_{n-1}(t)}{a_n(t)} \end{pmatrix},$$

$t \in [a, b]$ and

$$H(\tau) = -\big[\ell(W)\big]^{-1}\left(\int_\tau^b V(s)A(s)W(s)\,\mathrm{d}s \cdot W^{-1}(\tau) + V(\tau)\right), \quad \tau \in [a, b]. \tag{10}$$

If we denote by H_{ij} and ω_{ij} elements of the matrices H and W^{-1}, respectively, that is,

$$H(\tau) = \big(H_{ij}(\tau)\big)_{i,j=1}^n, \qquad W^{-1}(\tau) = \big(\omega_{ij}(\tau)\big)_{i,j=1}^n, \tag{11}$$

we can define functions g_j, $j = 1, \dots, n$, as

$$g_j(t, \tau) = \sum_{k=1}^n \tilde{u}_k(t)\big(H_{kj}(\tau) + \chi_{(\tau,b]}(t)\omega_{kj}(\tau)\big), \quad t, \tau \in [a, b]. \tag{12}$$

For each fixed $\tau \in [a, b]$ the functions $\frac{\partial^k g_j(t,\tau)}{\partial \tau^k}$, $k = 0, 1, \dots, n-1$, will be understood as right-continuous extensions at $t = a$ and left-continuous extensions at $t = \tau$ and $t = b$. In this way the Green's function of problem (6) is built (*cf.* Remark 6).

Remark 3 In order to state one of the main results of [20] we introduce the set of all functions u continuous on the intervals $[a, t_1], (t_1, t_2], \dots, (t_p, b]$, with t_i from (5), having their derivatives $u', \dots, u^{(n-1)}$ continuously extendable onto these intervals. This set is denoted by $\mathbb{PC}^{n-1}([a, b])$. For $u \in \mathbb{PC}^{n-1}([a, b])$ we define

$$u^{(k)}(a) = u^{(k)}(a+), \qquad u^{(k)}(t_i) = u^{(k)}(t_i-) \quad \text{for } k = 1, \dots, n-1, i = 1, \dots, p.$$

Equipped with the standard addition, scalar multiplication, and with the norm

$$\|u\| = \sum_{k=0}^{n-1} \big\|u^{(k)}\big\|_\infty, \quad u \in \mathbb{PC}^{n-1}\big([a, b]\big),$$

$\mathbb{PC}^{n-1}([a, b])$ forms a Banach space.

Now we are ready to state the operator representation theorem for the problem with impulses at fixed times $a < t_1 < \cdots < t_p < b$ which has the form

$$\sum_{j=0}^n a_j(t)u^{(j)}(t) = h\big(t, u(t), \dots, u^{(n-1)}(t)\big), \quad \text{a.e. } t \in [a, b], \tag{13}$$

$$u^{(j)}(t_i+) - u^{(j)}(t_i) = J_{ij}\big(u(t_i), u'(t_i), \dots, u^{(n-1)}(t_i)\big), \quad i = 1, \dots, p, j = 0, \dots, n-1, \tag{14}$$

$$\ell_k\big(u,u',\dots,u^{(n-1)}\big) = c_k, \quad k=1,\dots,n. \tag{15}$$

Theorem 4 [20, Theorem 17] *Let* (4), (9) *hold, and let* W, w, $\ell(W)$ *and* g_j, $j=1,\dots,n$ *be defined in* (7), (8), *and* (12). *Then* $u \in \mathbb{PC}^{n-1}([a,b])$ *is a fixed point of an operator* $\mathcal{H}: \mathbb{PC}^{n-1}([a,b]) \to \mathbb{PC}^{n-1}([a,b])$ *defined by*

$$\left.\begin{aligned}(\mathcal{H}u)(t) = {}& \int_a^b \frac{g_n(t,s)}{a_n(s)} h\big(s,u(s),\dots,u^{(n-1)}(s)\big)\,\mathrm{d}s \\ &+ \sum_{j=1}^{n}\sum_{i=1}^{p} g_j(t,t_i) J_{i,j-1}\big(u(t_i),\dots,u^{(n-1)}(t_i)\big) \\ &+ w(t)\big[\ell(W)\big]^{-1}(c_1,\dots,c_n)^T,\end{aligned}\right\} \tag{16}$$

$t \in [a,b]$, *if and only if* u *is a solution of problem* (13)-(15). *Moreover, the operator* $\mathcal{H}$ *is completely continuous*.

Remark 5 Let us note that the row vector

$$w(t)\big[\ell(W)\big]^{-1}$$

does not depend on the choice of a fundamental system of solutions $\tilde{u}_1,\dots,\tilde{u}_n$, but only on the data of problem (6).

Remark 6 Let us put

$$J_{ij} = 0, \quad i=1,\dots,p, j=0,\dots,n-1, \qquad c_k = 0, \quad k=1,\dots,n$$

and

$$h(t,x) = h_0(t) \in \mathbb{L}^1\big([a,b];\mathbb{R}\big) \quad \text{for } x \in \mathbb{R}^n.$$

Then the operator $\mathcal{H}$ in Theorem 4 can be written as

$$(\mathcal{H}_0 u)(t) = \int_a^b \frac{g_n(t,s)}{a_n(s)} h_0(s)\,\mathrm{d}s.$$

Theorem 4 implies that u is a fixed point of $\mathcal{H}_0$ if and only if u is a solution of the problem

$$\sum_{j=0}^{n} a_j(t)u^{(j)}(t) = h_0(t), \qquad \ell_j\big(u,u',\dots,u^{(n-1)}\big) = 0, \quad j=1,\dots,n. \tag{17}$$

Therefore a (unique) solution of problem (17) has the form

$$u(t) = \int_a^b \frac{g_n(t,s)}{a_n(s)} h_0(s)\,\mathrm{d}s,$$

and consequently $\frac{g_n(t,s)}{a_n(s)}$ is the Green's function of (6).

Remark 7 Under the assumption (9) we are allowed using (11) to define the functions

$$\left.\begin{aligned} g_j^{[1]}(t,\tau) &= \sum_{k=1}^{n} \tilde{u}_k(t) H_{kj}(\tau), \\ g_j^{[2]}(t,\tau) &= \sum_{k=1}^{n} \tilde{u}_k(t)\big(H_{kj}(\tau) + \omega_{kj}(\tau)\big) \end{aligned}\right\} \tag{18}$$

for $t, \tau \in [a,b]$, $j = 1, \dots, n$. Obviously, due to (12),

$$g_j(t,\tau) = \begin{cases} g_j^{[1]}(t,\tau) & \text{for } a \le t \le \tau \le b, \\ g_j^{[2]}(t,\tau) & \text{for } a \le \tau < t \le b, \end{cases} \tag{19}$$

for $j = 1, \dots, n$. Let us stress that $g_j^{[\nu]}$, as well as g_j, do not depend on the choice of fundamental system $\tilde{u}_1, \dots, \tilde{u}_n$, but only on the data of problem (6). The functions $g_j^{[\nu]}$ possess crucial properties for our approach. From their definition it follows that for each $\tau \in [a,b]$

$$\frac{\partial^k g_j^{[\nu]}}{\partial t^k}(\cdot,\tau) \in \mathbb{AC}\big([a,b];\mathbb{R}\big) \tag{20}$$

for $\nu = 1,2$, $j = 1, \dots, n$, $k = 0, \dots, n-1$. Moreover, for each $\nu = 1,2$, $j = 1, \dots, n$, $k = 0, \dots, n-1$, there exists a constant $C_{\nu jk} > 0$ such that

$$\left|\frac{\partial^k g_j^{[\nu]}}{\partial t^k}(t,\tau)\right| \le C_{\nu jk} \quad \text{and} \quad \left|\frac{\partial^k g_j}{\partial t^k}(t,\tau)\right| \le \max_{\nu=1,2} C_{\nu jk} \quad t,\tau \in [a,b]. \tag{21}$$

This follows from the definition of $g_j^{[\nu]}$ ($\nu = 1,2$), from the fact $w \in \mathbb{C}^{n-1}([a,b];\mathbb{R}^{1\times n})$ and from the boundedness of the matrices W^{-1} and H (*cf.* (7), (10) and (11)).

3 Transversality conditions

The most results for differential equations with state-dependent impulses concern initial value problems. Theorems about the existence, uniqueness or extension of solutions of initial value problems, and about intersections of such solutions with barriers γ_i can be found for example in [35, Chapter 5].

A different approach has to be used when boundary value problems with state-dependent impulses are discussed and boundary conditions are imposed on a solution anywhere in the interval $[a,b]$ including unknown points of impulses. This is the case of problem (1)-(3).

Our approach is based on the existence of a fixed point of an operator $\mathcal{F}$ in some set $\bar{\Omega} = \bar{\mathcal{B}}^{p+1}$ (*cf.* Lemma 12), where $\bar{\mathcal{B}} \subset \mathbb{C}^{n-1}([a,b];\mathbb{R})$ is a ball defined in (28). In order to get a fixed point, we need to prove for functions of $\bar{\mathcal{B}}$ assertions about their transversality through barriers. Such assertions are contained in Lemmas 9 and 10 and it is important that they are valid for all functions in $\bar{\mathcal{B}}$ and not only for solutions of problem (1), (2).

Remark 8 Having the lemmas about the transversality, we will prove in Section 4 the existence of a solution u of problem (1)-(3), which has the following property:

$$\left.\begin{array}{l}\text{for each } i \in \{1,\dots,p\} \text{ there exists a unique } \tau_i \in (a,b) \text{ such that}\\ \tau_i = \gamma_i(u(\tau_i-), u'(\tau_i-),\dots,u^{(n-2)}(\tau_i-)), a < \tau_1 < \cdots < \tau_p < b,\\ \text{and the restrictions } u|_{[a,\tau_1]}, u|(\tau_1,\tau_2],\dots,u|_{(\tau_p,b]} \text{ have absolutely}\\ \text{continuous derivatives of the } (n-1)\text{th order.}\end{array}\right\} \tag{22}$$

Consider real numbers K_j, $j = 0,1,\dots,n-1$, and denote

$$\mathcal{A}_n = \{(x_0,x_1,\dots,x_{n-1}) \in \mathbb{R}^n : |x_0| \le K_0,\dots,|x_{n-1}| \le K_{n-1}\}. \tag{23}$$

Now, we are ready to formulate the following transversality conditions:

$$a < \min_{\mathcal{A}_{n-1}} \gamma_1 \le \max_{\mathcal{A}_{n-1}} \gamma_{i-1} < \min_{\mathcal{A}_{n-1}} \gamma_i \le \max_{\mathcal{A}_{n-1}} \gamma_p < b, \quad i = 2,\dots,p, \tag{24}$$

$$\left.\begin{array}{l}\text{for each } i = 1,\dots,p, k = 0,\dots,n-2 \text{ there exists } L_{ik} \in [0,\infty) \text{ such that}\\ \text{if } (x_0,x_1,\dots,x_{n-2}), (y_0,y_1,\dots,y_{n-2}) \text{ belong to } \mathcal{A}_{n-1}, \text{ then}\\ |\gamma_i(x_0,x_1,\dots,x_{n-2}) - \gamma_i(y_0,y_1,\dots,y_{n-2})| \le \sum_{j=0}^{n-2} L_{ij}|x_j - y_j|,\\ i = 1,\dots,p,\end{array}\right\} \tag{25}$$

$$\sum_{j=0}^{n-2} L_{ij}K_{j+1} < 1 \quad \text{for } i = 1,\dots,p, \tag{26}$$

$$\left.\begin{array}{l}\gamma_i(x_0 + J_{i0}(x_0,\dots,x_{n-1}),\dots,x_{n-2} + J_{i,n-2}(x_0,\dots,x_{n-1}))\\ \quad \le \gamma_i(x_0,\dots,x_{n-2}), \quad (x_0,\dots,x_{n-1}) \in \mathcal{A}_n, i = 1,\dots,p.\end{array}\right\} \tag{27}$$

Let us define the set

$$\mathcal{B} = \{u \in \mathbb{C}^{n-1}([a,b];\mathbb{R}) : \|u^{(j)}\|_\infty < K_j \text{ for } j = 0,\dots,n-1\}. \tag{28}$$

Our current goal is to find a continuous functional $\mathcal{P}_i$ for $i = 1,\dots,p$, which maps each function u from $\overline{\mathcal{B}}$ to some time instant τ_i of (2).

Lemma 9 *Let* K_j, $j = 0,\dots,n-1$, L_{ik}, $i = 1,\dots,p$, $k = 0,\dots,n-2$, *be real numbers satisfying* (26), *and let* $\mathcal{A}_n$ *and* $\mathcal{B}$ *be given by* (23) *and* (28), *respectively. Finally, assume that* γ_i, $i = 1,\dots,p$, *satisfy* (24), (25), *and choose* $u \in \overline{\mathcal{B}}$. *Then the function*

$$\sigma(t) = \gamma_i(u(t), u'(t),\dots,u^{(n-2)}(t)) - t, \quad t \in [a,b], \tag{29}$$

is continuous and decreasing on $[a,b]$ *and it has a unique root in the interval* (a,b), *i.e. there exists a unique solution of the equation*

$$t = \gamma_i(u(t),\dots,u^{(n-2)}(t)). \tag{30}$$

Proof Let $u \in \overline{\mathcal{B}}$, $i \in \{1,\dots,p\}$. By (24),

$$\sigma(a) = \gamma_i(u(a), u'(a),\dots,u^{(n-2)}(a)) - a > 0,$$

$$\sigma(b) = \gamma_i\big(u(b), u'(b), \dots, u^{(n-2)}(b)\big) - b < 0$$

is valid. This together with the fact that σ is continuous shows that σ has at least one root in (a, b). Now, we will prove that σ is decreasing, by a contradiction. Let $s_1, s_2 \in (a, b)$, $s_1 < s_2$ be such that

$$\sigma(s_1) = \sigma(s_2),$$

i.e.

$$\gamma_i\big(u(s_1), \dots, u^{(n-2)}(s_1)\big) - \gamma_i\big(u(s_2), \dots, u^{(n-2)}(s_2)\big) = s_1 - s_2.$$

From (25), (26), (28), and the Mean Value Theorem we obtain

$$\begin{aligned} 0 < |s_1 - s_2| &= \big|\gamma_i\big(u(s_1), \dots, u^{(n-2)}(s_1)\big) - \gamma_i\big(u(s_2), \dots, u^{(n-2)}(s_2)\big)\big| \\ &\le \sum_{j=0}^{n-2} L_{ij}\big|u^{(j)}(s_1) - u^{(j)}(s_2)\big| \le \sum_{j=0}^{n-2} L_{ij}K_{j+1}|s_1 - s_2| < |s_1 - s_2|, \end{aligned}$$

which is a contradiction.

According to Lemma 9, we can define a functional $\mathcal{P}_i : \overline{\mathcal{B}} \to (a, b)$ by

$$\mathcal{P}_i u = \tau_i, \quad u \in \overline{\mathcal{B}}, \tag{31}$$

where τ_i is a solution of (30), *i.e.* a unique root of the function σ from Lemma 9, for $i = 1, \dots, p$. □

Lemma 10 *Let the assumptions of Lemma* 9 *be satisfied. The functionals* $\mathcal{P}_i$, $i = 1, \dots, p$, *are continuous.*

Proof Let $u_m, u \in \overline{\mathcal{B}}$, for $m \in \mathbb{N}$ such that

$$u_m \to u \quad \text{in } \mathbb{C}^{n-1}\big([a, b]; \mathbb{R}\big) \text{ as } m \to \infty. \tag{32}$$

Let us choose $i \in \{1, \dots, p\}$ and prove that $\mathcal{P}_i u_m \to \mathcal{P}_i u$ as $m \to \infty$. We denote

$$\tau = \mathcal{P}_i u, \qquad \tau_m = \mathcal{P}_i u_m, \quad m \in \mathbb{N}.$$

From Lemma 9 it follows that $\tau, \tau_m \in (a, b)$ are the unique roots of the functions

$$\sigma(t) = \gamma_i\big(u(t), \dots, u^{(n-2)}(t)\big) - t, \qquad \sigma_m(t) = \gamma_i\big(u_m(t), \dots, u_m^{(n-2)}(t)\big) - t, \quad t \in [a, b],$$

and these functions are strictly decreasing. Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$ be such that $\tau - \epsilon, \tau + \epsilon \in (a, b)$. Then $\sigma(\tau - \epsilon) > 0$ and $\sigma(\tau + \epsilon) < 0$. According to (32) we see that $\sigma_m \to \sigma$ uniformly on $[a, b]$, in particular $\sigma_m(\tau - \epsilon) \to \sigma(\tau - \epsilon)$ and $\sigma_m(\tau + \epsilon) \to \sigma(\tau + \epsilon)$ as $m \to \infty$. These facts imply that

$$\sigma_m(\tau - \epsilon) > 0 \quad \text{and} \quad \sigma_m(\tau + \epsilon) < 0 \quad \text{for a.e. } m \in \mathbb{N}.$$

From the continuity of σ_m and the Intermediate Value Theorem it follows that

$$\mathcal{P}_i u_m = \tau_m \in (\tau - \epsilon, \tau + \epsilon) = (\mathcal{P}_i u - \epsilon, \mathcal{P}_i u + \epsilon) \quad \text{for a.e. } m \in \mathbb{N},$$

which completes the proof. □

Our next step is to define an appropriate operator representation of the BVP with state-dependent impulses. The first idea would be a direct exploitation of the operator $\mathcal{H}$ from Theorem 4, putting $\mathcal{P}_i u$ in place of t_i. This is not possible for many reasons. First, each $\mathcal{P}_i$ acts on the space of functions having continuous derivatives - but we need functions having p discontinuities. Even if we would overcome this difficulty we arrive at a problem of choosing an appropriate Banach space on which $\mathcal{H}$ would be acting. According to Remark 8, we search a solution u of problem (1)-(3), which has its jumps (together with $u, u', \dots, u^{(n-1)}$) at the points $\tau_i = \mathcal{P}_i u \in (a,b)$, $i = 1, \dots, p$ (see (31)). In general, these points are different for different solutions. Consequently, such solutions have to be searched in the Banach space $\mathbb{G}_L([a,b];\mathbb{R}^n)$. But then there is a difficulty with the continuity of such operator. In fact the operator $\mathcal{H}$ from (16) having $\mathcal{P}_i u$ in place of t_i is not continuous in the space $\mathbb{G}_L([a,b];\mathbb{R}^n)$ (*cf.* Remark 6.2 and Example 6.3 in [36]).

Therefore, we choose the way here, which we have developed in our joint papers [8–10]. The main idea of our approach lies in representing the solution u of problem (1)-(3) by an ordered $(p+1)$-tuple $(u_1, \dots, u_{p+1}) \in [\mathbb{C}^{n-1}([a,b];\mathbb{R})]^{p+1}$ as follows:

$$u(t) = \begin{cases} u_1(t), & t \in [a, \mathcal{P}_1 u_1], \\ u_2(t), & t \in (\mathcal{P}_1 u_1, \mathcal{P}_2 u_2], \\ \dots & \dots \\ u_{p+1}(t), & t \in (\mathcal{P}_p u_p, b]. \end{cases} \tag{33}$$

Consequently, we work with the space

$$X = \left[\mathbb{C}^{n-1}\left([a,b];\mathbb{R}\right)\right]^{p+1}$$

equipped with the norm

$$\left\|(u_1, \dots, u_{p+1})\right\| = \sum_{i=1}^{p+1} \sum_{j=0}^{n-1} \left\|u_i^{(j)}\right\|_\infty \quad \text{for } (u_1, \dots, u_{p+1}) \in X.$$

It is well known that X is a Banach space.

4 Main results

Let us turn our attention to problem (1)-(3) with state-dependent impulses under the assumptions (4) and (9). In our approach we will make use of the tools introduced in the previous sections.

In addition we assume

$$\left.\begin{array}{l} \text{there exists } m \in \mathbb{L}^1([a,b];\mathbb{R}), A_{ij} \in \mathbb{R} \text{ such that} \\ \left|\frac{h(t,x)}{a_n(t)}\right| \le m(t) \text{ for a.e. } t \in [a,b] \text{ and all } x \in \mathbb{R}^n, \\ |J_{ij}(x)| \le A_{ij} \text{ for each } i = 1, \dots, p, j = 0, \dots, n-1. \end{array}\right\} \tag{34}$$

Consider $c_1, \dots, c_n$ from (3), w from (7) and $\ell(W)$ from (8), and denote

$$M = \int_a^b m(t)\,\mathrm{d}t, \qquad c_0 = (c_1, \dots, c_n)^T, \qquad D_r = \max_{t\in[a,b]} w^{(r)}(t)\big[\ell(W)\big]^{-1} c_0, \tag{35}$$

and

$$K_r = M \max_{\nu=1,2}\{C_{\nu nr}\} + \sum_{j=1}^{n}\sum_{k=1}^{p} \max_{\nu=1,2}\{C_{\nu jr}\} A_{k,j-1} + D_r, \tag{36}$$

for $r = 0, \dots, n-1$, where $C_{\nu jr}$ are constants from (21).

Remark 11 Let us note that the constants D_r from (35) do not depend on the choice of the fundamental system of solutions $\tilde{u}_1, \dots, \tilde{u}_n$, but only on the coefficients a_i of the differential equation (1) and on the operators ℓ_j from (3) (and, of course, on the constants c_j).

Now, we are ready to construct a convenient operator for a representation of problem (1)-(3). Let us choose its domain as the closure of the set

$$\Omega = \mathcal{B}^{p+1} \subset X,$$

where $\mathcal{B}$ is defined in (28) with K_j from (36).

Now, we have to modify the operator $\mathcal{H}$ from Theorem 4 using $g_j^{[1]}$ and $g_j^{[2]}$ instead of the Green's functions g_j, that is, we define an operator $\mathcal{F} : \overline{\Omega} \to X$ by $\mathcal{F}(u_1, \dots, u_{p+1}) = (x_1, \dots, x_{p+1})$ with

$$\left.\begin{aligned} x_i(t) = {} & \sum_{k=1}^{p+1} \int_{\tau_{k-1}}^{\tau_k} g_n(t,s) \frac{h(s, u_k(s), \dots, u_k^{(n-1)}(s))}{a_n(s)}\,\mathrm{d}s \\ & + \sum_{j=1}^{n} \Bigg(\sum_{i\le k\le p} g_j^{[1]}(t,\tau_k) J_{k,j-1}\big(u_k(\tau_k), \dots, u_k^{(n-1)}(\tau_k)\big) \\ & + \sum_{1\le k<i} g_j^{[2]}(t,\tau_k) J_{k,j-1}\big(u_k(\tau_k), \dots, u_k^{(n-1)}(\tau_k)\big) \Bigg) + w(t)\big[\ell(W)\big]^{-1} c_0 \end{aligned}\right\} \tag{37}$$

for $i = 1, \dots, p+1$, $t \in [a,b]$, where

$$\tau_k = \mathcal{P}_k u_k \quad \text{for } k = 1, \dots, p, \tau_0 = a, \tau_{p+1} = b,$$

and W, w, g_j, $g_j^{[1]}$, $g_j^{[2]}$, $j = 1, \dots, n$, and c_0 are from (7), (12), (18), and (35), respectively.

Let us compare (16) for the operator $\mathcal{H}$ with (37) for the operator $\mathcal{F}$. The first term in (16) expresses a solution of homogeneous boundary value problem without impulses. This term is decomposed in (37) on subintervals which depend on the choice of $(p+1)$-tuple $(u_1, \dots, u_{p+1})$. The second term in (16) caused (according to the discontinuity of functions g_j) needed impulses of solutions of the fixed-time impulsive problem (13)-(15). We significantly modify this term in (37) in such a way that, instead of discontinuous functions g_j which have jumps at the points $\tau_k = P_k u_k$, we use smooth functions $g_j^{[1]}$, $g_j^{[2]}$ defined in (18). Due to this modification the operator $\mathcal{F}$ maps one tuple of smooth functions

$u_1,\dots,u_{p+1}$ onto another tuple of smooth functions $x_1,\dots,x_{p+1}$, and we will be able to prove the compactness of $\mathcal{F}$ on $\overline{\Omega}$.

In the next lemma we arrive at a justification of our definition.

Lemma 12 *Let assumptions* (4), (9), (23)-(27), (34)-(36) *be satisfied. If* $(u_1,\dots,u_{p+1})$ *is a fixed point of the operator* $\mathcal{F}$, *then the function* u *defined by* (33) *is a solution of problem* (1)-(3) *satisfying* (22).

Proof Let $\mathcal{B}$ be defined by (28) and $\Omega = \mathcal{B}^{p+1}$. Further, let $(u_1,\dots,u_{p+1}) \in \overline{\Omega}$ be such that $\mathcal{F}(u_1,\dots,u_{p+1}) = (u_1,\dots,u_{p+1})$. For each $i \in \{1,\dots,p+1\}$, we have $u_i \in \overline{\mathcal{B}}$, and hence by Lemma 9 and (31), there exists a unique solution $\tau_i = P_i u_i$ of the equation $t = \gamma_i(u_i(t),\dots,u_i^{(n-2)}(t))$. Due to (24), the inequalities $a < \tau_1 < \cdots < \tau_p < b$ are valid and u can be defined by (33). We will prove that u is a fixed point of the operator $\mathcal{H}$ from Theorem 4, taking the space $\mathbb{PC}^{n-1}([a,b])$ from Remark 3 with

$$t_i = \tau_i, \quad i = 1,\dots,p.$$

Denote

$$\tau_0 = a, \qquad \tau_{p+1} = b, \qquad \mathcal{I}_1 = [\tau_0,\tau_1], \qquad \mathcal{I}_2 = (\tau_1,\tau_2],$$
$$\mathcal{I}_3 = (\tau_2,\tau_3], \qquad \dots, \qquad \mathcal{I}_{p+1} = (\tau_p,\tau_{p+1}],$$

and choose $i \in \{1,\dots,p+1\}$, $t \in \mathcal{I}_i$. Then, according to (33), we have

$$\begin{aligned}
u(t) = u_i(t) &= \sum_{k=1}^{p+1} \int_{\mathcal{I}_k} \frac{g_n(t,s)}{a_n(s)} h\bigl(s, u_k(s),\dots,u_k^{(n-1)}(s)\bigr)\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \biggl(\sum_{i\le k\le p} g_j^{[1]}(t,\tau_k) J_{k,j-1}\bigl(u_k(\tau_k),\dots,u_k^{(n-1)}(\tau_k)\bigr) \\
&\quad + \sum_{1\le k<i} g_j^{[2]}(t,\tau_k) J_{k,j-1}\bigl(u_k(\tau_k),\dots,u_k^{(n-1)}(\tau_k)\bigr) \biggr) + w(t)\bigl[\ell(W)\bigr]^{-1} c_0 \\
&= \sum_{k=1}^{p+1} \int_{\mathcal{I}_k} \frac{g_n(t,s)}{a_n(s)} h\bigl(s, u(s),\dots,u^{(n-1)}(s)\bigr)\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \biggl(\sum_{i\le k\le p} g_j^{[1]}(t,\tau_k) J_{k,j-1}\bigl(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\bigr) \\
&\quad + \sum_{1\le k<i} g_j^{[2]}(t,\tau_k) J_{k,j-1}\bigl(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\bigr) \biggr) + w(t)\bigl[\ell(W)\bigr]^{-1} c_0.
\end{aligned}$$

Of course we have

$$\sum_{k=1}^{p+1} \int_{\mathcal{I}_k} \frac{g_n(t,s)}{a_n(s)} h\bigl(s, u(s),\dots,u^{(n-1)}(s)\bigr)\,\mathrm{d}s = \int_a^b \frac{g_n(t,s)}{a_n(s)} h\bigl(s, u(s),\dots,u^{(n-1)}(s)\bigr)\,\mathrm{d}s.$$

Let $k \in \mathbb{N}$ be such that $i \le k \le p$. Then $t \le \tau_i \le \tau_k$ and therefore (19) gives

$$g_j^{[1]}(t,\tau_k) = g_j(t,\tau_k) \quad \text{for } j = 1,\dots,n.$$

Let $k \in \mathbb{N}$ be such that $1 \le k < i$ (such k exists only if $i > 1$). Then $t > \tau_{i-1} \ge \tau_k$ and therefore we get by (19)

$$g_j^{[2]}(t,\tau_k) = g_j(t,\tau_k) \quad \text{for } j = 1,\dots,n.$$

These facts imply that

$$\begin{aligned}
&\sum_{i\le k\le p} g_j^{[1]}(t,\tau_k)J_{k,j-1}\bigl(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\bigr) \\
&\qquad + \sum_{1\le k<i} g_j^{[2]}(t,\tau_k)J_{k,j-1}\bigl(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\bigr) \\
&\quad = \sum_{i\le k\le p} g_j(t,\tau_k)J_{k,j-1}\bigl(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\bigr) \\
&\qquad + \sum_{1\le k<i} g_j(t,\tau_k)J_{k,j-1}\bigl(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\bigr) \\
&\quad = \sum_{k=1}^{p} g_j(t,\tau_k)J_{k,j-1}\bigl(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\bigr),
\end{aligned}$$

for $j = 1,\dots,n$. Consequently, by virtue of (16) and Theorem 4, u is a solution of problem (13)-(15). Clearly u fulfils equation (1) a.e. on $[a,b]$ and satisfies the boundary conditions (3). In addition, since u fulfils the impulse conditions (14) with $t_i = \tau_i$, where $\tau_i = \gamma_i(u_i(\tau_i),\dots,u_i^{(n-2)}(\tau_i)) = \gamma_i(u(\tau_i),\dots,u^{(n-2)}(\tau_i-))$, $i = 1,\dots,p$, we see that u also fulfils the state-dependent impulse conditions (2). According to Remark 8, it remains to prove that $\tau_1,\dots,\tau_p$ are the only instants at which the function u crosses the barriers $\gamma_1,\dots,\gamma_p$, respectively. To this aim, due to (24) and (33), it suffices to prove that

$$t \ne \gamma_i\bigl(u_{i+1}(t),u'_{i+1}(t),\dots,u_{i+1}^{(n-2)}(t)\bigr) \quad \text{for } t \in (\tau_i,b], i = 1,\dots,p. \tag{38}$$

Choose an arbitrary $i \in \{1,\dots,p\}$ and consider σ from (29). Since u fulfils (2), we have

$$\sigma(\tau_i-) = 0.$$

Let us denote

$$\psi(t) = \gamma_i\bigl(u_{i+1}(t),u'_{i+1}(t),\dots,u_{i+1}^{(n-2)}(t)\bigr) - t, \quad t \in [a,b].$$

From Lemma 9 it follows that ψ is decreasing. So, by virtue of (38), it suffices to prove that

$$\psi(\tau_i) \le 0. \tag{39}$$

Using (33), (2), and (27), we have

$$\begin{aligned}\psi(\tau_i) &= \gamma_i\big(u_{i+1}(\tau_i),\dots,u_{i+1}^{(n-2)}(\tau_i)\big) - \tau_i = \gamma_i\big(u(\tau_i+),\dots,u^{(n-2)}(\tau_i+)\big) - \tau_i \\ &= \gamma_i\big(u(\tau_i-) + J_{i0}\big(u(\tau_i-),\dots,u^{(n-1)}(\tau_i-)\big),\dots,u^{(n-2)}(\tau_i-) \\ &\quad + J_{i,n-2}\big(u(\tau_i-),\dots,u^{(n-1)}(\tau_i-)\big)\big) - \tau_i \\ &\le \gamma_i\big(u(\tau_i-),\dots,u^{(n-2)}(\tau_i-)\big) - \tau_i = 0,\end{aligned}$$

which yields (39). This completes the proof. □

Lemma 13 *Let assumptions* (4), (9), (23)-(27), (34)-(36) *be satisfied. Then the operator* $\mathcal{F}$ *from* (37) *has a fixed point in* $\overline{\Omega}$.

Proof The last term $\omega(t)[\ell(W)]^{-1}c_0$ in (37) is the same as in (16) for the compact operator $\mathcal{H}$. Therefore it suffices to prove the compactness of the operator $\mathcal{F}$ on $\overline{\Omega}$ for $c_0 = 0$. To do it we can use the same arguments as in the proof of Lemma 6 in [9], where the second-order state-dependent impulsive problem is investigated. In particular, the compactness of $\mathcal{F}$ on $\overline{\Omega}$ is a consequence of the following properties of functions and functionals contained in (37):

- the first term in (37) can be written in the form

$$\begin{aligned}&\sum_{k=1}^{p+1}\int_{\tau_{k-1}}^{\tau_k} g_n(t,s)\frac{h(s,u_k(s),\dots,u_k^{(n-1)}(s))}{a_n(s)}\,\mathrm{d}s \\ &\quad= \int_a^b g_n(t,s)\sum_{k=1}^{p+1}\frac{h(s,u_k(s),\dots,u_k^{(n-1)}(s))}{a_n(s)}\chi_{(\tau_{k-1},\tau_k)}(s)\,\mathrm{d}s,\end{aligned}$$

  where $\tau_k = \mathcal{P}_k u_k$ for $k = 1,\dots,p$, $\tau_0 = a$, $\tau_{p+1} = b$,
- $\mathcal{P}_k$ are continuous on $\overline{\mathcal{B}}$ (due to Lemma 10),
- $\frac{h(t,x)}{a_n(t)} \in \mathrm{Car}([a,b]\times\mathbb{R}^n;\mathbb{R})$,
- $g_j^{[1]}, g_j^{[2]}$ satisfy (20), g_n satisfies (19),
- J_{kj} are continuous on $\mathbb{R}^n$.

For the application of the Schauder Fixed Point Theorem it remains to prove that

$$\mathcal{F}(\overline{\Omega}) \subset \overline{\Omega}. \tag{40}$$

Let $(x_1,\dots,x_{p+1}) = \mathcal{F}(u_1,\dots,u_{p+1})$ for some $(u_1,\dots,u_{p+1}) \in \overline{\Omega}$. Then, by (21), (34), (35), and (37), we have

$$\big|x_i^{(r)}(t)\big| \le M\max_{\nu=1,2}\{C_{\nu nr}\} + \sum_{j=1}^{n}\sum_{k=1}^{p}\max_{\nu=1,2}\{C_{\nu jr}\}A_{k,j-1} + D_r$$

for $i = 1,\dots,p+1$, $r = 0,\dots,n-1$, $t \in [a,b]$. From (36) we get

$$\big\|x_i^{(r)}\big\|_\infty \le K_r, \quad i = 1,\dots,p+1, r = 0,\dots,n-1,$$

and so $\mathcal{F}(u_1,\dots,u_{p+1}) \in \overline{\Omega}$. We have proved (40), and consequently there exists at least one fixed point of $\mathcal{F}$ in $\overline{\Omega}$. □

Theorem 14 *Let assumptions* (4), (9), (23)-(27), (34)-(36) *be satisfied. Then there exists at least one solution to problem* (1)-(3) *satisfying* (22).

Proof The assertion follows directly from Lemma 12 and Lemma 13. □

Remark 15 The existence result from Theorem 14 can be extended to unbounded functions h and J_{ij} by means of the method of *a priori* estimates. This can be found for the special case $n = 2$ in [10].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the manuscript and read and approved the final draft.

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