The structure of a set of positive solutions to Dirichlet BVPs with time and space singularities

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Abstract: The paper discusses the solvability of the singular Dirichlet boundary value problem

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \quad u(0) = 0, \; u(T) = 0.$$

Here $a \in (-\infty, -1)$ and f satisfies the local Carathéodory conditions on $[0, T] \times \mathcal{D}$, where $\mathcal{D} = (0, \infty) \times \mathbb{R}$. It is shown that the cardinality of the set $\mathcal{L}$ of all positive solutions to the problem is a continuum. In addition, the structure and the properties of the set $\mathcal{L}$ are described. Applications and numerical simulations of the results are presented.

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1 Introduction

We consider the singular Dirichlet boundary value problem

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \tag{1.1a}$$

$$u(0) = 0, \quad u(T) = 0, \tag{1.1b}$$

where $a \in (-\infty, -1)$. Here, f satisfies the local Carathéodory conditions on $[0,T] \times \mathcal{D}$, where $\mathcal{D} = (0,\infty) \times \mathbb{R}$.

We recall that a function $h : [0,T] \times \mathcal{A} \to \mathbb{R}$, $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}$, *satisfies the local Carathéodory conditions on* $[0,T] \times \mathcal{A}$, if

(i) $h(\cdot, x, y) : [0,T] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathcal{A}$,

(ii) $h(t, \cdot, \cdot) : \mathcal{A} \to \mathbb{R}$ is continuous for a.e. $t \in [0,T]$,

(iii) for each compact set $\mathcal{U} \subset \mathcal{A}$ there exists a function $m_{\mathcal{U}} \in L^1[0,T]$ such that

$$|h(t,x,y)| \le m_{\mathcal{U}}(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathcal{U}.$$

For such functions we use the notation $h \in Car([0,T] \times \mathcal{A})$.

We see that $(0,y) \notin \mathcal{D}$ for each $y \in \mathbb{R}$, and hence $f(t,x,y)$ may be singular (unbounded in our case) at $x = 0$. Equation (1.1a) has a time singularity at $t = 0$ due to the structure of the differential operator on its left hand side. This operator has the equivalent form $(t^{-a}(t^a u)')'$ and, after the substitution $v(t) = t^a u(t)$ it takes the form $(t^{-a} v'(t))'$. Therefore, results derived for equation (1.1a) also apply for the modified equation $(t^{-a} v'(t))' = g(t, v(t), v'(t))$. Such type of models arises in the study of phase transitions of Van der Waals fluids [3], [8], [12], [14], [18], in population genetics, in models for the spatial distribution of the genetic composition of a population [6], [7], in the homogenenous nucleation theory [1], in relativistic cosmology in description of particles which can be treated as domains in the universe [15], and in the nonlinear field theory [9], in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [5].

Problem (1.1), where f has no singularity at $x = 0$, i.e. f satisfies the local Carathéodory conditions on $[0,T] \times \overline{\mathcal{D}}$, where $\overline{\mathcal{D}} = [0,\infty) \times \mathbb{R}$, has been investigated in [16]. This paper provides a comprehensive study of the set of all positive solutions of problem (1.1).

Systems of the form

$$u''(t) - \frac{A_1}{t} u'(t) - \frac{A_0}{t^2} u(t) = f(t, u(t), u'(t)), \quad t \in (0,T], \tag{1.2a}$$

$$G(u(0), u'(0), u(1), u'(1)) = 0, \quad u \in C^1[0,1], \tag{1.2b}$$

where, A_1 and A_0 are real valued $n \times n$ matrices, $f : (0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ are *smooth* functions, $m \le 2n$, have been studied in [19]. The missing $2n - m$ conditions have to be formulated in such a way that the requirement $u \in C^1[0,1]$ is satisfied. The main aim in [19] was to investigate the structure of boundary conditions which yield a well-posed boundary value problem. Moreover, in linear case, the existence and uniqueness theory was provided and the smoothness of u was studied. In the nonlinear case, sufficient conditions for u to be *isolated*, or locally unique, have been specified.

The approach taken in [19] is based on a technique developed in [10]. Instead of investigating directly the second order system (1.2a), its first order form obtained after the so called Euler transformation $y(t) = (y_1(t), y_2(t))^T := (u(t), tu'(t)^T$ is analyzed,

$$y'(t) - \frac{M}{t} y(t) = F(t, y(t)), \quad t \in (0, T], \tag{1.3}$$

where

$$M = \begin{pmatrix} 0 & I \\ A_0 & A_1 + I \end{pmatrix}, \quad F(t, y(t)) = \begin{pmatrix} 0 \\ tf(t, y_1(t), \frac{y_2(t)}{t}) \end{pmatrix}.$$

It turns out that the eigenvalues of M play crucial role in describing the solution structure and therefore, the structure of boundary conditions necessary for the solution to be continuous on $[0, 1]$. This is clear, because the fundamental matrix solution reads $Y(t) = e^{M \ln t}$.

In case of the homogeneous differential equation (1.1a), we have

$$y'(t) - \frac{M}{t} y(t) = 0, \quad t \in (0, T], \quad M = \begin{pmatrix} 0 & 1 \\ a & -a + 1 \end{pmatrix}, \tag{1.4}$$

and the eigenvalues of M are $\lambda_1 = -a$ and $\lambda_2 = 1$. By decoupling (1.4), we conclude that the general solution of the homogeneous problem (1.1a) is $u(t) = c_1 t^{\lambda_1} + c_2 t^{\lambda_2} = c_1 t^{-a} + c_2 t$ with arbitrary constants[1] $c_1, c_2 \in \mathbb{R}$. Since both eigenvalues are positive, it follows immediately from [19] that the problem (1.1) is not well-posed and has infinitely many solutions. By prescribing finial conditions, $u(T) = 0$, $u'(T) = -c$, instead of (1.1b), the problem becomes well-posed and can be solved numerically, cf. Section 6. Since we are interested in positive solutions, we choose $c \geq 0$.

The aim of this paper is to extend results from [16] and [19] to problem (1.1) having space singularities. We discuss its solvability and describe the structure of the set $\mathcal{L}$ of all its positive solutions. The existence results are proved by the combination of regularization and sequential techniques with the Leray-Schauder nonlinear alternative. We also show the interesting result stating that for each $c \geq 0$ there exists a function $u \in \mathcal{L}$ such that $u'(T) = -c$, and hence, the cardinality of the set $\mathcal{L}$ is a continuum. Finally, by means of three nonlinear test examples, we illustrate the theoretical findings. These examples are solved using a MATLAB code `bvpsuite` [13] based on collocation.

We start by introducing the necessary notions.

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[1] Note, that we obtain the same solution if in

$$u''(t) + \frac{a}{t} u'(t) - \frac{a}{t^2} u(t) = 0$$

the substitution $u(t) := t^\lambda$ is made. Clearly, in the scalar case, the roots of the so called characteristic polynomial $\lambda(\lambda - 1) + a\lambda - a = 0$ coincide with the eigenvalues of M.

Let us denote by $L^1[0,T]$ the set of functions which are Lebesgue integrable on $[0,T]$ equipped with the norm $\|x\|_1 = \int_0^T |x(t)|\,\mathrm{d}t$. Moreover, let us denote by $C[0,T]$ and $C^1[0,T]$ the set of functions being continuous on $[0,T]$, and having continuous first derivative on $[0,T]$, respectively. The norm on $C[0,T]$ and $C^1[0,T]$ is defined as $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$ and $\|x\|_{C^1} = \|x\|_\infty + \|x'\|_\infty$, respectively. Finally, we denote by $AC^1[0,T]$ the set of functions which have absolutely continuous first derivatives on $[0,T]$, while $AC^1_{loc}(0,T]$ is the set of functions having absolutely continuous derivatives on each compact subinterval of $(0,T]$.

We say that $u : [0,T] \to \mathbb{R}$ is *a positive solution of problem* (1.1) if $u \in AC^1[0,T]$, $u > 0$ on $(0,T)$, u satisfies the boundary conditions (1.1b) and (1.1a) holds for a.e. $t \in [0,T]$.

We work with the following conditions on f in (1.1a).

(H_1) $f \in Car([0,T] \times \mathcal{D})$, where $\mathcal{D} = (0,\infty) \times \mathbb{R}$.

(H_2) There exists $\Delta > 0$ such that

$$\Delta \le f(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathcal{D}.$$

(H_3) For a.e. $t \in [0,T]$ and all $(x,y) \in \mathcal{D}$ the estimate

$$f(t,x,y) \le h(t,x,|y|) + g(x),$$

holds, where $h \in Car([0,T] \times \mathcal{A})$, $\mathcal{A} = [0,\infty) \times [0,\infty)$ and $g \in C(0,\infty)$ are positive, $h(t,x,y)$ is nondecreasing in the variables x, y, g is nonincreasing, and

$$\lim_{x\to\infty} \frac{1}{x} \int_0^T h(t,x,x)\,\mathrm{d}t = 0, \quad \int_0^1 g(s^2)\,\mathrm{d}s < \infty.$$

Remark 1.1 Let g satisfy the conditions given in (H_3). Then $\int_0^b g(cs^2)\,\mathrm{d}s < \infty$ for each $b, c \in (0,\infty)$, and it follows from the inequality

$$t(T-t)^2 \ge \begin{cases} \dfrac{T}{2}t^2, & t \in \left[0, \frac{T}{2}\right], \\ \dfrac{T}{2}(T-t)^2, & t \in \left[\frac{T}{2}, T\right], \end{cases}$$

that

$$\int_0^T g\left(ct(T-t)^2\right)\,\mathrm{d}t < \infty \quad \text{for each } c \in (0,\infty).$$

The paper is organized as follows. Section 2 contains inequalities which we will require in the next three sections. Section 3 is devoted to the study of limit properties as $t \to 0+$ of solutions to equations of the following type:

$$u''(t) + \frac{}{t}u'(t) - \frac{a}{t^2}u(t) = r(t, u(t), u'(t)),$$

where the function r satisfies the global Carathéodory conditions on $[0, T] \times \mathbb{R}^2$. In Section 4, we investigate auxiliary regular problems associated with the singular problem (1.1). We show their solvability and properties of their solutions. Existence results for singular problem (1.1) are given in Section 5. Here, in addition, the properties of the set $\mathcal{L}$ of all positive solutions to the problem are derived together with some applications. Finally, in Section 6, we illustrate the theoretical findings by means of numerical experiments.

Throughout the paper $a \in (-\infty, -1)$.

2 Preliminaries

This section contains inequalities required for the proofs in Sections 3 to 5.

Lemma 2.1 *Let $p \in L^1[0, T]$. Then the inequalities*

$$\left| t^{-a-1} \int_t^T s^{a+1} p(s) \,\mathrm{d}s \right| \le \int_t^T |p(s)| \,\mathrm{d}s, \tag{2.1}$$

$$\left| \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} p(\xi) \,\mathrm{d}\xi \right) \mathrm{d}s \right| \le \frac{1}{|a+1|} \int_t^T |p(s)| \,\mathrm{d}s \tag{2.2}$$

hold for $t \in [0, T]$.

Proof. See [16, Lemma 1]. □

Lemma 2.2 *The inequality*

$$\int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} \,\mathrm{d}\xi \right) \mathrm{d}s \ge \begin{cases} \dfrac{(T-t)^2}{2T}, & a \in [-3, -1), \\[2mm] \dfrac{(T-t)^2}{2T(a+2)^2}, & a \in (-\infty, -3) \end{cases} \tag{2.3}$$

holds for $t \in [0, T]$.

Proof. Let $a \in [-2, -1)$, then

$$\begin{aligned}\int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1}\,\mathrm{d}\xi \right) \mathrm{d}s \;&\geq\; T^{a+1} \int_t^T (T-s) s^{-a-2}\,\mathrm{d}s \\ &\geq \frac{1}{T} \int_t^T (T-s)\,\mathrm{d}s = \frac{(T-t)^2}{2T}.\end{aligned}$$

In particular,

$$\int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1}\,\mathrm{d}\xi \right) \mathrm{d}s \geq \frac{(T-t)^2}{2T} \quad \text{for } t \in [0,T] \text{ and } a \in [-2,-1). \tag{2.4}$$

Let $a \in (-\infty, -2)$. Then

$$\begin{aligned}\int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1}\,\mathrm{d}\xi \right) \mathrm{d}s \;&=\; \frac{1}{|a+2|} \int_t^T \left(1 - \left(\frac{s}{T}\right)^{-a-2} \right) \mathrm{d}s \\ &= \frac{T}{|a+2|} \int_{t/T}^1 (1-s)^{-a-2}\,\mathrm{d}s\end{aligned} \tag{2.5}$$

for $t \in [0,T]$. Choose $p(x) := 1 - x^\beta - \beta(1-x)$ for $x \in [0,1]$, where $\beta \in (0,1)$. Then $p(0) = 1 - \beta > 0$, $p(1) = 0$, and since $p'(x) = \beta\left(1 - x^{\beta-1}\right) < 0$ for $x \in (0,1)$, we have $p > 0$ on $[0,1)$. Consequently, $1 - x^\beta \geq \beta(1-x)$ for $x \in [0,1]$ and $\beta \in (0,1]$. This gives for $\beta = -a-2$,

$$1 - x^{-a-2} \geq |a+2|(1-x) \quad \text{for } x \in [0,1] \text{ and } a \in [-3,-2).$$

Hence, by (2.5), the relation

$$\int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1}\,\mathrm{d}\xi \right) \mathrm{d}s \geq T \int_{t/T}^1 (1-s)\,\mathrm{d}s = \frac{(T-t)^2}{2T} \tag{2.6}$$

is satisfied for $t \in [0,1]$ and $a \in [-3,-2)$.

In order to verify (2.3) for $a \in (-\infty,-3)$, let

$$r(x) := 1 - x^\gamma - \frac{1-x}{\gamma} \quad \text{for } x \in [0,1],$$

where $\gamma > 1$. Then $r(0) = 1 - \frac{1}{\gamma} > 0$, $r(1) = 0$, $r'(x) = -\gamma x^{\gamma-1} + \frac{1}{\gamma}$, and $r''(x) = -\gamma(\gamma-1)x^{\gamma-2}$. Hence, $r'' < 0$ on $(0,1]$, and since $r'(0) = \frac{1}{\gamma} > 0$ and $r'(1) = -\gamma + \frac{1}{\gamma} < 0$, we conclude that $r \geq 0$ on $[0,1]$. That is $1 - x^\gamma \geq \frac{1-x}{\gamma}$ for $x \in [0,1]$ and $\gamma > 1$. Therefore, for $\gamma = -a-2$,

$$1 - x^{-a-2} \geq \frac{1-x}{|a+2|} \quad \text{for } x \in [0,1] \text{ and } a \in (-\infty,-3),$$

and so, by (2.5),

$$\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\,\mathrm{d}\xi\right)\mathrm{d}s \ge \frac{T}{(a+2)^2}\int_{t/T}^1 (1-s)\,\mathrm{d}s = \frac{(T-t)^2}{2T(a+2)^2}, \tag{2.7}$$

for $t \in [0,T]$ and $a \in (-\infty,-3)$.

Inequality (2.3) now follows from (2.4), (2.6) and (2.7). □

3 Limit properties of solutions

In this section we consider the differential equation

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = r(t,u(t),u'(t)), \quad t \in (0,T], \tag{3.1}$$

where r satisfies *the global Carathéodory condition on* $[0,T]\times\mathbb{R}^2$, that is,

(H_4) $r(\cdot,x,y) : [0,T] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathbb{R}\times\mathbb{R}$, $r(t,\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in [0,T]$, and there exists $\mu \in L^1[0,T]$ such that

$$|r(t,x,y)| \le \mu(t) \ \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2. \tag{3.2}$$

We now describe the analytical form and the asymptotic behavior for $t \to 0+$ of functions u satisfying (3.1) a.e. on $[0,T]$.

Lemma 3.1 *Let condition* (H_4) *hold. Let the function* $u \in AC^1_{loc}(0,T]$ *satisfy* (3.1) *for a.e.* $t \in [0,T]$. *Then* u *can be extended on* $[0,T]$ *with* $u \in AC^1[0,T]$ *and the representation*

$$u(t) = c_1 t + c_2 t^{-a} + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} r(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s, \tag{3.3}$$

where $c_1, c_2 \in \mathbb{R}$, *holds for* $t \in [0,T]$.

Proof. Keeping in mind that u is fixed, consider the Euler linear differential equation

$$v''(t) + \frac{a}{t}v'(t) - \frac{a}{t^2}v(t) = r(t,u(t),u'(t)). \tag{3.4}$$

Each function $v \in AC^1_{loc}(0,T]$ satisfying (3.4) a.e. on $[0,T]$ has the form

$$v(t) = c_1^* t + c_2^* t^{-a} + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} r(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s,$$

where $c_1^*, c_2^* \in \mathbb{R}$. Since, by the assumption $u \in AC^1_{loc}(0,T]$ satisfies (3.4) for a.e. $t \in [0,T]$, there exists $c_1, c_2 \in \mathbb{R}$ such that equality (3.3) holds for $t \in (0,T]$. In order to prove that u can be extended on $[0,T]$ as a function in $AC^1[0,T]$, and consequently, that (3.3) is satisfied for $t \in [0,T]$, we have to show that

$$\int_0^T |u''(t)| \,\mathrm{d}s < \infty. \tag{3.5}$$

By (3.3),

$$\frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = -at^{-a-2}\left(c_2(a+1) + \int_t^T s^{a+1} r(s,u(s),u'(s))\,\mathrm{d}s\right) \text{ for } t \in (0,T],$$

and then using (3.2) we obtain

$$\left|\frac{a}{t}u'(t) - \frac{a}{t^2}u(t)\right| \le |a| t^{-a-2}\left(|c_2(a+1)| + \int_t^T s^{a+1}\mu(s)\,\mathrm{d}s\right) \text{ for } t \in (0,T]. \tag{3.6}$$

Hence, by (2.2),

$$\begin{aligned}
\int_0^T |u''(t)|\,\mathrm{d}s \le\ & \int_0^T \left|\frac{a}{t}u'(t) - \frac{a}{t^2}u(t)\right| \mathrm{d}t + \int_0^T |r(t,u(t),u'(t))|\,\mathrm{d}t \\
\le\ & |a|\left(|c_2(a+1)|\int_0^T t^{-a-2}\,\mathrm{d}t + \int_0^T t^{-a-2}\left(\int_t^T s^{a+1}\mu(s)\,\mathrm{d}s\right)\mathrm{d}t\right) \\
& + \int_0^T \mu(s)\,\mathrm{d}s \\
\le\ & |ac_2| T^{-a-1} + \frac{(2a+1)\|\mu\|_1}{a+1}.
\end{aligned}$$

Consequently, (3.5) holds and this completes the proof. □

The following corollaries extend the statement of Lemma 3.1 for $r \in Car([0,T]\times\mathbb{R}^2)$, that is for r satisfying only the local Carathéodory conditions on $[0,T]\times\mathbb{R}^2$.

Corollary 3.2 *Let $r \in Car([0,T]\times\mathbb{R}^2)$ and let $u \in AC^1_{loc}(0,T]$ satisfy* (3.1) *a.e. on $[0,T]$. Assume also that*

$$L := \sup\{|u(t)| + |u'(t)| : t \in (0,T]\} < \infty$$

holds. Then, the assertion of Lemma 3.1 *is satisfied.*

Proof. Let

$$\rho(z) := \begin{cases} L, & z > L, \\ z, & |z| \le L, \\ -L, & z < -L, \end{cases}$$

and let $r^*(t, x, y) := r(t, \rho(x), \rho(y))$ for a.e. $t \in [0, T]$ and all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then r^* satisfies the global Carathéodory conditions on $[0, T] \times \mathbb{R}^2$ and the equality

$$u''(t) + \frac{a}{t} u'(t) - \frac{a}{t^2} u(t) = r^*(t, u(t), u'(t))$$

holds for a.e. $t \in [0, T]$. The result now follows from Lemma 3.1, where r is replaced by r^* in equation (3.1). □

Corollary 3.3 *Let $r \in Car([0, T] \times \mathbb{R}^2)$ and let $u \in AC^1[0, T]$ be a solution of equation* (3.1)*. Then there exist $c_1, c_2 \in \mathbb{R}$ such that equality* (3.3) *is satisfied for $t \in [0, T]$.*

Proof. We can apply Corollary 3.2, since $u \in AC^1[0, T]$ yields

$$\sup\{|u(t)| + |u'(t)| : t \in [0, T]\} < \infty.$$

□

Remark 3.4 Corollary 3.3 shows that each solution $u \in AC^1[0, T]$ of equation (3.1) with $r \in Car([0, T] \times \mathbb{R}^2)$ has the form given in (3.3), where $c_1, c_2 \in \mathbb{R}$, and therefore, it satisfies $u(0) = 0$. Consequently, when discussing solutions $u \in AC^1[0, T]$ of equation (3.1) together with boundary conditions, especially including the condition $u(0) = u_0$, then, necessarily, $u_0 = 0$.

4 Auxiliary regular problems

Since equation (1.1a) is singular, we use the regularization and sequential techniques for solving problem (1.1). To this end, we define $f_n : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, $n \in \mathbb{N}$, by the formula

$$f_n(t, x, y) = \begin{cases} f(t, x, y), & x \geq \dfrac{1}{n}, \\ f\left(t, \dfrac{1}{n}, y\right), & x < \dfrac{1}{n}. \end{cases}$$

Under conditions $(H_1) - (H_3)$, $f_n \in Car([0, T] \times \mathbb{R}^2)$ and

$$\Delta \leq f_n(t, x, y) \quad \text{for a.e. } t \in [0, T] \text{ and all } (x, y) \in \mathbb{R}^2, \tag{4.1}$$

$$\left.\begin{array}{c} f_n(t, x, y) \leq h(t, 1 + |x|, |y|) + g(|x|) \\ \text{for a.e. } t \in [0, T] \text{ and all } (x, y) \in \mathbb{R}_0 \times \mathbb{R}. \end{array}\right\} \tag{4.2}$$

Here $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. Hence,

$$\left.\begin{array}{r}\Delta \le \lambda f_n(t,x,y) + (1-\lambda)\Delta \le h(t, 1+|x|, |y|) + g(|x|)\\ \text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in\mathbb{R}_0\times\mathbb{R},\ \lambda\in[0,1].\end{array}\right\} \tag{4.3}$$

We consider the differential equations

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f_n(t, u(t), u'(t)), \quad n\in\mathbb{N}, \tag{4.4}$$

and

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = \lambda f_n(t, u(t), u'(t)) + (1-\lambda)\Delta, \ \ \lambda\in[0,1],\ n\in\mathbb{N}. \tag{4.5}$$

A function $u : [0,T] \to \mathbb{R}$ is called *a solution of* (4.4) if $u \in AC^1[0,T]$ and u satisfies (4.4) for a.e. $t \in [0,T]$. Solutions of (4.5) are defined analogously.

Let us now define the boundary value problem (4.6) consisting of the differential equation specified in (4.4) subject to the boundary condition (1.1b),

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f_n(t, u(t), u'(t)), \quad n\in\mathbb{N}, \tag{4.6a}$$

$$u(0) = 0, \quad u(T) = 0. \tag{4.6b}$$

Lemma 4.1 *Let condition* (H_1) *hold. Then, all solutions* $u \in AC^1[0,T]$ *of problem* (4.6) *form a one-parameter system* $\mathcal{A}$*, where*

$$\mathcal{A} = \Bigg\{c_2 t(t^{-a-1} - T^{-a-1}) + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f_n(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s : c_2\in\mathbb{R}\Bigg\}.$$

Proof. Let $u \in AC^1[0,T]$ be a solution of problem (4.6). Since u is a solution of (4.6a), the equality

$$u(t) = c_1 t + c_2 t^{-a} + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f_n(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s \tag{4.7}$$

holds for $t \in [0,T]$ by Corollary 3.3, where $c_1, c_2 \in \mathbb{R}$. Then $u(0) = 0$ and the condition $u(T) = 0$ yields $c_1 = -c_2 T^{-a-1}$. Hence, by (4.7), $u \in \mathcal{A}$.

Let $u \in \mathcal{A}$, that is,

$$u(t) = c_2 t(t^{-a-1} - T^{-a-1}) + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f_n(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s \tag{4.8}$$

for $t \in [0,T]$, where $c_2 \in \mathbb{R}$. Then u satisfies condition (4.6b) and

$$\begin{aligned} u'(t) &= -c_2(at^{-a-1} + T^{-a-1}) \\ &\quad + \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} f_n(\xi, u(\xi), u'(\xi))\, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad - t^{-a-1} \int_t^T s^{a+1} f_n(s, u(s), u'(s))\, \mathrm{d}s, \quad t \in [0,T], \end{aligned} \tag{4.9}$$

$$\begin{aligned} u''(t) &= a(a+1)c_2 t^{-a-2} + at^{-a-2} \int_t^T s^{a+1} f_n(s, u(s), u'(s))\, \mathrm{d}s \\ &\quad + f_n(t, u(t), u'(t)), \text{ for a.e. } t \in [0,T]. \end{aligned} \tag{4.10}$$

By (H_1), $f_n(t, u(t), u'(t)) \in L^1[0,T]$ and consequently, (2.2) implies

$$t^{-a-2} \int_t^T s^{a+1} f_n(s, u(s), u'(s))\, \mathrm{d}s \in L^1[0,T].$$

As a result, $u \in AC^1[0,T]$. Using (4.8), (4.9) and (4.10), we can verify that u satisfies (4.6a) for a.e. $t \in [0,T]$. □

In the following lemma, we discuss solutions u of the boundary value problem:

$$u''(t) + \frac{a}{t} u'(t) - \frac{a}{t^2} u(t) = f_n(t, u(t), u'(t)), \quad n \in \mathbb{N}, \tag{4.11a}$$

$$u(0) = 0, \quad u(T) = 0, \quad u'(T) = -c, \quad c \geq 0. \tag{4.11b}$$

In this problem u satisfies, besides the Dirichlet conditions (4.6b), the additional condition

$$u'(T) = -c, \tag{4.12}$$

for a fixed $c \geq 0$. Note that condition (4.12) together with (4.9) yields

$$c_2 = \frac{c}{a+1} T^{a+1} \leq 0 \tag{4.13}$$

in (4.8).

Lemma 4.2 *Let* (H_1) *hold. Then a function* $u \in AC^1[0,T]$ *is a solution of problem* (4.11) *if and only if* u *is a solution of the integral equation*

$$\begin{aligned} u(t) &= t \frac{cT^{a+1}}{|a+1|} (T^{-a-1} - t^{-a-1}) \\ &\quad + t \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} f_n(\xi, u(\xi), u'(\xi))\, \mathrm{d}\xi \right) \mathrm{d}s \end{aligned} \tag{4.14}$$

in the set $C^1[0,T]$.

Proof. ($\Rightarrow$) Let us first assume that $u \in AC^1[0,T]$ is a solution of problem (4.11). Then $u \in \mathcal{A}$, that is u satisfies (4.8), with c_2 given by (4.13). As a result, u is a solution of (4.14) in $C^1[0,T]$.

($\Leftarrow$) Let now $u \in C^1[0,T]$ be a solution of (4.14). Then u satisfies (4.8), (4.9) and (4.10) with c_2 given by (4.13). Therefore, u satisfies boundary conditions (4.11b). The same reasoning as in the proof of Lemma 4.1 implies that $u \in AC^1[0,T]$ and u satisfies equation (4.11a) for a.e. $t \in [0,T]$. $\square$

Let

$$M = \begin{cases} \dfrac{\Delta}{2T}, & a \in [-3,-1), \\ \dfrac{\Delta}{2T(a+2)^2}, & a \in (-\infty,-3), \end{cases} \tag{4.15}$$

with Δ specified in condition (H_2).

Consider a fixed $c \geq 0$. We now derive bounds for solutions of the boundary value problem

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = \lambda f_n(t,u(t),u'(t)) + (1-\lambda)\Delta, \quad \lambda \in [0,1], \tag{4.16a}$$

$$u(0)=0, \quad u(T)=0, \quad u'(T) = -\lambda c. \tag{4.16b}$$

Note, that this time u satisfies additionally, besides the Dirichlet conditions (4.6b), the following condition

$$u'(T) = -\lambda c, \quad \lambda \in [0,1]. \tag{4.17}$$

Lemma 4.3 *Let conditions $(H_1)-(H_3)$ hold. Then, there exists a positive constant S independent of n and λ such that for all solutions u of problems* (4.16) *the estimates*

$$u(t) \geq Mt(T-t)^2, \quad t \in [0,t], \tag{4.18}$$

$$\|u\|_\infty < ST, \quad \|u'\|_\infty < S, \tag{4.19}$$

hold.

Proof. Let u be a solution of problem (4.16) for some $n \in \mathbb{N}$ and $\lambda \in [0,1]$. Applying Lemma 4.2 to this problem we obtain the equality

$$\begin{aligned} u(t) &= \lambda t \frac{cT^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) \\ &\quad + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}[\lambda f_n(\xi,u(\xi),u'(\xi)) + (1-\lambda)\Delta]\,\mathrm{d}\xi\right)\mathrm{d}s \end{aligned} \tag{4.20}$$

for $t \in [0,T]$. Since $c \ge 0$, $\lambda \ge 0$, we have due to (2.3), (4.1) and (4.15),

$$u(t) \ge \Delta t \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1}\, \mathrm{d}\xi \right) \mathrm{d}s \ge M t (T-t)^2 \quad \text{for } t \in [0,T],$$

and so (4.18) is true. Furthermore, $u > 0$ on $(0,T)$ and $g(u(t)) \le g\left(Mt(T-t)^2\right)$ for $t \in (0,T)$ since g is nonincreasing on $(0,\infty)$ by (H_3). Hence,

$$\|g(u(t))\|_1 \le \left\| g\left(Mt(T-t)^2\right)\right\|_1 =: W_1, \tag{4.21}$$

where $W_1 < \infty$ by Remark 1.1. Note that the value of W_1 neither depends on the choice of solution u to problem (4.16) nor on n and λ. Since

$$\begin{aligned} u'(t) \;&=\; \frac{\lambda c T^{a+1}}{|a+1|}(T^{-a-1} + a t^{-a-1}) \\ &\quad + \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1}[\lambda f_n(\xi, u(\xi), u'(\xi)) + (1-\lambda)\Delta]\, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad - t^{-a-1} \int_t^T s^{a+1}[\lambda f_n(s, u(s), u'(s)) + (1-\lambda)\Delta]\, \mathrm{d}s, \quad t \in [0,T], \end{aligned} \tag{4.22}$$

it follows from (2.1), (2.2), (4.3) and (4.21) that the relation

$$\begin{aligned} |u'(t)| \;&\le\; c\left(\frac{1}{|a+1|} + 1 \right) \\ &\quad + \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1}[h(\xi, 1+u(\xi), |u'(\xi)|) + g(u(\xi))]\, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad + t^{-a-1} \int_t^T s^{a+1}[h(s, 1+u(s), |u'(s)|) + g(u(s))]\, \mathrm{d}s \\ &\le \left(\frac{1}{|a+1|} + 1 \right) \left(\int_0^T h(s, 1+u(s), |u'(s)|)\, \mathrm{d}s + W_1 + c \right) \\ &\le \left(\frac{1}{|a+1|} + 1 \right) \left(\int_0^T h(s, 1+\|u\|_\infty, \|u'\|_\infty)\, \mathrm{d}s + W \right) \end{aligned}$$

is satisfied for $t \in [0,T]$ and $W := W_1 + c$.

In particular,

$$\|u'\|_\infty \le \left(\frac{1}{|a+1|} + 1 \right) \left(\int_0^T h(s, 1+\|u\|_\infty, \|u'\|_\infty)\, \mathrm{d}s + W \right).$$

Since $u(t) = \int_0^t u'(s)\, \mathrm{d}s$, we have

$$\|u\|_\infty \le T \|u'\|_\infty, \tag{4.23}$$

and therefore

$$\|u'\|_\infty \le \left(\frac{1}{|a+1|} + 1\right) \left(\int_0^T h(s, 1 + T\|u'\|_\infty, \|u'\|_\infty)\,\mathrm{d}t + W\right). \tag{4.24}$$

By (H_3), $\lim_{x\to\infty} \frac{1}{x}\int_0^T h(t, 1+Tx, x)\,\mathrm{d}t = 0$, and consequently, there exists $S > 0$ such that

$$\left(\frac{1}{|a+1|} + 1\right)\left(\int_0^T h(s, 1+Tx, x)\,\mathrm{d}t + W\right) < x \quad \text{for all } x \ge S.$$

Now we conclude from the last relation and from (4.24) that $\|u'\|_\infty < S$, and therefore, by (4.23), $\|u\|_\infty < ST$. Hence (4.19) holds and this completes the proof. □

We are now in the position to show the existence of a solution of problem (4.11). This result is proved by the following nonlinear alternative of Leray-Schauder type which follows, for example, from [2, Theorem 5.1].

Lemma 4.4 *Let X be a Banach space, Ω an open bounded subset of X and $p \in \Omega$. Assume that $\mathcal{F} : \overline{\Omega} \to X$ is a compact operator. Then, either*
(i) $\mathcal{F}$ has a fixed point in $\overline{\Omega}$, or
(ii) There exists a $u \in \partial\Omega$ and $\lambda \in (0,1)$ such that $u = \lambda\mathcal{F}u + (1-\lambda)p$.

Theorem 4.5 *Let (H_1)–(H_3) hold. Let S be the positive constant from Lemma 4.3. Then problems (4.11) are solvable. If u is a solution of (4.11) for some $n \in \mathbb{N}$, then u satisfies (4.18) and (4.19) with the positive constant M given in (4.15).*

Proof. Let

$$\Omega := \{x \in C^1[0,T] : \|x\|_\infty < ST,\ \|x'\|_\infty < S\}.$$

Then Ω is an open bounded subset of the Banach space $C^1[0,T]$. Choose $n \in \mathbb{N}$ and consider an operator $\mathcal{K} : [0,1] \times \overline{\Omega} \to C^1[0,T]$,

$$\mathcal{K}(\lambda, x) = \lambda\mathcal{F}x + (1-\lambda)p, \tag{4.25}$$

where

$$\begin{aligned}(\mathcal{F}x)(t) &= t\frac{cT^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) \\ &\quad + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f_n(\xi, x(\xi), x'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s, \\ p &= t\Delta\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\,\mathrm{d}\xi\right)\mathrm{d}s.\end{aligned}$$

By Lemma 4.2, any fixed point of the operator $\mathcal{K}(1,\cdot) = \mathcal{F}$ is a solution of problem (4.11). Hence, to show the result we need to prove that $\mathcal{K}(1,\cdot)$ has a fixed point. Applying Lemma 4.4 for $X = C^1[0,T]$, we have to show that

(i) $\mathcal{K}(1,\cdot) : \overline{\Omega} \to C^1[0,T]$ is a compact operator, and

(ii) $\mathcal{K}(\lambda, x) \neq x$ for each $\lambda \in (0,1)$ and $x \in \partial\Omega$.

We begin by proving the continuity of $\mathcal{K}(1,\cdot)$. To this end let $\{x_m\} \subset \overline{\Omega}$ be a convergent sequence and let $\lim_{m\to\infty} x_m = x$. Let

$$r_m(t) := f_n(t, x_m(t), x_m'(t)) - f_n(t, x(t), x'(t)) \text{ for a.e. } t \in [0,T].$$

Then (2.1) and (2.2) yield

$$\begin{aligned}
|\mathcal{K}(1,x_m)(t) - \mathcal{K}(1,x)(t)| &= \left| t \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} r_m(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \right| \le \frac{T\|r_m\|_1}{|a+1|}, \\
|\mathcal{K}(1,x_m)'(t) - \mathcal{K}(1,x)'(t)| &= \left| \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} r_m(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \right. \\
&\qquad \left. - t^{-a-1} \int_t^T s^{a+1} r_m(s)\,\mathrm{d}s \right| \\
&\le \left(\frac{1}{|a+1|} + 1 \right) \|r_m\|_1,
\end{aligned}$$

for $t \in [0,T]$ and $m \in \mathbb{N}$. Here, $\mathcal{K}(1,x)' = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(1,x)$. In particular,

$$\begin{aligned}
\|\mathcal{K}(1,x_m) - \mathcal{K}(1,x)\|_\infty &\le \frac{T\|r_m\|_1}{|a+1|}, \\
\|\mathcal{K}(1,x_m)' - \mathcal{K}(1,x)'\|_\infty &\le \left(\frac{1}{|a+1|} + 1 \right) \|r_m\|_1,
\end{aligned}$$

for $m \in \mathbb{N}$. If we show that $\lim_{m\to\infty} \|r_m\|_1 = 0$, then the above inequalities guarantee that $\mathcal{K}(1,\cdot)$ is a continuous operator. From

$$\lim_{m\to\infty} f_n(t, x_m(t), x_m'(t)) = f_n(t, x(t), x'(t)) \text{ for a.e. } t \in [0,T],$$

and from the fact that $f_n \in Car([0,T] \times \mathbb{R}^2)$ and $\{x_m\}$ is bounded in $C^1[0,T]$, it follows that

$$|f_n(t, x_m(t), x_m'(t)| \le \rho(t) \text{ for a.e. } t \in [0,T] \text{ and all } m \in \mathbb{N},$$

where $\rho \in L^1[0,T]$. Finally, $\lim_{m\to\infty} \|r_m\|_1 = 0$ follows by the Lebesgue dominated convergence theorem.

Now, we show that the set $\mathcal{K}(1,\overline{\Omega})$ is relatively compact in $C^1[0,T]$. From $f_n \in Car([0,T] \times \mathbb{R}^2)$ we conclude that

$$\Delta \le f_n(t, x(t), x'(t)) \le \mu(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in \overline{\Omega}, \tag{4.26}$$

where $\mu \in L^1[0,T]$. Then, by (2.1), (2.2) , (4.25) and (4.26),

$$\begin{aligned} 0 \le \mathcal{K}(1,x)(t) &\le \frac{cT}{|a+1|} + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\mu(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \le \frac{T}{|a+1|}(c+\|\mu\|_1), \\ |\mathcal{K}(1,x)'(t)| &\le \left|\frac{cT^{a+1}}{|a+1|}(T^{-a-1}+at^{-a-1})\right| \\ &\quad + \int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\mu(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s + t^{-a-1}\int_t^T s^{a+1}\mu(s)\,\mathrm{d}s \\ &\le \left(\frac{1}{|a+1|}+1\right)(c+\|\mu\|_1) \end{aligned}$$

for $t \in [0,T]$ and $x \in \overline{\Omega}$. Therefore, the set $\mathcal{K}(1,\overline{\Omega})$ is bounded in $C^1[0,T]$.

We now show that the set $\{\mathcal{K}(1,x)' \colon x \in \overline{\Omega}\}$ is equicontinuous on $[0,T]$. For a.e. $t \in [0,T]$ and all $x \in \overline{\Omega}$ we have, by (4.26),

$$\begin{aligned} |\mathcal{K}(1,x)''(t)| &= \Bigg| acT^{a+1}t^{-a-2} + at^{-a-2}\int_t^T s^{a+1} f_n(s,x(s),x'(s))\,\mathrm{d}s \\ &\qquad + f_n(t,x(t),x'(t))\Bigg| \\ &\le |a|cT^{a+1}t^{-a-2} + |a|t^{-a-2}\int_t^T s^{a+1}\mu(s)\,\mathrm{d}s + \mu(t). \end{aligned}$$

Hence, by (2.1),

$$\int_0^T |\mathcal{K}(1,x)''(t)|\mathrm{d}t \le \left|\frac{a}{a+1}\right| c + \left|\frac{a}{a+1}\right| \|\mu\|_1 + \|\mu\|_1$$

for all $x \in \overline{\Omega}$, which guarantees the equicontinuity of the set $\{\mathcal{K}(1,x)' \colon x \in \overline{\Omega}\}$ on $[0,T]$. Therefore, the set $\mathcal{K}(1,\overline{\Omega})$ is relatively compact in $C^1[0,T]$ by the Arzelà-Ascoli theorem and consequently, $\mathcal{K}(1,\cdot)$ is a compact operator and (i) follows.

It remains to prove (ii), that is, $\mathcal{K}(\lambda,x) \neq x$ for each $\lambda \in (0,1)$ and $x \in \partial\Omega$. Let u be a fixed point of $\mathcal{K}(\lambda,\cdot)$ for some $\lambda \in (0,1)$. Then equalities (4.20) and (4.22) hold for $t \in [0,T]$. We see that u satisfies (4.16b) and, as in the proof of Lemma 4.1, we conclude that u is a solution of equation (4.16a). Therefore, Lemma 4.3 guarantees that $\mathcal{K}(\lambda,x) \neq x$ for $\lambda \in (0,1)$ and $x \in \partial\Omega$. By Lemma 4.4, problem (4.11) has a solution $u \in \overline{\Omega}$. Lemma 4.3 guarantees that u satisfies (4.18) and (4.19). □

The following result provides an important property of solutions of problem (4.11) which will be used in the proof of Theorem 5.1 in Section 5.

Lemma 4.6 *Let* $(H_1)-(H_3)$ *hold. Let* u_n *be a solution of problem* (4.11). *Then, the sequence* $\{u_n'\}$ *is equicontinuous on* $[0,T]$.

Proof. By Theorem 4.5,

$$u_n(t) \geq Mt(T-t)^2 \quad \text{for } t \in [0,T] \text{ and } n \in \mathbb{N}, \tag{4.27}$$

and

$$\|u_n\|_\infty < ST, \quad \|u_n'\|_\infty < S \quad \text{for } n \in \mathbb{N}, \tag{4.28}$$

where S is a positive constant and M is given in (4.15). Since u_n is a fixed point of $\mathcal{K}(1,\cdot)$, cf. (4.25), the equality

$$u_n''(t) = acT^{a+1}t^{-a-2} + at^{-a-2}\int_t^T s^{a+1} f_n(s, u_n(s), u_n'(s))\,\mathrm{d}s + f_n(t, u_n(t), u_n'(t))$$

holds for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$. Owing to (4.1), (4.2), (4.27) and (4.28) we have

$$\Delta \leq f_n(t, u_n(t), u_n'(t)) \leq h(t, 1+ST, S) + \rho(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } n \in \mathbb{N},$$

where $\rho(t) = g\left(Mt(T-t)^2\right)$ for $t \in (0,T)$. Let us choose $\chi(t) := h(t, 1+ST, S) + \varrho(t)$ for a.e. $t \in [0,T]$. By (H_3) and Remark 1.1, χ is positive and $\chi \in L^1[0,T]$. Hence,

$$|u_n''(t)| \leq |a|cT^{a+1}t^{-a-2} + |a|t^{-a-2}\int_t^T s^{a+1}\chi(s)\,\mathrm{d}s + \chi(t)$$

for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$. Consequently, by (2.2), the inequality

$$\int_0^T |u_n''(t)|\,\mathrm{d}s \leq \left|\frac{a}{a+1}\right| c + \left|\frac{a}{a+1}\right| \|\chi\|_1 + \|\chi\|_1$$

holds for all $n \in \mathbb{N}$, which means that that the sequence $\{u_n'\}$ is equicontinuous on $[0,T]$. □

5 Analytical properties of solutions to problem (1.1)

In this section, we denote by $\mathcal{L}$ the set of all positive solutions of the singular Dirichlet problem (1.1). For $c \geq 0$, we denote by $\mathcal{S}_c$ the set of all positive solutions of problem (1.1) satisfying condition (4.12). Our aim is to describe the structure of $\mathcal{L}$. In particular, we show that $\mathcal{L}$ is a one parameter set.

Theorem 5.1 *Let $(H_1)-(H_3)$ hold. Then, for each $c \geq 0$, the set $\mathcal{S}_c$ is nonempty and*

$$\mathcal{L} = \bigcup_{0 \leq c} \mathcal{S}_c.$$

Hence the cardinality of the set $\mathcal{L}$ is a continuum. Moreover,

$$u(t) \geq Mt(T-t)^2, \quad t \in [0,T], \tag{5.1}$$

with M given in (4.15)*, holds for each $u \in \mathcal{L}$.*

Proof. Let us fix $c \geq 0$. Theorem 4.5 guarantees that the regular problem (4.11) has a solution u_n satisfying inequalities (4.27) and (4.28), where S is a positive constant and M is given in (4.15). Furthermore, $\{u_n'\}$ is equicontinuous on $[0,T]$ by Lemma 4.6. Consequently, by Arzelà-Ascoli theorem, there is a subsequence $\{u_{\ell_n}\}$ of $\{u_n\}$ converging in $C^1[0,T]$. Denote $\lim_{n\to\infty} u_{\ell_n} =: u$. Then u satisfies the boundary conditions (4.11b) and taking the limit $n \to \infty$ in (4.27) and (4.28), with u_n replaced by u_{ℓ_n}, we obtain

$$\|u\|_\infty \leq ST, \ \|u'\|_\infty \leq S \text{ and } u(t) \geq Mt(T-t)^2 \ \text{ for } t \in [0,T].$$

Hence $u > 0$ on $(0,T)$ and

$$\lim_{n\to\infty} f_{\ell_n}\left(t, u_{\ell_n}(t), u_{\ell_n}'(t)\right) = f(t, u(t), u'(t)) \ \text{ for a.e. } t \in [0,T].$$

By (H_3) and Remark 1.1,

$$\left|f_{\ell_n}\left(t, u_{\ell_n}(t), u_{\ell_n}'(t)\right)\right| \leq h(t, 1+ST, S) + g\left(Mt(T-t)^2\right) \in L^1[0,T].$$

Therefore, $f(t, u(t), u'(t)) \in L^1[0,T]$ and

$$\lim_{n\to\infty} \left\|f_{\ell_n}\left(t, u_{\ell_n}(t), u_{\ell_n}'(t)\right) - f(t, u(t), u'(t))\right\|_1 = 0 \tag{5.2}$$

by the Lebesgue dominated convergence theorem. It follows from the inequality (2.2), that it holds

$$\left|t \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} \left[f_{\ell_n}\left(\xi, u_{\ell_n}(\xi), u_{\ell_n}'(\xi)\right) - f(\xi, u(\xi), u'(\xi))\right] \mathrm{d}\xi\right) \mathrm{d}s\right|$$
$$\leq \frac{T}{|a+1|} \left\|f_{\ell_n}\left(t, u_{\ell_n}(t), u_{\ell_n}'(t)\right) - f(t, u(t), u'(t))\right\|_1.$$

Now, from (5.2) we conclude that

$$\lim_{n\to\infty} \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} f_{\ell_n}\left(\xi, u_{\ell_n}(\xi), u_{\ell_n}'(\xi)\right) \mathrm{d}\xi\right) \mathrm{d}s$$
$$= \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} f(\xi, u(\xi), u'(\xi) \, \mathrm{d}\xi\right) \mathrm{d}s$$

is satisfied for $t \in [0,T]$. Letting $n \to \infty$ in

$$u_{\ell_n}(t) = t\frac{cT^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f_{\ell_n}\left(\xi, u_{\ell_n}(\xi), u'_{\ell_n}(\xi)\right)\,\mathrm{d}\xi\right)\mathrm{d}s$$

yields

$$u(t) = t\frac{cT^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f(\xi, u(\xi), u'(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \tag{5.3}$$

for $t \in [0,T]$. A direct computation shows that u is a solution of (1.1a). This means that u is a positive solution of problem (1.1) and (4.12), that is $u \in \mathcal{S}_c$. Consequently $\mathcal{S}_c \neq \emptyset$. Since for each positive solution u of problem (1.1) there exists $c = c(u) \geq 0$ such that $u'(T) = -c$, we see that $\mathcal{L} = \bigcup_{0 \leq c} \mathcal{S}_c$ is the set of all positive solutions of problem (1.1). □

For $K \geq 0$, let us denote

$$\mathcal{L}_K := \bigcup_{0 \leq c \leq K} \mathcal{S}_c.$$

Then we have the following theorem.

Theorem 5.2 *Let $(H_1) - (H_3)$ hold. Then, for each $K \geq 0$, the set $\mathcal{L}_K$ is compact in $C^1[0,T]$.*

Proof. Let us choose $K \geq 0$. Then inequality (5.1) with M given in (4.15) holds for each $u \in \mathcal{L}_K$. Consider an arbitrary $u \in \mathcal{L}_K$. Then (5.3) is satisfied for some $c = c(u) \in [0, K]$ and $t \in [0,T]$. Therefore,

$$\begin{aligned} u'(t) &= \frac{cT^{a+1}}{|a+1|}(T^{-a-1} + at^{-a-1}) \\ &\quad + \int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s \\ &\quad\quad - t^{-a-1}\int_t^T s^{a+1} f(s, u(s), u'(s))\,\mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

It follows from (2.1) and (2.2),

$$|u'(t)| \leq K\left(\frac{1}{|a+1|} + 1\right) + \left(\frac{1}{|a+1|} + 1\right)\int_t^T f(s, u(s), u'(s))\,\mathrm{d}s, \quad t \in [0,T].$$

Hence, by (H_3) and $\|u\|_\infty \leq T\|u'\|_\infty$, since $u(t) = \int_0^t u'(s)\,\mathrm{d}s$, we have

$$\begin{aligned} |u'(t)| &\leq \left(\frac{1}{|a+1|} + 1\right)\left(K + \int_t^T (h(s, u(s), |u'(s)|) + g(u(s)))\,\mathrm{d}s\right) \\ &\leq \left(\frac{1}{|a+1|} + 1\right)\left(K + \int_0^T h(s, T\|u'\|_\infty, \|u'\|_\infty)\,\mathrm{d}s + W\right), \ t \in [0,T]. \end{aligned}$$

Note that by Remark 1.1,

$$\int_0^T g(u(t))\,\mathrm{d}t \le \int_0^T g(Mt(T-t)^2)dt =: W < \infty. \tag{5.4}$$

In particular,

$$1 \le \frac{1}{\|u'\|_\infty}\left(\frac{1}{|a+1|}+1\right)\left(K+\int_0^T h(s,T\|u'\|_\infty,\|u'\|_\infty)\,\mathrm{d}s + W\right). \tag{5.5}$$

Due to (H_3),

$$\lim_{w\to\infty}\frac{1}{w}\int_0^T h(\xi,Tw,w)d\xi = 0,$$

and so

$$\lim_{w\to\infty}\frac{1}{w}\left(\frac{1}{|a+1|}+1\right)\left(K+\int_0^T h(s,Tw,w)\,\mathrm{d}s + W\right) = 0,$$

which implies that there exists $\varrho^* > 0$ such that

$$\frac{1}{w}\left(\frac{1}{|a+1|}+1\right)\left(K+\int_0^T h(s,Tw,w)\,\mathrm{d}s + W\right) < 1$$

for each $w \ge \varrho^*$. This together with (5.5) results in

$$\|u'\|_\infty < \varrho^*, \quad \|u\|_\infty < \varrho^* T$$

for each $u \in \mathcal{L}_K$ and therefore, $\mathcal{L}_K$ is bounded in $C^1[0,T]$.

We now verify that the set $\{u' : u \in \mathcal{L}_K\}$ is equicontinuous on $[0,T]$. For any $u \in \mathcal{L}_K$ we have

$$\begin{aligned} u''(t) &= acT^{a+1}t^{-a-2} + at^{-a-2}\int_t^T s^{a+1} f(s,u(s),u'(s)\,\mathrm{d}s \\ &\quad + f(t,u(t),u'(t)) \quad \text{for a.e. } t \in [0,T] \quad \text{and } c = c(u) \in [0,K]. \end{aligned}$$

Therefore,

$$|u''(t)| \le |a|KT^{a+1}t^{-a-2} \;+\; |a|t^{-a-2}\int_t^T s^{a+1}m(s)\,\mathrm{d}s + m(t)$$

for a.e. $t \in [0,T]$, where $m(t) = h(t,T\varrho^*,\varrho^*) + g(Mt(T-t)^2)$. Since, cf. (5.4) and (H_3),

$$\int_0^T m(t)\,\mathrm{d}t \le \int_0^T h(t,T\varrho^*,\varrho^*)\,\mathrm{d}t + W < \infty,$$

we see that $m \in L^1[0,T]$. Therefore, by (2.2),

$$t^{-a-2}\left(\int_t^T s^{a+1} m(s)\,\mathrm{d}s\right) \in L^1[0,T].$$

Consequently, there exists a majorant function $p^* \in L^1[0,T]$ satisfying

$$|u''(t)| \le p^*(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } u \in \mathcal{L}_K.$$

As a result the set $\{u' : u \in \mathcal{L}_K\}$ is equicontinuous on $[0,T]$.

In order to complete the proof, we need to show that the set $\mathcal{L}_K$ is closed in $C^1[0,T]$. To this end, we consider a sequence $\{u_n\} \subset \mathcal{L}_K$ converging in $C^1[0,T]$ to a function $u \in C^1[0,T]$. Therefore, there exists a sequence $\{c_n\} \subset [0,K]$ such that, due to (5.3),

$$\begin{aligned} u_n(t) \;=\; & t\frac{c_n T^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) \\ & + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f(\xi, u_n(\xi), u_n'(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \text{ for } t \in [0,T] \text{ and } n \in \mathbb{N}. \end{aligned}$$

Since $u_n'(T) = -c_n$, we see that $\{c_n\}$ is convergent. Let us define $\lim_{n\to\infty} c_n =: c \in [0,K]$ and let $n \to \infty$ in the above equality for $u_n(t)$. Then, by the Lebesgue dominated convergence theorem, arguing as in the proof of Theorem 5.1, we obtain

$$\begin{aligned} u(t) \;=& t\frac{c T^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) \\ & + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f(\xi, u(\xi), u'(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \quad \text{for } t \in [0,T]. \end{aligned} \tag{5.6}$$

Therefore u satisfies (1.1) and (4.12) and hence $u \in \mathcal{S}_c \subset \mathcal{L}_K$. □

Remark 5.3 It follows from the proof of Theorem 5.2 that for each fixed $c \ge 0$ the set $\mathcal{S}_c$ is compact in $C^1[0,T]$.

Remark 5.4 Consider a sequence $\{u_n\} \subset \mathcal{L}$. Since $\mathcal{L} = \cup_{0\le c}\mathcal{S}_c$, we see that $u_n \in \mathcal{S}_{c_n}$ for some $c_n \ge 0$ and $u_n'(T) = -c_n$, $n \in \mathbb{N}$. We now can show that

$$\lim_{n\to\infty} \|u_n\|_{C^1} = \infty \quad \Longleftrightarrow \quad \lim_{n\to\infty} c_n = \infty \tag{5.7}$$

holds. First, assume that

$$\lim_{n\to\infty} \|u_n\|_{C^1} = \infty, \tag{5.8}$$

and further, assume in a contrary, that there exist $K > 0$ and a subsequence $\{c_{m_n}\} \subset \{c_n\}$ such that

$$c_{m_n} \le K \quad \text{for } n \in \mathbb{N}.$$

Since $\mathcal{L}_K = \cup_{0 \le c \le K} \mathcal{S}_c$, we have $\{u_{m_n}\} \subset \mathcal{L}_K$. By Theorem 5.2, $\mathcal{L}_K$ is compact in $C^1[0,T]$ which implies that the sequence $\{u_{m_n}\}$ is bounded in $C^1[0,T]$, which contradicts (5.8).

If we assume that $\lim_{n\to\infty} c_n = \infty$, then (5.8) immediately follows, because $\|u_n\|_{C^1} \ge \|u_n'\|_\infty \ge c_n$, $n \in \mathbb{N}$.

Remark 5.5 It follows from (5.7) that the set $\mathcal{L}$ is unbounded in $C^1[0,T]$. In particular, if $u \in \mathcal{L}$, then $u \in \mathcal{S}_c$ for some $c \ge 0$ and, keeping in mind that f is positive, we get by (5.6),

$$u\left(\frac{T}{2}\right) \ge \frac{Tc}{2|a+1|}\left(1 - \frac{1}{2^{-a-1}}\right).$$

That is $\|u\|_\infty \ge cL$, where

$$L = \frac{T}{2|a+1|}\left(1 - \frac{1}{2^{-a-1}}\right) > 0. \tag{5.9}$$

Hence,

$$\sup\{\|u\|_\infty : u \in \mathcal{L}\} \ge \sup\{cL : c \in [0,\infty)\} = \infty,$$

and Theorem 5.2 implies the following results concerning minimal values of functionals defined on the set $\mathcal{L}$ of all positive solutions to problem (1.1).

Let $\mathcal{M}$ be the set of continuous functionals $\Phi : C^1[0,T] \to [0,\infty)$, which are coercive on $\mathcal{L}$, that is,

$$\lim_{x\in\mathcal{L},\, \|x\|_{C^1}\to\infty} \Phi(x) = \infty. \tag{5.10}$$

Theorem 5.6 *Let* $(H_1)-(H_3)$ *hold and let* $\Phi \in \mathcal{M}$. *Then there exists a positive solution* u_* *of problem* (1.1) *such that*

$$\min\{\Phi(x) : x \in \mathcal{L}\} = \Phi(u_*). \tag{5.11}$$

Proof. Choose $u_0 \in \mathcal{S}_0$ and $A := \Phi(u_0)$. By (5.10), there exists $B > 0$ such that

$$x \in \mathcal{L},\ \|x\|_{C^1} > B \quad \Rightarrow \quad \Phi(x) > A. \tag{5.12}$$

Let $K := B/(L+1)$, with $L > 0$ from (5.9). According to Theorem 5.2, $\mathcal{L}_K$ is compact in $C^1[0,T]$ and consequently, the continuity of Φ implies the existence of $u_* \in \mathcal{L}_K$ such that

$$\min\{\Phi(x) : x \in \mathcal{L}_K\} = \Phi(u_*) \le A. \tag{5.13}$$

Now, assume that $c > K$. Then, for each $x \in \mathcal{S}_c$, we have $\|x'\|_\infty \ge |x'(T)| = c$ and, by Remark 5.5, $\|x\|_\infty \ge cL$. Therefore,

$$\|x\|_{C^1} \ge c(L+1) > K(L+1) = B.$$

This together with (5.12) yields $\Phi(x) > A$. Consequently, $\Phi(x) > A$ for each $x \in \mathcal{L} \setminus \mathcal{L}_K$ and (5.11) follows from (5.13). □

We now present applications of Theorem 5.6. Assume $(H_1) - (H_3)$ to hold. Choose $\alpha \in (0,\infty)$ and consider functionals $\Phi_1, \Phi_2 : C^1[0,T] \to [0,\infty)$ given by

$$\Phi_1(x) = \int_0^T |x(t)|^\alpha \mathrm{d}t, \qquad \Phi_2(x) = \int_0^T \sqrt{1 + x'^2(t)}\mathrm{d}t. \tag{5.14}$$

Then, Φ_1, Φ_2 are continuous on $C^1[0,T]$.

Let us now show that Φ_1 and Φ_2 satisfy (5.10), where $\mathcal{L}$ is the set of all positive solutions of problem (1.1). Choose an arbitrary sequence $\{u_n\} \subset \mathcal{L}$ such that $\lim_{n\to\infty} \|u_n\|_{C^1} = \infty$. For $c_n = -u_n'(T)$, $n \in \mathbb{N}$, we obtain using (5.7), $\lim_{n\to\infty} c_n = \infty$. Since $u_n(t) = \int_0^t u_n'(s)\mathrm{d}s$, it follows from Remark 5.5 that

$$c_n L \le \|u_n\|_\infty \le \int_0^T |u_n'(t)|\mathrm{d}t \le \Phi_2(u_n), \quad n \in \mathbb{N}.$$

Consequently, Φ_2 is coercive on $\mathcal{L}$.

Furthermore, from (5.6) and the positivity of f, we have

$$u_n(t) \ge c_n \varphi(t), \quad t \in [0,T],$$

where

$$\varphi(t) := t\frac{T^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) > 0, \quad t \in (0,T).$$

According to (5.14),

$$\Phi_1(u_n) = \int_0^T |u_n(t)|^\alpha \mathrm{d}t \ge c_n^\alpha \int_0^T \varphi^\alpha(t)\mathrm{d}t = c_n^\alpha M_0, \ n \in \mathbb{N},$$

where $M_0 := \int_0^T \varphi^\alpha(t)\mathrm{d}t > 0$. Hence, we have shown that $\lim_{n\to\infty} \Phi_1(u_n) = \infty$ and therefore, Φ_1 is coercive on $\mathcal{L}$.

Consequently, Theorem 5.6 is applicable to both, Φ_1 and Φ_2. We can easily see a geometrical meaning of this result. For example, dealing with Φ_2, we get that among all positive solutions of (1.1) there exists a solution having a graph with the shortest lenght. Note, that values of Φ_1 with $\alpha = 1$ are discussed in Example 3.

6 Numerical simulations

For the numerical simulation, we use an alternative formulation of problem (1.1),

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \quad t \in [0,1], \tag{6.1a}$$

$$u(1) = 0, \quad u'(1) = -c, \tag{6.1b}$$

where $c \geq 0$. According to [19], the above boundary value problem (6.1) is well-posed and therefore it is suitable for the numerical treatment. For all examples, the calculations have been carried for the values of $a = -2$ and $c = 0, 0.1, 0.2, 0.3, 0.5, 1, 2, 5, 10, 100$.

6.1 MATLAB Code `bvpsuite`

To illustrate the analytical results discussed in the previous section, we solved numerically Examples (6.2), (6.4) and (6.5) using a MATLAB™ software package `bvpsuite` designed to solve boundary value problems in ordinary differential equations and differential algebraic equations. The solver routine is based on a class of collocation methods whose orders may vary from 2 to 8. Collocation has been investigated in context of singular differential equations of first and second order in [11] and [20], respectively. This method could be shown to be robust with respect to singularities in time and for retains its high convergence order in case that the analytical solution is appropriately smooth. The code also provides an asymptotically correct estimate for the global error of the numerical approximation. To enhance the efficiency of the method, a mesh adaptation strategy is implemented, which attempts to choose grids related to the solution behavior, in such a way that the tolerance is satisfied with the least possible effort. Error estimate procedure and the mesh adaptation work dependably provided that the solution of the problem and its global error are appropriately smooth[2]. The code and the manual can be downloaded from `http://www.math.tuwien.ac.at/∼ewa`. For further information see [13]. This software is useful for the approximation of numerous singular boundary value problems important for applications, see e.g. [4], [9], [12], [17].

6.2 Example 1

We first investigate the following problem:

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = u(t)^{\frac{2}{3}} + t + 1, \quad t \in [0,1], \tag{6.2a}$$

$$u(1) = 0, \quad u'(1) = -c, \tag{6.2b}$$

[2]The required smoothness of higher derivatives is related to the order of the used collocation method.

where $a = -2$. In Figures 1 and 2, solutions to problem (6.2) for different values of c are shown.

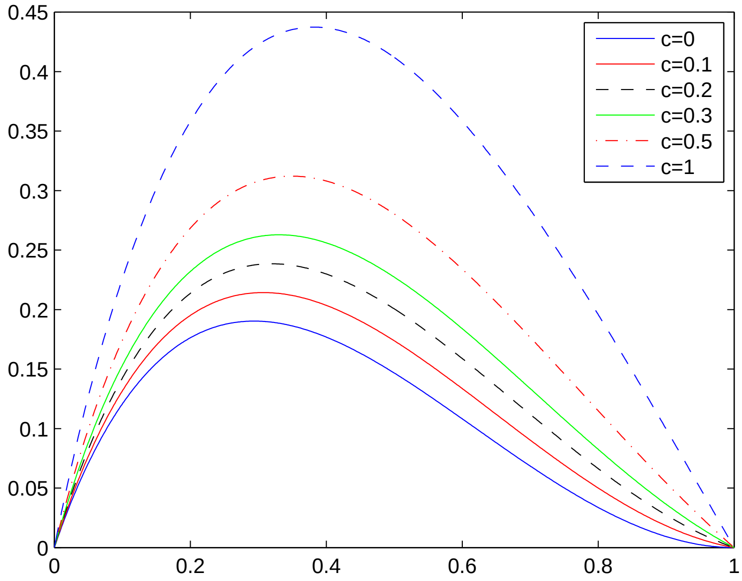

Figure 1: Problem (6.2): Parameter $a = -2$.

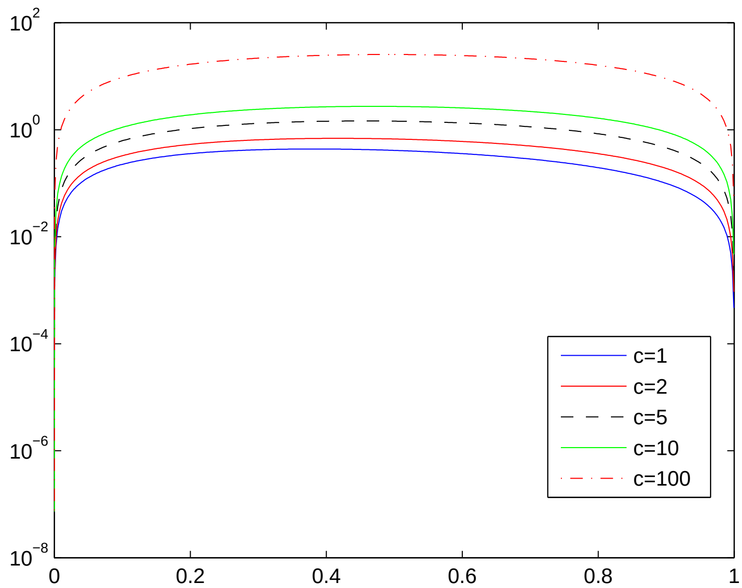

Figure 2: Problem (6.2): Parameter $a = -2$.

For a given $c \geq 0$ denote by u_c a solution of problem (6.2). We can see in Figures 1 and 2 that graphs of solutions u_c are ordered, that is

$$c_1 < c_2 \quad \Longrightarrow \quad u_{c_1}(t) < u_{c_2}(t), \quad t \in (0,1). \tag{6.3}$$

This corresponds to the theory in Section 4 of [16], where the special case of problem (1.1) with $f \in Car([0,T] \times [0,\infty))$ and $f(t,u)$ increasing in u, has been investigated.

6.3 Example 2

Here, we study the influence of u' in $f(t,u,u') = u^{\frac{2}{3}} + u'^{\frac{2}{3}} + t + 1$. The boundary value problem has now the form

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = u(t)^{\frac{2}{3}} + u'(t)^{\frac{2}{3}} + t + 1, \quad t \in [0,1], \tag{6.4a}$$

$$u(1) = 0, \quad u'(1) = -c, \tag{6.4b}$$

where $a = -2$. Note that since $u'(t)$ may become negative, we replace $u'(t)^{\frac{2}{3}}$ by $u'(t)\,|u'(t)|^{-\frac{1}{3}}$ in numerical simulations.

The solutions of the boundary value problem (6.4) for different values of c, can be found in Figures 3 and 4.

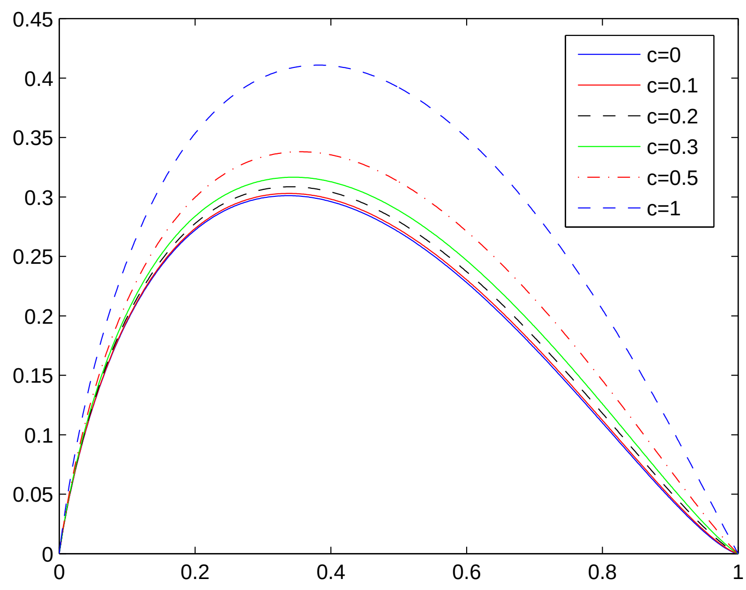

Figure 3: Problem (6.4): Parameter $a = -2$.

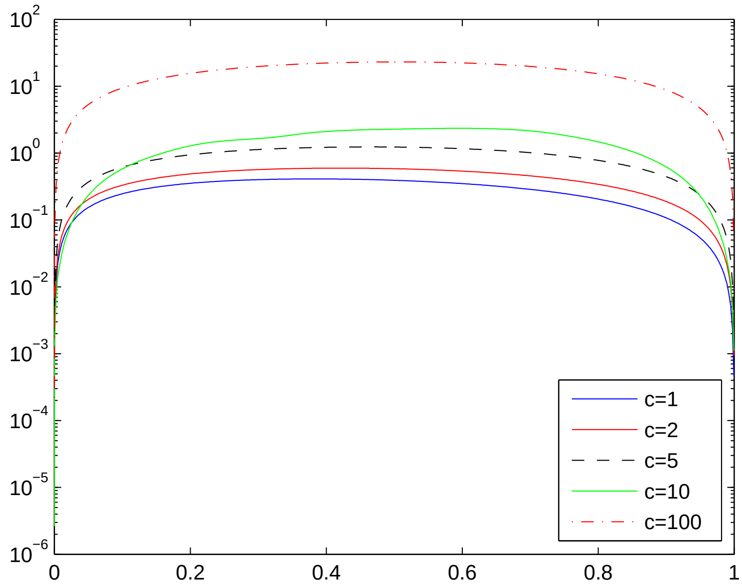

Figure 4: Problem (6.4): Parameter $a = -2$.

Figure 3 shows ordered graphs of solutions u_c of problem (6.4) with c changing from 0 to 1 but Figure 4 demonstrates that, for c having values from 1 to 100, the graphs of solutions u_c do not keep order (6.3).

The solutions of problem (6.4) for $a = -3$ and $a = -10$ show similar behavior as for $a = -2$ and hence, they are not displayed here. All results for the above class have been obtained using the same starting guess: the numerical solution for $c = 0$ obtained with the piecewise hat function as an initial profile, see Figure 5.

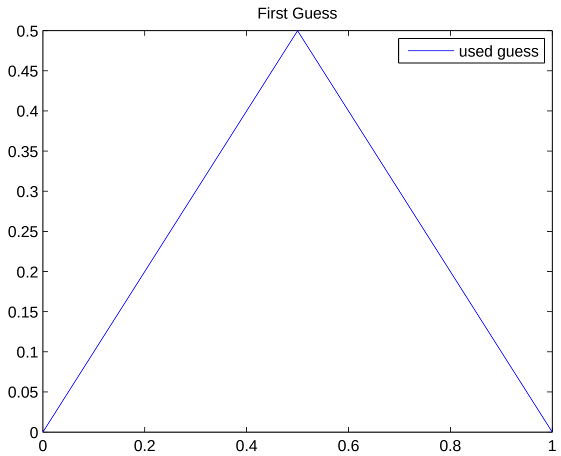

Figure 5: Starting guess for $c \neq 0$ has been obtained by solving the problem for $c = 0$ and the initial profile shown above.

6.4 Example 3

In order to discuss the influence of a possible space singularity in f, we put $f(t,u) = u^{-\frac{1}{3}} + t + 1$ and look at the following problem:

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = u(t)^{-\frac{1}{3}} + t + 1, \quad t \in [0,1], \tag{6.5a}$$

$$u(1) = 0, \quad u'(1) = -c, \tag{6.5b}$$

where $a = -2$. The solutions to the above boundary value problem can be found in Figures 6 and 7.

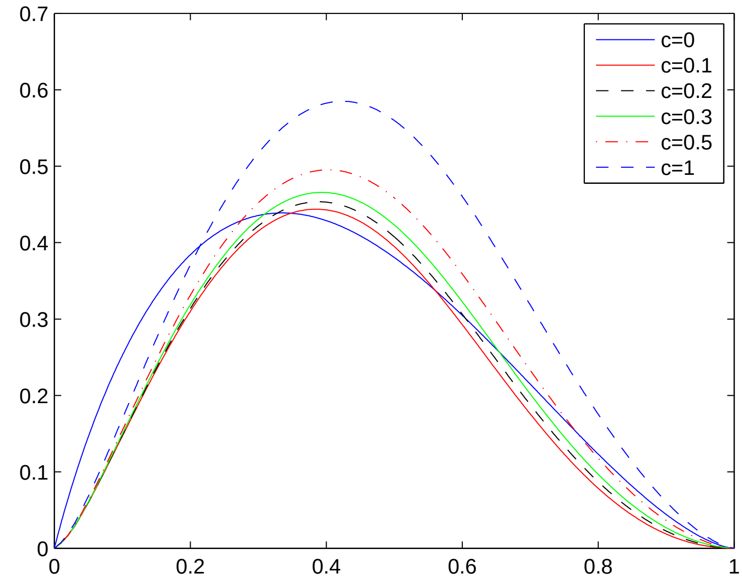

Figure 6: Problem (6.5): Parameter $a = -2$.

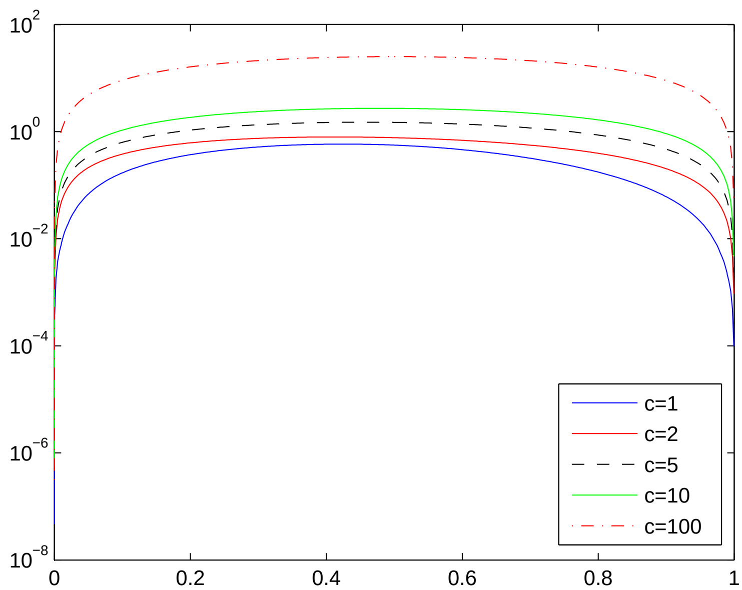

Figure 7: Problem (6.5): Parameter $a = -2$.

For the values of $c \geq 1$ the starting guess mentioned above has been used. For larger values of c providing alternative starting guesses was necessary. For example, for $c = 5$ the earlier computed solution for $c = 2$ has been used.

Figure 6 illustrates that solutions u_c of problem (6.5) with c changing from 0 to 1 do not fulfil the order given by (6.3). This together with Figure 4 leads to the hypothesis that the order (6.3) cannot be proved for solutions of (1.1), where $f(t, x, y)$ depends on y or f has a singularity at $x = 0$.

Now, consider the functional Φ_1 of (5.14) with $\alpha = 1$ and $T = 1$, that is $\Phi_1(x) = \int_0^1 |x(t)|\mathrm{d}t$ for $x \in C^1[0,1]$. Let $\mathcal{L}$ be the set of all positive solutions of problem (6.5). Using Theorem 5.6, we have proved that there exists a positive solution of problem (6.5) giving a minimal value of Φ_1 on $\mathcal{L}$. To illustrate this result, we have approximated the values of the integrals

$$\Phi_1(u_c) = \int_0^1 u_c(t)\mathrm{d}t.$$

Here, we put $a = -2$ and by u_c we denote a positive solution of (6.5) for a specific nonnegative value of c. In order to approximate $\Phi_1(u_c)]$, we introduce a partition of the interval $[0,1]$ into equidistant subintervals of length 10^{-2}. As a quadrature formula, we use the composed Gaussian rule with five evaluation points in each subinterval of $[0,1]$. The results can be found in Table 1 below.

c	$\Phi_1(u_c)$
0.0	0.259842454338672
0.1	0.228105737635487
0.2	0.235421595255397
0.3	0.244242057208884
0.5	0.264432168174144
1.0	0.322297787747369
2.0	0.521359670723535
5.0	1.000102652113081
10.0	1.819813323209159
100.0	16.79012750689064

Table 1: Problem (6.5).

Table 1 shows that $\Phi_1(u_c)$ is not monotonous for $c \in [0, \infty)$, that is the inequality $c_1 < c_2$ need not imply $\Phi_1(u_{c_1}) \le \Phi_1(u_{c_2})$. But we know that there exists at least one $c_* \in [0, \infty)$ such that $\Phi_1(u_c)$ reaches its minimum at u_{c^*}.

Acknowledgements

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