General existence principle for singular BVPs and its applications *

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Abstract: This paper provides a general existence principle which can be used for a large class of singular boundary value problems of the form

$$u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t)), \quad u \in \mathcal{S},$$

where f satisfies the local Carathéodory conditions on $[0,T] \times \mathcal{D}$, the set $\mathcal{D} \subset \mathbb{R}^n$ is not closed, f has singularities in its phase variables on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ and $\mathcal{S}$ is a closed subset in $C^{n-1}([0,T])$. The proof is based on the regularization and sequential techniques. An application of the general existence principle to singular conjugate $(p, n-p)$ BVPs is given here, as well.

Keywords: General existence principle, singular BVP, conjugate BVP, regularization, Vitali's convergence theorem.
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1 General existence principle for singular BVPs

1.1 Introduction

Let $n \in \mathbb{N}$, $[0,T] \subset \mathbb{R}$ and $\mathcal{D} \subset \mathbb{R}^n$. As usual, $C^{n-1}([0,T])$ and $AC^{n-1}([0,T])$ denotes the set of functions having $(n-1)$-th derivatives continuous and absolutely continuous on $[0,T]$, respectively. $L_1([0,T])$ is the set of Lebesgue integrable functions on $[0,T]$.

Assume $k \in \mathbb{N}$ and $\mathbb{Y} \subset \mathbb{R}^k$. $Car([0,T] \times \mathbb{Y})$ stands for the set of functions $f : [0,T] \times \mathbb{Y} \to \mathbb{R}$ fulfilling *the local Carathéodory conditions* on $[0,T] \times \mathbb{Y}$, i.e.: (i) for each $y \in \mathbb{Y}$ the function $f(\cdot, y) : [0,T] \to \mathbb{R}$ is measurable; (ii) for a.e.

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$t \in [0,T]$ the function $f(t,\cdot) : \mathbb{Y} \to \mathbb{R}$ is continuous; (iii) for each compact set $K \subset \mathbb{Y}$ the function $m_K(t) = \sup\{|f(t,y)| : y \in K\}$ is Lebesgue integrable on $[0,T]$.

Having a Banach space and its subset M, then $cl(M)$ and ∂M stand for the closure and the boundary of M. In what follows we assume that $C^{n-1}([0,T])$ and $L_1([0,T])$ is respectively equipped with the norm

$$\|x\|_{C^{n-1}} = \max\{\sum_{i=0}^{n-1} |x^{(i)}(t)| : t \in [0,T]\}, \quad \text{and} \quad \|y\|_{L_1} = \int_0^T |y(t)|dt.$$

Then $C^{n-1}([0,T])$ and $L_1([0,T])$ become Banach spaces. For any measurable set $\mathcal{M} \subset \mathbb{R}$, $\mu(\mathcal{M})$ denotes the Lebesgue measure of $\mathcal{M}$.

We study the singular BVP

$$u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t)), \tag{1.1}$$

$$u \in \mathcal{S}, \tag{1.2}$$

where f satisfies the local Carathéodory conditions on $[0,T] \times \mathcal{D}$, the set $\mathcal{D}$ is not closed, f has singularities in its phase variables on the boundary $\partial\mathcal{D}$ and $\mathcal{S}$ is a closed subset in $C^{n-1}([0,T])$. Denote $\mathcal{D}_i = \{x_i \in \mathbb{R} : (x_0, \ldots, x_{n-1}) \in \mathcal{D}\}$, $0 \le i \le n-1$.

We say that f has *a singularity on $\partial\mathcal{D}$ in its phase variable* x_j $(0 \le j \le n-1)$, if there exists $d_j \in \partial\mathcal{D}_j$ such that

$$\limsup_{x_j \to d_j, x_j \in \mathcal{D}_j} |f(t, x_0, \ldots, x_j, \ldots, x_{n-1})| = \infty \tag{1.3}$$

for a.e. $t \in [0,T]$ and all $x_i \in \mathcal{D}_i$, $0 \le i \le n-1, i \ne j$.

We say that u is *a solution of singular BVP* (1.1), (1.2) if $u \in AC^{n-1}([0,T]) \cap \mathcal{S}$ and u satisfies (1.1) for a.e. $t \in [0,T]$.

A point $t_0 \in [0,T]$ is called *a singular point* corresponding to a solution u of BVP (1.1), (1.2) if there is a $j \in \{0, \ldots, n-1\}$ such that $u^{(j)}(t_0) = d_j$ where $d_j \in \partial\mathcal{D}_j$ fulfils (1.3).

Only singular BVPs with singular points $t = 0$ or (and) $t = T$ have been studied in the mathematical literature, till now. See e.g. [1], [2], [6], [7], [8], [10], [11], [15]. Such singular BVPs can be considered under weaker assumptions imposed on f provided we look for their solutions in the set $AC^{n-1}_{loc}((0,T)) \cap C^{n-2}([0,T])$. But this approach cannot be used in the case of the general BVP (1.1), (1.2) which can have singular points anywhere in $[0,T]$. Moreover, singular points corresponding to different solutions of BVP (1.1), (1.2) can be different, as well. The first existence results for higher order BVPs having singular points inside $(0,T)$ have been recently achieved by the authors in [3] and [12].

In this paper we present the general existence principle which can be used for a large class of singular problems including those, considered in [3] and [12]. Our existence principle is formulated for the singular BVP (1.1), (1.2) and is based on the regularization and sequential techniques. Such techniques consist in a construction of a proper sequence of auxiliary regular problems and in limiting processes.

Having auxiliary regular BVPs, we first need to prove their solvability. Such proofs are often based on the Nonlinear Fredholm Alternative (see e.g. [9], Theorem 4 or [14], p. 25) which we formulate in the form convenient for the application to the problems mentioned above. Particularly, we consider the differential equation

$$u^{(n)}(t)+\sum_{i=0}^{n-1} a_i(t)u^{(i)}(t)=g(t,u(t),\dots,u^{(n-1)}(t)) \tag{1.4}$$

and the corresponding linear homogeneous equation

$$u^{(n)}(t)+\sum_{i=0}^{n-1} a_i(t)u^{(i)}(t)=0, \tag{1.5}$$

where $a_i\in L_1([0,T])$, $0\le i\le n-1$, $g\in Car([0,T]\times\mathbb{R}^n)$. Further we deal with the boundary conditions

$$\mathcal{L}_j(u)=r_j,\ 1\le j\le n, \tag{1.6}$$

with $r_j\in\mathbb{R}$ and continuous linear functionals $\mathcal{L}_j : C^{n-1}([0,T])\to\mathbb{R}$, $1\le j\le n$ and with the corresponding homogeneous conditions

$$\mathcal{L}_j(u)=0,\ 1\le j\le n. \tag{1.7}$$

By *a solution of BVP* (1.4), (1.6) we understand a function $u\in AC^{n-1}([0,T])$ which satisfies conditions (1.6) and for a.e. $t\in[0,T]$ fulfils (1.4).

Theorem 1.1. (Nonlinear Fredholm Alternative) *Let the linear homogeneous problem* (1.5), (1.7) *have only the trivial solution and let there exist a function* $\psi\in L_1([0,T])$ *such that*

$$|g(t,x_0,\dots,x_{n-1})|\le\psi(t)\quad \textit{for a.e. } t\in[0,T] \textit{ and all } x_0,\dots,x_{n-1}\in\mathbb{R}.$$

Then the nonlinear problem (1.4), (1.6) *has a solution.*

The classical tool which has been often used in limiting processes is the Lebesgue Dominated Convergence Theorem. Note that in our case of the general boundary condition (1.2) we need not be able to find a Lebesgue integrable majorant function to any auxiliary sequence of regular functions relevant to problem (1.1), (1.2), because we do not know positions of singular points t corresponding to solutions of (1.1), (1.2). Therefore our limiting processes are based on the Vitali's Convergence Theorem, where the assumption about the existence of a Lebesgue integrable majorant function is replaced with a more general assumption about the uniform absolute continuity.

A collection $\mathcal{A} \subset L_1([0,T])$ is called *uniformly absolutely continuous* on $[0,T]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi \in \mathcal{A}$ and $\mathcal{M} \subset [0,T]$ with $\mu(\mathcal{M}) < \delta$, then

$$\int_{\mathcal{M}} |\varphi(t)| dt < \varepsilon.$$

Theorem 1.2. (Vitali's convergence theorem, [5], p. 178-180.) *Let $\{f_n\}$ be a sequence in $L_1([0,T])$ which is convergent to f for a.e. $t \in [0,T]$. Then the following statements are equivalent:*

(a) *$f \in L_1([0,T])$ and $\lim_{n\to\infty} \|f_n - f\|_{L_1} = 0$.*

(b) *The sequence $\{f_n\}$ is uniformly absolutely continuous on $[0,T]$.*

1.2 Main result

Consider an auxiliary sequence of regular differential equations

$$u^{(n)}(t) = f_m(t, u(t), \dots, u^{(n-1)}(t)), \tag{1.8}$$

where $f_m \in Car([0,T] \times \mathbb{R}^n)$, $m \in \mathbb{N}$.

Theorem 1.3. (General existence principle) *Let us suppose that there is a bounded set $\Omega \subset C^{n-1}([0,T])$ such that*

(i) *for each $m \in \mathbb{N}$, the regular BVP (1.8), (1.2) has a solution $u_m \in \Omega$;*

(ii) *the sequence $\{f_m(t, u_m(t), \dots, u_m^{(n-1)}(t))\}_{m=1}^{\infty}$ is uniformly absolutely continuous on $[0,T]$.*

Then we have

(I) *there exist $u \in cl(\Omega)$ and a subsequence $\{u_{m'}\} \subset \{u_m\}$ such that*

$$\lim_{m'\to\infty} \|u_{m'} - u\|_{C^{n-1}} = 0,$$

(II) *u is a solution of the singular BVP (1.1), (1.2), if*

$$\lim_{m'\to\infty} f_{m'}(t, u_{m'}(t), \dots, u_{m'}^{(n-1)}(t)) = f(t, u(t), \dots, u^{(n-1)}(t))$$

for a.e. $t \in [0,T]$.

Proof. By (i), for each $m \in \mathbb{N}$ there exists a solution u_m of BVP (1.8), (1.2) in Ω. Since $\Omega \subset C^{n-1}([0,T])$ is bounded, there is an $r > 0$ such that

$$\|u_m\|_{C^{n-1}} \le r \quad \text{for } m \in \mathbb{N}. \tag{1.9}$$

Further, according to (ii), for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t, \tau \in [0,T]$ and each $m \in \mathbb{N}$ we have

$$|u_m^{(n-1)}(t) - u_m^{(n-1)}(\tau)| \le \left| \int_\tau^t |f_m(s, u_m(s), \dots, u_m^{(n-1)}(s))| \, ds \right| < \varepsilon$$

whenever $|t - \tau| < \delta$. Hence $\{u_m^{(n-1)}(t)\}$ is equicontinuous on $[0,T]$ and since $\{u_m\}$ satisfies (1.9), the Arzelà-Ascoli theorem guarantees the existence of a subsequence $\{u_{m'}\}$ converging in $C^{n-1}([0,T])$ to a function $u \in cl(\Omega)$. Let

$$\lim_{m' \to \infty} f_{m'}(t, u_{m'}(t), \dots, u_{m'}^{(n-1)}(t)) = f(t, u(t), \dots, u^{(n-1)}(t))$$

for a.e. $t \in [0,T]$. By (ii), the sequence $\{f_{m'}(t, u_{m'}(t), \dots, u_{m'}^{(n-1)}(t))\}$ is uniformly absolutely continuous on $[0,T]$. Therefore we can use the Vitali's convergence theorem by which $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1([0,T])$ and letting $m' \to \infty$ in the equalities

$$u_{m'}^{(n-1)}(t) = u_{m'}^{(n-1)}(0) + \int_0^t f_{m'}(s, u_{m'}(s), \dots, u_{m'}^{(n-1)}(s)) \, ds, \quad t \in [0,T], \ m' \in \mathbb{N},$$

we get

$$u^{(n-1)}(t) = u^{(n-1)}(0) + \int_0^t f(s, u(s), \dots, u^{(n-1)}(s)) \, ds, \quad t \in [0,T].$$

Consequently $u \in AC^{n-1}([0,T])$ and satisfies (1.1) a.e. on $[0,T]$. In addition, since $\{u_{m'}\} \subset \mathcal{S}$ and $\mathcal{S}$ is closed, we have $u \in \mathcal{S}$. We have proved that $u \in cl(\Omega)$ is a solution of the singular BVP (1.1), (1.2).

□

Remark 1.4. Assumption (ii) in Theorem 1.3 is equivalent to the following condition:

(iii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{(\tau_j, t_j)\}_{j=1}^\infty$ is a system of mutually disjoint intervals $(\tau_j, t_j) \subset (0,T)$ and if

$$\sum_{j=1}^\infty (t_j - \tau_j) < \delta,$$

then

$$\sum_{j=1}^\infty \int_{\tau_j}^{t_j} |f_m(t, u_m(t), \dots, u_m^{(n-1)}(t))| dt < \varepsilon$$

for each $m \in \mathbb{N}$.

For the proof see e.g. [4], Lemma 3.

Remark 1.5. The absolute continuity of the Lebesgue integral yields that condition (ii) in Theorem 1.3 is satisfied if there exists a function $\varphi \in L_1([0,T])$ such that

$$|f_m(t, u_m(t), \ldots, u_m^{(n-1)}(t))| \le \varphi(t)$$

for a.e. $t \in [0,T]$ and each $m \in \mathbb{N}$.

2 Application to conjugate BVPs

2.1 Formulation of problems

Consider the $(p, n-p)$ conjugate boundary value problem of the form

$$u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t)), \tag{2.1}$$

$$u(0) = \ldots = u^{(n-p-1)}(0) = 0, \quad u(T) = \ldots = u^{(p-1)}(T) = 0, \tag{2.2}$$

where $n > 2$, $1 \le p \le n-1$ are fixed natural numbers. Here, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_0^{n-1}$ and $f \in Car([0,T] \times \mathcal{D})$. We assume that f has singularities on $\partial\mathcal{D}$ in all its phase variables, i.e.

$$\limsup_{x_j \to 0} |f(t, x_0, \ldots, x_j, \ldots, x_{n-1})| = \infty, \quad 0 \le j \le n-1,$$

for a.e. $t \in [0,T]$ and all $(x_0, \ldots, x_{n-1}) \in \mathcal{D}$.

By replacing t by $T-t$ if necessary, we may asssume that $p \in \{1, \ldots, n/2\}$ for n even and $p \in \{1, \ldots, (n+1)/2\}$ for n odd. Then we observe that the greater p is chosen, the more complicated structure of the set of all singular points of a solution to (2.1), (2.2) is obtained. Since singular problem (2.1), (2.2) for $p = 1$ has been studied in [3] and [6], we assume $p = 2$ and consider the $(n-2, 2)$ conjugate boundary conditions

$$u(0) = \ldots = u^{(n-3)}(0) = 0, \quad u(T) = u'(T) = 0. \tag{2.3}$$

Problem (2.1), (2.3) is general enough to demonstrate the profit of the application of Theorem 1.3 because it has solutions with singular points lying inside of $(0,T)$ and it has not been solved in this setting before. Hence, the application of Theorem 1.3 provides new existence results for problem (2.1), (2.3) which generalize earlier ones in [1] and [7].

In this paper we will use the following assumptions

$$\left.\begin{array}{l} f \in Car([0,T] \times \mathcal{D}) \text{ and there exists } c > 0 \text{ such that} \\ c \le f(t, x_0, \ldots, x_{n-1}) \text{ for a.e. } t \in [0,T] \text{ and all } (x_0, \ldots, x_{n-1}) \in \mathcal{D}; \end{array}\right\} \tag{2.4}$$

$$\left.\begin{array}{l} h \in Car([0,T]\times[0,\infty)) \text{ is nondecreasing in its second argument and} \\ \limsup\limits_{z\to\infty} \dfrac{1}{z}\displaystyle\int_0^T h(t,z)dt < \left(1+\sum_{i=0}^{n-2}\frac{T^{n-i-1}}{(n-i-2)!}\right)^{-1}; \end{array}\right\} \quad (2.5)$$

$$\left.\begin{array}{l} \omega_i : \mathbb{R}_+ \to \mathbb{R}_+ \text{ are nonincreasing and} \\ \displaystyle\int_0^T \omega_i(t^{n-i})dt < \infty,\ 0 \le i \le n-1; \end{array}\right\} \quad (2.6)$$

$$\left.\begin{array}{l} f(t,x_0,\ldots,x_{n-1}) \le h(t, \sum\limits_{i=0}^{n-1}|x_i|) + \sum\limits_{i=0}^{n-1}\omega_i(|x_i|) \\ \text{for a.e. } t\in[0,T] \text{ and all } (x_0,\ldots,x_{n-1})\in\mathcal{D}, \\ \text{where } h \text{ and } \omega_i \text{ satisfy (2.5) and (2.6).} \end{array}\right\} \quad (2.7)$$

The main result for the solvability of problem (2.1,),(2.3) is given in the following theorem.

Theorem 2.1. (Existence result for (2, n-2) conjugate BVP) *Let assumptions* (2.4) – (2.7) *hold. Then problem* (2.1), (2.3) *has a solution which is positive on* $(0,T)$.

2.2 Lemmas

Here, we prove *a priori* estimates of a certain class of functions (Lemmas 2.4 and 2.5) which will be needed for the construction of the set Ω in order to apply Theorem 1.3. To this aim we first provide a description of zeros for the class of functions under consideration.

Lemma 2.2. *Let* $c > 0$ *and let* $u \in AC^{n-1}([0,T])$ *satisfy* (2.3) *and*

$$u^{(n)}(t) \ge c \quad \text{for a.e. } t \in [0,T]. \tag{2.8}$$

Then we have

(*i*) $u > 0$ *on* $(0,T)$ *and there exists a unique point* $t_k \in (0,T)$ *such that* $u^{(k)}(t_k) = 0$ *for* $k = 1$ *and* $k = n-1$;

(*ii*) *if* $n \ge 4$, *then there exist just two points* $t_i, s_i \in (0,T)$, $t_i < s_i$, *such that*

$$u^{(i)}(t_i) = u^{(i)}(s_i) = 0 \text{ for } 2 \le i \le n-2;$$

(*iii*) *all these zeros are ordered in the following way:*
if $n = 3$, *then* $0 < t_1 < t_2 < T$,
if $n = 4$, *then* $0 < t_2 < t_1 < s_2 < T$, $0 < t_2 < t_3 < T$,
if $n > 4$, *then*

$$\begin{cases} 0 < t_{n-2} < t_{n-3} < \ldots < t_3 < t_2 < t_1 < T, \\ 0 < t_{n-2} < t_{n-1} < s_{n-2} < s_{n-3} < \ldots < s_3 < s_2 < T. \end{cases} \tag{2.9}$$

Proof. First, we prove that u' has just one zero in $(0,T)$. Since $u(0)=u(T)$, u' has at least one zero in $(0,T)$.
Let $n=3$. Assume that there are t_1, s_1 such that $0 \le t_1 < s_1 < T$ and $u'(t_1) = u'(s_1) = 0$. Then, by (2.3) and the Mean Value Theorem, we can find $t_2 \in (t_1, s_1)$, $s_2 \in (s_1, T)$ such that $u''(t_2) = u''(s_2) = 0$, which contradicts (2.8).
Let $n \ge 4$ and asssume that u' has zeros t_1, s_1 and $0 < t_1 < s_1 < T$. Then, by (2.3) as before, we can find $t_2 \in (0, t_1)$, $s_2 \in (t_1, s_1)$, $r_2 \in (s_1, T)$ such that

$$u''(t_2) = u''(s_2) = u''(r_2) = 0. \tag{2.10}$$

If $n=4$, then (2.10) implies that $u^{(n)}$ has a zero in $(0,T)$, contrary to (2.8). Let $n>4$. Then (2.10) with $u''(0)=0$ yields the existence of points $t_3 \in (0,t_2)$, $s_3 \in (t_2, s_2)$, $r_3 \in (s_2, T)$ such that $u'''(t_3) = u'''(s_3) = u'''(r_3) = 0$. Continue inductively we get points $t_{n-2} \in (0, t_{n-3})$, $s_{n-2} \in (t_{n-3}, s_{n-3})$, $r_{n-2} \in (s_{n-3}, T)$ such that $u^{(n-2)}(t_{n-2}) = u^{(n-2)}(s_{n-2}) = u^{(n-2)}(r_{n-2}) = 0$. Consequently we have at least two zeros of $u^{(n-1)}$ in $(0,T)$ and at least one zero of $u^{(n)}$ in $(0,T)$, contrary to (2.8).
Hence, for $n \ge 3$, u' has just one zero $t_1 \in (0,T)$. Therefore we can deduce by the same argument as before that $u^{(i)}$ has at least two zeros $t_i, s_i \in (0,T), t_i < s_i, 2 \le i \le n-2$, $u^{(n-1)}$ has at least one zero $t_{n-1} \in (0,T)$, and (2.9) holds. By (2.8), $u^{(n-1)}$ is increasing on $[0,T]$ which implies that $u^{(n-1)}$ has the only zero $t_{n-1} \in (0,T)$. So, $u^{(i)}$ has just two zeros $t_i, s_i \in (0,T), 2 \le i \le n-2$. Finally, by (2.8), $u^{(n-2)}$ is convex on $[0,T]$ which yields $u^{(n-2)} > 0$ on $(0, t_{n-2})$. Therefore,by (2.3), $u^{(i)} > 0$ on $(0, t_i)$, $1 \le i \le n-3$, and consequently $u > 0$ on $(0,T)$. This completes the proof. □

Remark 2.3. Lemma 2.2 describes a location of all singular points of an arbitrary solution u to problem (2.1),(2.3). Particularly, u always has the singular points 0 and T and, for $n>4$, u has two sets of singular points depending on u and lying inside $(0,T)$: $\{t_i\}_{i=1}^{n-1}$ and $\{s_j\}_{j=2}^{n-2}$. These two sets need not be disjoint and their location is given by (2.9).

Define

$$\mathcal{B} = \{u \in AC^{n-1}([0,T]) : u \text{ satisfies (2.3) and (2.8) with } c > 0\}. \tag{2.11}$$

Lemma 2.4. *Let ω_i fulfil* (2.6). *Then there exist constants $A_i \in \mathbb{R}_+$, $0 \le i \le n-1$, such that for each function $u \in \mathcal{B}$ the estimates*

$$\int_0^T \omega_i(|u^{(i)}(t)|)dt \le A_i, \quad 0 \le i \le n-1, \tag{2.12}$$

hold.

Proof. Let $u \in \mathcal{B}$. Then, by Lemma 2.2, there is just one zero t_{n-1} of $u^{(n-1)}$ in $(0,T)$. Integrating (2.8) we get

$$\left.\begin{aligned} -u^{(n-1)}(t) &\geq c(t_{n-1}-t) \quad \text{for} \quad t \in [0, t_{n-1}], \\ u^{(n-1)}(t) &\geq c(t-t_{n-1}) \quad \text{for} \quad t \in [t_{n-1}, T]. \end{aligned}\right\} \tag{2.13}$$

Step 1. We find estimates for $u^{(n-2)}$.
Integrate the first inequality in (2.13) from $t \in [0, t_{n-2})$ to t_{n-2}. Then

$$u^{(n-2)}(t) = -\int_t^{t_{n-2}} u^{(n-1)}(\tau)d\tau \geq \frac{c}{2}(-(t_{n-1}-t_{n-2})^2+(t_{n-1}-t)^2) \geq \frac{c}{2}(t_{n-2}-t)^2.$$

Hence,

$$u^{(n-2)}(t) \geq \frac{c}{2!}(t_{n-2}-t)^2 \text{ for } t \in [0, t_{n-2}]. \tag{2.14}$$

Let $n > 3$. Integrate the first inequality in (2.13) from t_{n-2} to $t \in (t_{n-2}, t_{n-1}]$ and then integrate the second inequality in (2.13) from $t \in [t_{n-1}, s_{n-2})$ to s_{n-2} and from s_{n-2} to $t \in (s_{n-2}, T]$. We get

$$\left.\begin{aligned} -u^{(n-2)}(t) &\geq \frac{c}{2!}(t-t_{n-1})^2 \quad \text{for} \quad t \in [t_{n-2}, t_{n-1}], \\ -u^{(n-2)}(t) &\geq \frac{c}{2!}(s_{n-2}-t)^2 \quad \text{for} \quad t \in [t_{n-1}, s_{n-2}], \\ u^{(n-2)}(t) &\geq \frac{c}{2!}(t-s_{n-2})^2 \quad \text{for} \quad t \in [s_{n-2}, T]. \end{aligned}\right\} \tag{2.15}$$

If $n = 3$, we put $s_{n-2} = T$ in (2.15).

Step 2. We find estimates for $u^{(n-3)}$.
Let $n > 4$. There are two cases to consider: (i) $t_{n-3} \leq t_{n-1}$ and (ii) $t_{n-3} > t_{n-1}$.

Case (i). Let $t_{n-3} \leq t_{n-1}$. After integration, (2.14) gives

$$u^{(n-3)}(t) \geq \frac{c}{3!}(-(t_{n-2}-t)^3 + t_{n-2}^3) \geq \frac{c}{3!}t^3 \text{ on } [0, t_{n-2}].$$

The first inequality in (2.15) implies, by integration,

$$\left.\begin{aligned} u^{(n-3)}(t) &\geq \frac{c}{3!}(t_{n-3}-t)^3 \quad \text{for} \quad t \in [t_{n-2}, t_{n-3}], \\ -u^{(n-3)}(t) &\geq \frac{c}{3!}(t-t_{n-3})^3 \quad \text{for} \quad t \in [t_{n-3}, t_{n-1}]. \end{aligned}\right\}$$

Since $-u^{(n-3)}(t_{n-1}) \geq 0$, we integrate the second inequality in (2.15) from t_{n-1} to $t \in (t_{n-1}, s_{n-2}]$ and deduce that

$$u^{(n-3)}(t) \geq \frac{c}{3!}(t-t_{n-1})^3 \text{ for } t \in [t_{n-1}, s_{n-2}].$$

Finally, integrating the third inequality in (2.15), we get

$$\left.\begin{aligned} -u^{(n-3)}(t) &\geq \frac{c}{3!}(s_{n-3}-t)^3 \quad \text{for} \quad t \in [s_{n-2}, s_{n-3}], \\ u^{(n-3)}(t) &\geq \frac{c}{3!}(t-s_{n-3})^3 \quad \text{for} \quad t \in [s_{n-3}, T]. \end{aligned}\right\}$$

Case (ii). Let $t_{n-3} > t_{n-1}$. Then by an analogous procedure we get

$$\left.\begin{array}{rll} u^{(n-3)}(t) \geq \dfrac{c}{3!}t^3 & \text{for} & t \in [0, t_{n-2}], \\ u^{(n-3)}(t) \geq \dfrac{c}{3!}(t_{n-1}-t)^3 & \text{for} & t \in [t_{n-2}, t_{n-1}], \\ u^{(n-3)}(t) \geq \dfrac{c}{3!}(t_{n-3}-t)^3 & \text{for} & t \in [t_{n-1}, t_{n-3}], \\ -u^{(n-3)}(t) \geq \dfrac{c}{3!}(t-t_{n-3})^3 & \text{for} & t \in [t_{n-3}, s_{n-2}], \\ -u^{(n-3)}(t) \geq \dfrac{c}{3!}(s_{n-3}-t)^3 & \text{for} & t \in [s_{n-2}, s_{n-3}], \\ u^{(n-3)}(t) \geq \dfrac{c}{3!}(t-s_{n-3})^3 & \text{for} & t \in [s_{n-3}, T]. \end{array}\right\}$$

If $n=4$ ($n=3$), we put $s_{n-3}=T$ ($t_{n-3}=s_{n-3}=T$) and argue as above.

Step 3. If $n \geq 4$ we find estimates for $u^{(i)}$, $0 \leq i \leq n-4$.
Using the estimates of *Step 2* we continue inductively and deduce that for any $i \in \{2, \ldots, n-4\}$ there are p_i disjoint intervals $(a_{k-1}, a_k), 1 \leq k \leq p_i,\ \ p_i \leq 2(n-i)$, such that

$$\bigcup_{k=1}^{p_i} [a_{k-1}, a_k] = [0, T], \tag{2.16}$$

and for each $k \in \{1, \ldots, p_i\}$ one of the inequalities

$$\left.\begin{array}{lll} |u^{(i)}(t)| \geq \dfrac{c}{(n-i)!}(t-a_{k-1})^{n-i} & \text{for} & t \in [a_{k-1}, a_k] \\ \text{or} & & \\ |u^{(i)}(t)| \geq \dfrac{c}{(n-i)!}(a_k-t)^{n-i} & \text{for} & t \in [a_{k-1}, a_k] \end{array}\right\} \tag{2.17}$$

is satisfied. If $i \in \{0, 1\}$, then (2.16) and (2.17) are valid with $p_i \leq 2n-3$.

Step 4. We prove the estimates (2.12).
To summarize, *Steps 1-3* provide that for each function $u \in \mathcal{B}$ and for each $i \in \{0, \ldots, n-1\}$ there exists a finite number $p_i < 2n$ of disjoint intervals (a_{k-1}, a_k), $1 \leq k \leq p_i$, such that (2.16) and (2.17) are satisfied. Now, choose an arbitrary $i \in \{0, \ldots, n-1\}$. Since ω_i is nonincreasing and (2.17) is true, we have

$$\int_0^T \omega_i(|u^{(i)}(t)|)dt = \sum_{k=1}^{p_i} \int_{a_{k-1}}^{a_k} \omega_i(|u^{(i)}(t)|)dt$$

$$< \sum_{k=1}^{p_i} \left[\int_{a_{k-1}}^{a_k} \omega_i\left(\frac{c}{(n-1)!}(t-a_{k-1})^{n-i}\right)dt + \int_{a_{k-1}}^{a_k} \omega_i\left(\frac{c}{(n-1)!}(a_k-t)^{n-i}\right)dt\right].$$

If we put $c_i^{n-i} = c/(n-i)!$ and $z = c_i(t-a_{k-1})$ in the first integral ($z = c_i(a_k-t)$ in the second integral), we get, by (2.6),

$$\int_0^T \omega_i(|u^{(i)}(t)|)dt < \frac{4n}{c_i} \int_0^{c_iT} \omega_i(z^{n-i})dz = A_i < \infty.$$

Lemma is proved. $\square$

Lemma 2.5. *Let* $\mathcal{B}$ *be defined by* (2.11) *and let* (2.5), (2.6) *be fulfilled. Then there exists* $r^* > 1$ *such that for each function* $u \in \mathcal{B}$ *satisfying*

$$u^{(n)}(t) \le h(t, n + \sum_{i=0}^{n-1} |u^{(i)}(t)|) + \sum_{i=0}^{n-1} [\omega_i(|u^{(i)}(t)|) + \omega_i(1)] \tag{2.18}$$

for a.e. $t \in [0, T]$, *the estimate*

$$\|u\|_{C^{n-1}} < r^* \tag{2.19}$$

holds.

Proof. Consider $u \in \mathcal{B}$ satisfying (2.18). By Lemma 2.2 we can find $t_{n-2} \in (0, T)$ such that $u^{(n-2)}(t_{n-2}) = 0$.

Put $\max\{|u^{(n-1)}(t)| : t \in [0, T]\} = \rho$. Then $-\rho \le u^{(n-1)}(t) \le \rho$ on $[0, T]$. Integrate this inequality from t_{n-2} to $t \in (t_{n-2}, T]$ and from $t \in [0, t_{n-2})$ to t_{n-2}. We get $-\rho T \le u^{(n-2)}(t) \le \rho T$ on $[0, T]$. Similarly, using (2.3) and repeating the integration from 0 to $t \in (0, T]$, we obtain step by step

$$|u^{(i)}(t)| \le \rho \frac{T^{n-i-1}}{(n-i-2)!} \quad \text{on } [0, T], \quad 0 \le i \le n-3.$$

Hence,

$$\|u\|_{C^{n-1}} = \max \sum_{i=0}^{n-1} \{|u^{(i)}(t)| : t \in [0, T]\} \le \rho K, \tag{2.20}$$

where

$$K = 1 + \sum_{i=0}^{n-2} \frac{T^{n-i-1}}{(n-i-2)!}.$$

Now, integrate (2.18) from t_{n-1} to $t \in (t_{n-1}, T]$, where $t_{n-1} \in (0, T)$ is the unique zero of $u^{(n-1)}$, by Lemma 2.2. Then, due to (2.8),

$$0 < u^{(n-1)}(t) \le \int_{t_{n-1}}^{t} h\Big(s, n + \sum_{i=0}^{n-1} |u^{(i)}(s)|\Big) ds + \sum_{i=0}^{n-1} \int_{t_{n-1}}^{t} [\omega_i(|u^{(i)}(s)|) + \omega_i(1)] ds$$

for $t \in (t_{n-1}, T]$. Similarly, integrating (2.18) from $t \in [0, t_{n-1})$ to t_{n-1} and using (2.8), we get

$$\begin{aligned} 0 < -u^{(n-1)}(t) \ &\le\ \int_{t}^{t_{n-1}} h\Big(s, n + \sum_{i=0}^{n-1} |u^{(i)}(s)|\Big) ds \\ &\quad + \sum_{i=0}^{n-1} \int_{t}^{t_{n-1}} [\omega_i(|u^{(i)}(s)|) + \omega_i(1)] ds \end{aligned}$$

for $t \in [0, t_{n-1})$. Hence, by (2.5) and (2.20),

$$|u^{(n-1)}(t)| \le \int_0^T h(s, n + \rho K) ds + \sum_{i=0}^{n-1} \Big[T\omega_i(1) + \int_0^T \omega_i(|u^{(i)}(s)|) ds \Big]$$

for $t \in [0, T]$. Further, by Lemma 2.4, we can find constants $A_i, 0 \le i \le n-1$, independent on u and satisfying (2.12). So, if we put

$$\sum_{i=0}^{n-1} [T\omega_i(1) + A_i] = A,$$

we have $\rho \le \int_0^T h(s, n + \rho K) ds + A$ and consequently

$$1 \le \frac{K}{n + \rho K} \int_0^T h(s, n + \rho K) ds + \frac{AK + n}{n + \rho K}. \tag{2.21}$$

According to (2.5) we can find $\varepsilon > 0$ and $\rho^* > 0$ such that for all $r > \rho^*$

$$\frac{1}{r} \int_0^T h(t, r) dt \le \frac{1 - \varepsilon}{K} \quad \text{and} \quad \frac{AK + n}{r} < \varepsilon.$$

Therefore, by (2.21), $n + \rho K \le \rho^*$. So, we can put $r^* = \rho^*/K$ and (2.19) is proved. □

According to the assumption (ii) of Theorem 1.3, the following lemma is devoted to the study of the uniform absolute continuity of the function set $\mathcal{W}$ defined below.

Lemma 2.6. *Let ω_i fulfil (2.6) and $\mathcal{B}$ be given by (2.11). Put*

$$\mathcal{W} = \{\omega_i(|u^{(i)}|) : u \in \mathcal{B},\ 0 \le i \le n-1\}. \tag{2.22}$$

Then the collection $\mathcal{W}$ is uniformly absolutely continuous on $[0, T]$; i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathcal{M}} \omega_i(|u^{(i)}(s)|) ds < \varepsilon$$

for each $u \in \mathcal{B}, 0 \le i \le n-1$, and for each $\mathcal{M} \subset [0, T], \mu(\mathcal{M}) < \delta$.

Proof. By Remark 1.4 it sufficies to prove that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any system of mutually disjoint intervals $(\alpha_j, \beta_j) \subset [0, T], j \in \mathbb{N}$, the condition

$$\sum_{j=1}^{\infty} (\beta_j - \alpha_j) < \delta \Longrightarrow \sum_{j=1}^{\infty} \int_{\alpha_j}^{\beta_j} \omega_i(|u^{(i)}(s)|) ds < \varepsilon, \tag{2.23}$$

is valid for $u \in \mathcal{B}$ and $0 \le i \le n-1$. Choose an arbitrary $\varepsilon > 0$, an arbitrary $u \in \mathcal{B}$ and an arbitrary $i_0 \in \{0, 1, \dots, n-1\}$. Put

$$\gamma_{i_0}(t) = \omega_{i_0}(t^{n-i_0}), \quad \Gamma_{i_0}(x) = \int_0^x \gamma_{i_0}(s)ds \quad \text{for } t, x \in [0, T].$$

Then, by (2.6), Γ_{i_0} is absolutely continuous on $[0, T]$. It means that if we put

$$c_{i_0}^{n-i_0} = \frac{c}{(n-i_0)!}, \quad \varepsilon_0 = \frac{1}{4} c_{i_0} \varepsilon, \tag{2.24}$$

then there exists $\delta_0 > 0$ such that for any system $\{(a_j, b_j)\}_{j=1}^{\infty}$ of mutually disjoint intervals $(a_j, b_j) \subset [0, T]$, the condition

$$\sum_{j=1}^{\infty} (b_j - a_j) < \delta_0 \Longrightarrow \sum_{j=1}^{\infty} (\Gamma_{i_0}(b_j) - \Gamma_{i_0}(a_j)) < \varepsilon_0 \tag{2.25}$$

is valid. By virtue of *Step 4* of the proof of Lemma 2.4, there are $p_{i_0} (< 2n)$ disjoint intervals $(a_{k-1}, a_k), 1 \le k \le p_{i_0}$, such that (2.16) and (2.17) hold. Choose a system $\{(\alpha_j, \beta_j)\}_{j=1}^{\infty}$, $(\alpha_j, \beta_j) \subset [0, T]$ satisfying

$$\sum_{j=1}^{\infty} (\beta_j - \alpha_j) < \delta, \tag{2.26}$$

where $\delta \in (0, \delta_0 / c_{i_0})$ is so small that for each $j \in \mathbb{N}$ there is a $k \in \{1, \dots, p_{i_0} - 1\}$ such that

$$[\alpha_j, \beta_j] \subset [a_{k-1}, a_{k+1}]. \tag{2.27}$$

Fix $j \in \mathbb{N}$ and find $k \in \{1, \dots, p_{i_0} - 1\}$ fulfilling (2.27). Put

$$I_j = \int_{\alpha_j}^{\beta_j} \omega_{i_0}(|u^{(i_0)}(t)|)dt. \tag{2.28}$$

There are two cases to consider. *Case (i)*: $\alpha_j < a_k < \beta_j$; *Case (ii)*: $[\alpha_j, \beta_j] \subset [a_{k-1}, a_k]$ or $[\alpha_j, \beta_j] \subset [a_k, a_{k+1}]$.

Case (i). Let $\alpha_j < a_k < \beta_j$. Since ω_{i_0} is nonincreasing, we get by (2.17), (2.24), (2.27) and (2.28)

$$\begin{aligned} I_j < & \int_{\alpha_j}^{a_k} \omega_{i_0}((c_{i_0}(t - a_{k-1}))^{n-i_0})dt + \int_{\alpha_j}^{a_k} \omega_{i_0}((c_{i_0}(a_k - t))^{n-i_0})dt \\ & + \int_{a_k}^{\beta_j} \omega_{i_0}((c_{i_0}(t - a_k))^{n-i_0})dt + \int_{a_k}^{\beta_j} \omega_{i_0}((c_{i_0}(a_{k+1} - t))^{n-i_0})dt. \end{aligned}$$

Hence,

$$\left.\begin{aligned} I_j < \frac{1}{c_{i_0}} & [\Gamma_{i_0}(c_{i_0}(a_k - a_{k-1})) - \Gamma_{i_0}(c_{i_0}(\alpha_j - a_{k-1})) \\ & + \Gamma_{i_0}(c_{i_0}(a_k - \alpha_j)) - \Gamma_{i_0}(0) + \Gamma_{i_0}(c_{i_0}(\beta_j - a_k)) - \Gamma_{i_0}(0) \\ & + \Gamma_{i_0}(c_{i_0}(a_{k+1} - a_k)) - \Gamma_{i_0}(c_{i_0}(a_{k+1} - \beta_j))]. \end{aligned}\right\} \tag{2.29}$$

Case (ii). If $[\alpha_j, \beta_j] \subset [a_{m-1}, a_m], m \in \{k, k+1\}$, we similarly get

$$\left.\begin{aligned} I_j < \frac{1}{c_{i_0}}[&\Gamma_{i_0}(c_{i_0}(\beta_j - a_{m-1})) - \Gamma_{i_0}(c_{i_0}(\alpha_j - a_{m-1})) \\ &+\Gamma_{i_0}(c_{i_0}(a_m - \alpha_j)) - \Gamma_{i_0}(c_{i_0}(a_m - \beta_j))]. \end{aligned}\right\} \tag{2.30}$$

By (2.25), (2.26), (2.29) and (2.30), we have

$$\sum_{j=1}^{\infty} I_j < \frac{4\varepsilon_0}{c_{i_0}} = \varepsilon$$

and so (2.23) is proved. □

2.3 Proof of the existence result and an example

Here, in the proof of the existence result for problem (2.1), (2.3) given in Theorem 2.1, we will demonstrate the application of Theorem 1.3.

Proof of Theorem 2.1.

Step 1. We construct auxiliary regular BVPs.
Let $r^* > 1$ be the constant by Lemma 2.5 satisfying (2.19). Put

$$\sigma_0(x) = \begin{cases} |x| & \text{if } |x| \le r^* \\ r^* & \text{if } |x| > r^*, \end{cases} \qquad \sigma(x) = \begin{cases} x & \text{if } |x| \le r^* \\ r^* \mathrm{sgn} x & \text{if } |x| > r^*. \end{cases}$$

Choose $m \in \mathbb{N}$ and first define an auxiliary function $h_m \in Car([0,T] \times [0,\infty) \times \mathbb{R}^{n-1})$ by the following recurrent formulas. For a.e. $t \in [0,T]$ and all $x_0 \in [0,\infty), x_1, \ldots, x_{n-1} \in \mathbb{R}$ we put

$$h_{m,0}(t, x_0, x_1, \ldots, x_{n-1}) = \begin{cases} f(t, x_0, x_1, \ldots, x_{n-1}) & \text{if } x_0 \ge \frac{1}{m} \\ f(t, \dfrac{1}{m}, x_1, \ldots, x_{n-1}) & \text{if } 0 \le x_0 \le \frac{1}{m}, \end{cases}$$

$$h_{m,i}(t, x_0, \ldots, x_i, \ldots, x_{n-1}) =$$

$$= \begin{cases} h_{m,i-1}(t, x_0, \ldots, x_i, \ldots, x_{n-1}) \quad \text{if } |x_i| \ge \dfrac{1}{m} \\ \dfrac{m}{2}[h_{m,i-1}(t, x_0, \ldots, x_{i-1}, \dfrac{1}{m}, x_{i+1}, \ldots, x_{n-1})(x_i + \dfrac{1}{m}) - \\ -h_{m,i-1}(t, x_0, \ldots, x_{i-1}, -\dfrac{1}{m}, x_{i+1}, \ldots, x_{n-1})(x_i - \dfrac{1}{m})] \\ \qquad\qquad\qquad\qquad \text{if } |x_i| < \dfrac{1}{m} \end{cases}$$

for $1 \le i \le n-1$, and

$$h_m(t, x_0, \dots, x_{n-1}) = h_{m,n-1}(t, x_0, \dots, x_{n-1}).$$

For a.e. $t \in [0,T]$ and for all $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$ put

$$f_m(t, x_0, x_1, \dots, x_{n-1}) = h_m(t, \sigma_0(x_0), \sigma(x_1), \dots, \sigma(x_{n-1})) \tag{2.31}$$

and define the set

$$\mathcal{S} = \{u \in C^{n-1}([0,T]) : u \text{ satisfies } (2.3)\}. \tag{2.32}$$

We see by (2.4) that $f_m \in Car([0,T] \times \mathbb{R}^n)$ for $m \in \mathbb{N}$. Hence, we have regular auxiliary problems (1.8),(1.2) with f_m and $\mathcal{S}$ given by (2.31) and (2.32), respectively. Since $\mathcal{S}$ is closed, we can apply Theorem 1.3 on problems (1.8),(1.2), $m \in \mathbb{N}$.

Step 2. We prove that the assumption (i) of Theorem 1.3 is satisfied.
By (2.4) and (2.31), for each $m \in \mathbb{N}$ there exists $g_m \in L([0,T])$ such that

$$\left.\begin{array}{l} c \le f_m(t, x_0, \dots, x_{n-1}) \le g_m(t) \\ \text{for a.e. } t \in [0,T] \text{ and for all } (x_0, \dots, x_{n-1}) \in \mathbb{R}^n. \end{array}\right\} \tag{2.33}$$

Further, for $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$ and $m \in \mathbb{N}$, we have $\max\{\sigma_0(x_0), 1/m\} \le |x_0| + 1$, $\omega_0(\max\{\sigma_0(x_0), 1/m\}) < \omega_0(|x_0|) + \omega_0(r^*) < \omega_0(|x_0|) + \omega_0(1)$, and similarly $\max\{|\sigma(x_i)|, 1/m\} \le |x_i| + 1$, $\omega_i(\max\{|\sigma(x_i)|, 1/m\}) < \omega_i(|x_i|) + \omega_i(1)$, $1 \le i \le n-1$. Therefore, by (2.7), for each $m \in \mathbb{N}$

$$\left.\begin{array}{l} f_m(t, x_0, \dots, x_{n-1}) \le h\Big(t, n + \sum\limits_{i=0}^{n-1} |x_i|\Big) + \sum\limits_{i=0}^{n-1} [\omega_i(|x_i|) + \omega_i(1)] \\ \text{for a.e. } t \in [0,T] \text{ and for all } (x_0, \dots, x_{n-1}) \in \mathbb{R}^n. \end{array}\right\} \tag{2.34}$$

We see that the problem $u^{(n)}(t) = 0$, (2.3) has only the trivial solution. Hence, by (2.33) and the Nonlinear Fredholm Alternative (see Theorem 1.1), there exists a solution u_m of problem (1.8),(1.2) for $m \in \mathbb{N}$. Define

$$\Omega = \{u \in \mathcal{B} : \|u\|_{C^{n-1}} < r^*\}.$$

According to (2.33) and (2.34), $u_m \in \mathcal{B}$ and satisfies (2.18). Hence, by Lemma 2.5, $u_m \in \Omega$ for $m \in \mathbb{N}$.

Step 3. We prove that the assumption (ii) of Theorem 1.3 holds.
Let us consider the sequence

$$\mathcal{A} = \{f_m(t, u_m(t), \dots, u_m^{(n-1)}(t))\}_{m=1}^{\infty}.$$

Then, by (2.33) and (2.34), for $m \in \mathbb{N}$

$$0 < f_m(t, u_m(t), \dots, u_m^{(n-1)}(t)) \le h\Big(t, n + \sum_{i=0}^{n-1} |u_m^{(i)}(t)|\Big) + \sum_{i=0}^{n-1} [\omega_i(|u_m^{(i)}(t)|) + \omega_i(1)].$$

Since $h \in Car([0,T] \times [0,\infty))$ and $u_m \in \Omega$ for $m \in \mathbb{N}$, there exists a function $h^* \in L_1([0,T])$ such that

$$h\Big(t, n + \sum_{i=0}^{n-1} |u_m^{(i)}(t)|\Big) + \sum_{i=0}^{n-1} \omega_i(1) \le h^*(t)$$

for a.e. $t \in [0,T]$ and $m \in \mathbb{N}$. So, we have

$$\left. \begin{array}{l} |f_m(t, u_m(t), \dots, u_m^{(n-1)}(t))| \le h^*(t) + \displaystyle\sum_{i=0}^{n-1} \omega_i(|u_m^{(i)}(t)|) \\ \text{for a.e. } t \in [0,T] \text{ and for } m \in \mathbb{N}. \end{array} \right\} \tag{2.35}$$

Let the collection $\mathcal{W}$ be defined by (2.22). Then, by Lemma 2.6, $\mathcal{W}$ is uniformly absolutely continuous on $[0,T]$, which implies that the sequence

$$\Big\{ \sum_{i=0}^{n-1} \omega_i(|u_m^{(i)}(t)|) \Big\}_{m=1}^{\infty}$$

is uniformly absolutely continuous on $[0,T]$. Therefore, by (2.35) and Remark 1.5, the sequence $\mathcal{A}$ is uniformly absolutely continuous on $[0,T]$, as well.

Step 4. By *Step 4* in the proof of Lemma 2.4, for each $u_m \in \mathcal{B}$ and for each $i \in \{0, \dots, n-1\}$ there exists a finite number $p_{i,m} (< 2n)$ of disjoint intervals $(a_{k-1,m}, a_{k,m}), 1 \le k \le p_{i,m}$, such that $\bigcup_{k=1}^{p_{i,m}} [a_{k-1,m}, a_{k,m}] = [0,T]$, and

$$\left. \begin{array}{l} |u_m^{(i)}(t)| \ge \dfrac{c}{(n-i)!}(t - a_{k-1,m})^{n-i} \text{ for } t \in [a_{k-1,m}, a_{k,m}] \\ \text{or} \\ |u_m^{(i)}(t)| \ge \dfrac{c}{(n-i)!}(a_{k,m} - t)^{n-i} \text{ for } t \in [a_{k-1,m}, a_{k,m}]. \end{array} \right\}$$

By virtue of the assertion (I) of Theorem 1.3 there is an $u \in cl(\Omega)$ and a subsequence $\{u_{m'}\} \subset \{u_m\}$ such that $\lim_{m' \to \infty} \|u_{m'} - u\|_{C^{n-1}} = 0$.

Letting $m' \to \infty$ and working with subsequences if necessary, we get

$$\lim_{m' \to \infty} p_{i,m'} = p_i, \quad p_i < 2n,$$

and

$$\lim_{m' \to \infty} a_{k,m'} = a_k, \quad 0 \le k \le p_i,$$

where $0 = a_0 \le a_1 \le \ldots \le a_{p_i} = T$. Moreover (2.16) and (2.17) hold. Therefore the set $U_1 \subset [0,T]$ of all zeros of $u^{(i)}, 0 \le i \le n-1$, is finite. Denote the set of all $t \in [0,T]$ such that $f(t,\cdot,\dots,\cdot) : \mathcal{D} \to \mathbb{R}$ is not continuous by U_2. Then $\mu(U_1 \cup U_2) = 0$ and

$$\lim_{m' \to \infty} f_{m'}(t, u_{m'}(t), \dots, u_{m'}^{(n-1)}(t)) = f(t, u(t), \dots, u^{(n-1)}(t))$$

for all $t \in [0,T] \setminus (U_1 \cup U_2)$.

Therefore, by Theorem 1.3, problem (2.1), (2.3) has a solution u. By (2.4) and Lemma 2.2 we get $u > 0$ on $(0,T)$. □

Consider the problem (2.1), (2.2) with a general p, $1 \le p \le n-1$. Arguing similarly as in the proof of Lemma 2.2 (for details see [13]) we can describe for any solution of BVP (2.1), (2.2) the set of all its singular points, and then prove Lemmas 2.4 - 2.6 as before. In such a way we get the following theorem the proof of which is similar to that of Theorem 2.1.

Theorem 2.7. **(Existence result for $(p, n-p)$ conjugate BVP)** *Let assumptions* (2.4) – (2.7) *hold. Then problem* (2.1), (2.2) *has a solution which is positive on* $(0,T)$.

Example 2.8. Consider the function

$$f(t, x_0, \dots, x_{n-1}) = a(t) + \sum_{i=0}^{n-1} \left[b_i(t)|x_i|^{\alpha_i} + c_i(t)\frac{1}{|x_i|^{\beta_i}}, \right]$$

where $1 \le i \le n-1$, $\alpha_i \in (0,1), \beta_i \in (0, 1/(n-i))$, the functions b_i, c_i are nonnegative and essentialy bounded on $[0,T]$ and the function $a \in L_1[0,T]$ fulfils $0 < c \le a(t)$ for a.e. $t \in [0,T]$ and for some $c \in (0,\infty)$. Then f satisfies the conditions (2.4)-(2.7) and so Theorem 2.1 guarantees the existence of a solution u of the differential equation

$$u^{(n)}(t) = a(t) + \sum_{i=0}^{n-1} \left[b_i(t)|u^{(i)}(t)|^{\alpha_i} + c_i(t)\frac{1}{|u^{(i)}(t)|^{\beta_i}} \right]$$

satisfying the boundary conditions (2.3).

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