Connections between types of singularities in differential equations and smoothness of solutions for Dirichlet BVPs

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Abstract: We present sufficient conditions for the existence of smooth sign-changing solutions of the singular Dirichlet boundary value problem with a positive parameter μ

$$(r(x)x')' = \mu q(t) f(t,x), \quad x(0) = x(T) = 0,$$

where f is singular at the point $x = 0$ of the phase variable x. By means of the study of connections between a type of the singularity and the smoothness of solutions we get conditions for non-existence of solutions of the above problem. Finally, we introduce a notion of w-solutions and give exact multiplicity results.
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1 Introduction

We will study the singular Dirichlet boundary value problem with a positive parameter μ

$$(r(x(t))x'(t))' = \mu q(t) f(t, x(t)), \quad t \in (0,T), \tag{1.1}$$

$$x(0) = x(T) = 0, \ \max\{x(t) : 0 \le t \le T\} \cdot \min\{x(t) : 0 \le t \le T\} < 0, \tag{1.2}$$

where $T \in (0,\infty)$ and f is singular at the point $x = 0$ of the phase variable x in the following sense

$$\lim_{x\to 0^-} f(t,x) = -\infty, \ \lim_{x\to 0^+} f(t,x) = \infty \quad \text{for } t \in [0,T]. \tag{1.3}$$

To give a finer classification of the singularity of f at $x = 0$ we first assume that f satisfies the condition

$$0 < f(t,x)\operatorname{sign} x \le g(x) \quad \text{for } (t,x) \in [0,T] \times D, \tag{1.4}$$

where $D = (-\infty, 0) \cup (0, \infty)$ and $g \in C^0(D)$.

We say that f has the *weak singularity at* $x = 0$ if f satisfies (1.3), (1.4) and g fulfils

$$\int^0 g(x)\,dx < \infty, \quad \int_0 g(x)\,dx < \infty. \tag{1.5}$$

Similarly, we say that f has the *weak left singularity at* $x = 0$ if f and g satisfy the first condition in (1.3) and (1.5), respectively and f fulfils (1.4) on $[0,T] \times (-\infty, 0)$. Finally, f has the *weak right singularity at* $x = 0$ provided f and g satisfy the second condition in (1.3) and (1.5), respectively and f fulfils (1.4) on $[0,T] \times (0,\infty)$.

A simple example of a function f having the weak singularity at $x = 0$ is

$$f(t,x) = \frac{\operatorname{sign} x}{|x|^\alpha}, \qquad \alpha \in (0,1).$$

In accordance with [10] we say that a function $x \in C^1([0,T])$ is a *solution of problem* (1.1), (1.2) if x has precisely one zero t_0 in $(0,T)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, x fulfils (1.2) and there exists $\mu_0 > 0$ such that (1.1) is satisfied for $\mu = \mu_0$ and $t \in (0,T) \setminus \{t_0\}$.

In [10], under the assumptions

(H1) $r \in C^0(\mathbb{R})$, $r(x) \ge r_0 > 0$ for $x \in \mathbb{R}$,

(H2) $q \in C^0((0,T))$, $q(t) < 0$ for $t \in (0,T)$, $Q = \sup\{|q(t)| : t \in [0,T]\} < \infty$,

(H3) $f \in C^0([0,T] \times D)$, $f(t,\cdot)$ is nonincreasing on D for $t \in [0,T]$, f has the weak singularity at $x = 0$

we have proved the following results for problem (1.1), (1.2).

Theorem 1.1. *Let* (H1)-(H3) *be satisfied. Then for each* $A \in (0,\infty)$ *there exists a solution* x *of problem* (1.1), (1.2) *with the unique zero* $t_0 \in (0,T)$ *such that*

$$\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : 0 \le t \le t_0\} = A \quad \text{if} \quad t_0 \in \Big[\frac{T}{2}, T\Big),$$

$$\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : 0 \le t \le t_0\} \le A \quad \text{if} \quad t_0 \in \Big(0, \frac{T}{2}\Big).$$

Theorem 1.2. *Suppose that* (H1)-(H3) *hold. Then for each* $B \in (-\infty, 0)$ *there exists a solution* x *of problem* (1.1), (1.2) *with the unique zero* $t_0 \in (0,T)$ *such that*

$$\min\{x(t) : 0 \le t \le T\} = \min\{x(t) : t_0 \le t \le T\} = B \quad \text{if} \quad t_0 \in \Big(0, \frac{T}{2}\Big],$$

$$\min\{x(t) : 0 \le t \le T\} = \min\{x(t) : t_0 \le t \le T\} \ge B \quad if \quad t_0 \in \Big(\frac{T}{2}, T\Big).$$

Theorem 1.3. *Let* (H1)-(H3) *be true. Then for each* $A \in (0, \infty)$ *there exists a solution* x *of problem* (1.1), (1.2) *with the unique zero* $t_0 \in (0, T)$ *such that*

$$\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : t_0 \le t \le T\} = A \quad if \quad t_0 \in \Big(0, \frac{T}{2}\Big],$$

$$\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : t_0 \le t \le T\} \le A \quad if \quad t_0 \in \Big(\frac{T}{2}, T\Big).$$

Theorem 1.4. *If* (H1)-(H3) *are valid, then for each* $B \in (-\infty, 0)$ *there is a solution* x *of problem* (1.1), (1.2) *with the unique zero* $t_0 \in (0, T)$ *such that*

$$\min\{x(t) : 0 \le t \le T\} = \min\{x(t) : 0 \le t \le t_0\} = B \quad if \quad t_0 \in \Big[\frac{T}{2}, T\Big),$$

$$\min\{x(t) : 0 \le t \le T\} = \min\{x(t) : 0 \le t \le t_0\} \ge B \quad if \quad t_0 \in \Big(0, \frac{T}{2}\Big).$$

Remark 1.5. From (H3) it follows that $f(t, x) \operatorname{sign} x$ is continuous and positive on $J \times D$ and $\lim_{x \to 0} f(t, x) \operatorname{sign} x = \infty$ for $t \in [0, T]$. Hence, for each $M > 0$, there is a positive function $k_M \in C^0([0, T])$ such that

$$0 < k_M(t) \le f(t, x) \operatorname{sign} x \quad \text{for } (t, x) \in [0, T] \times [-M, 0) \cup (0, M]. \tag{1.6}$$

Example 1.6. Let $\gamma, A \in (0, \infty)$ and $\alpha \in (0, 1)$. Consider the differential equation

$$((1 + |x(t)|)^\gamma x'(t))' + \mu A\Big(2 + \sin \frac{1}{t(T - t)}\Big) \frac{|x(t)|^\alpha}{e^{x(t)} - e^{-x(t)}} = 0. \tag{1.7}$$

The assumptions (H1)–(H3) and conditions (1.4) and (1.5) are satisfied with $r(u) = (1 + |u|)^\gamma \ge 1$, $Q = 3A$ and $g(x) = \frac{|x|^\alpha}{|e^x - e^{-x}|}$. Consequently, Theorems 1.1–1.4 can be applied to problem (1.7), (1.2).

The idea of the proofs of Theorems 1.1–1.4 is based on "gluing" of positive and negative parts of solutions and on smoothing them. At the same time positive and negative parts of solutions are positive and negative solutions of (1.1) satisfying the Dirichlet boundary conditions on intervals $[0, t_0]$ and $[t_0, T]$ with a suitable $t_0 \in (0, T)$. We note that positive (negative) solutions on $(0, T)$ of the Dirichlet problems with nonlinearities having the singularity at the point $x = 0$ were studied in many papers (see, e.g., [1]–[9], [11]–[17] and references therein). Solutions were considered either in the set $C^0([0, T]) \cap C^2((0, T))$ ([1]–[3], [8], [9], [13], [14]) or $C^1([0, T]) \cap C^2((0, T))$ ([4], [9], [11]–[14], [17]) or $C^0([0, T]) \cap AC^1_{loc}((0, T))$ ([5]–[7], [15], [16]). Here $AC^1_{loc}((0, T))$ denotes the set of functions having absolutely continuous first derivatives on each $[a, b] \subset (0, T)$. The nonlinearities of equations have been usually nonpositive ([1]–[5], [8], [9], [11]–[15], [17]) but in ([3], [6], [7], [16]) this assumption was overcome.

The aim of this paper is the consideration of connections between the weak and strong singularities of f at $x = 0$, where the notation of the strong singularity of f at $x = 0$ will be defined in Section 3, and smoothness of solutions to problem (1.1), (1.2). By Theorems 1.1–1.4, we know that the weak singularity of f at $x = 0$ implies the existence of solutions to problem (1.1), (1.2) in the space $C^1([0, T])$. For other papers giving the existence results for the Dirichlet problems with the weak singularities we can refer to [1], [3], [4], [9], [11], [12], [13], [14], [17]. On the other hand the strong singularity (even one-sided) of f at $x = 0$ implies the non-existence of solutions for problem (1.1), (1.2) in $C^1([0, T])$ (see Theorems 3.3 and 3.5). Therefore solutions of the Dirichlet problems with the strong singularity at $x = 0$ can be smooth on $(0, T)$, only. Such existence results we can also find in [2], [3], [5]-[9], [13]–[16].

Finally, in Section 3, we introduce a notion of a w-solution to problem (1.1), (1.2) which can be nonsmooth at its zero $t_0 \in (0, T)$ and we give exact multiplicity results for such solutions in Theorems 3.7 and 3.8. Moreover, in Theorem 3.10 we prove the existence of a w-solution having its maximum and minimum given.

2 Lemmas

In this section we suppose that conditions (H1)-(H3) are satisfied. Moreover, assume that $A \in (0, \infty)$, $B \in (-\infty, 0)$ and $0 \le a < b \le T$. In our further consideration we will work with the following auxiliary boundary conditions

$$x(a) = x(b) = 0,\ x(t) > 0 \text{ for } t \in (a, b), \tag{2.1}$$

$$x(a) = x(b) = 0,\ x(t) < 0 \text{ for } t \in (a, b) \tag{2.2}$$

and we will use the function $H : (-\infty, 0] \to [0, \infty)$ defined by

$$H(u) = \int_u^0 r(s)\, ds, \tag{2.3}$$

where r is the function from (H1). Of course, H is continuous decreasing function. The inverse functions to H is denoted by $H^{-1} : [0, \infty) \to (-\infty, 0]$.

Let $j \in \{1, 2\}$ and μ be a positive fixed number. We say that $x \in C^1([a, b])$ is a *solution of problem* (1.1), (2.j), if x satisfies (2.j), $r(x)x' \in C^1((a, b))$ and (1.1) with this fixed μ is fulfilled for $t \in (a, b)$.

Existence and uniqueness results for problems (1.1), (2.j), $j \in \{1, 2\}$ are formulated in Lemmas 2.1 and 2.2 and follow from results in [10] and [12].

Lemma 2.1. *Let a, $b \in [0, T]$, $a < b$. Then for each $\mu > 0$ problem* (1.1), (2.1) *has a unique solution.*

Suppose moreover that $A > 0$ *and put*

$$m_+(a,b;A) = \frac{2\Big(\displaystyle\int_0^A r(s)\,ds\Big)^2}{(b-a)^2 Q \displaystyle\int_0^A r(s)g(s)\,ds}. \tag{2.4}$$

Then there is just one value μ_0 *of the parameter* μ *such that problem* (1.1), (2.1) *with* $\mu = \mu_0$ *has a solution* u *satisfying* $\max\{u(t) : a \le t \le b\} = A$. *This solution is unique and* $\mu_0 \in [m_+(a,b;A), \infty)$.

Lemma 2.2. *Let* a, $b \in [0,T]$, $a < b$. *Then for each* $\mu > 0$ *problem* (1.1), (2.2) *has a unique solution.*

Moreover, for any $M \in (0,\infty)$ *and any solution* v *of problem* (1.1), (2.2) *such that* $|v(t)| \le M$ *for* $t \in [a,b]$,

$$v(t) \le \begin{cases} H^{-1}\Big(\frac{2\mu(t-a)K_M(a,b)}{b-a}\Big) & \text{for } t \in [a, \frac{a+b}{2}] \\ H^{-1}\Big(\frac{2\mu(b-a)K_M(a,b)}{b-a}\Big) & \text{for } t \in (\frac{a+b}{2}, b], \end{cases}$$

where

$$K_M(a,b) = \min\Big\{\int_a^{\frac{a+b}{2}} (s-a)|q(s)|k_M(s)\,ds,\ \int_{\frac{a+b}{2}}^b (b-s)|q(s)|k_M(s)\,ds\Big\} \tag{2.5}$$

with $k_M \in C^0([0,T])$ *satisfying* (1.6).

Lemma 2.3. *Let* $\mu > 0$ *and* $a_1, a_2 \in [0,T)$, $a_1 < a_2$, $b = T$ *and* u_i *be a (unique) solution of problem* (1.1), (2.2) *with* $a = a_i$, $i = 1,2$. *Then*

$$u_1(t) \le u_2(t) \quad \text{for } t \in [a_2, T]. \tag{2.6}$$

Proof. Since $u_1(T) = u_2(T) = 0$ and $u_2(a_2) = 0 > u_1(a_2)$, there is a $\xi \in (a_2, T]$ such that $u_1(t) < u_2(t)$ for $t \in [a_2, \xi)$ and $u_1(\xi) = u_2(\xi)$. If $\xi = T$, then (2.6) is true. Assume that $\xi < T$ and let (2.6) not be satisfied. Then there exist $a_2 < \alpha < \beta \le T$ such that $u_1(\alpha) = u_2(\alpha)$, $u_1'(\alpha) \ge u_2'(\alpha)$, $u_1(\beta) = u_2(\beta)$, $u_1'(\beta) \le u_2'(\beta)$ and $u_1(t) > u_2(t)$ for $t \in (\alpha, \beta)$. Then according to (H3) we have $f(t, u_1(t)) \le f(t, u_2(t))$ for $t \in (\alpha, \beta)$. Set $p(t) = \int_{u_1(t)}^{u_2(t)} r(s)\,ds$ for $t \in [\alpha, \beta]$. Then $p(\alpha) = p(\beta) = 0$, $p < 0$ on (α, β), $p'(\alpha) \le 0$ and

$$p''(t) = (r(u_2(t))u_2'(t))' - (r(u_1(t))u_1'(t))' = \mu q(t)(f(t,u_2(t)) - f(t,u_1(t))) \le 0$$

for $t \in (\alpha, \beta)$. Therefore $p'(t) \le 0$ for $t \in [\alpha, \beta]$ and $p(\alpha) = p(\beta) = 0$ implies $p = 0$ on $[\alpha, \beta]$, contrary to $p < 0$ on (α, β). □

Let $A > 0$. Then, by Lemma 2.1, for each $c \in (0,T]$ there exists just one value of the parameter μ, which will be denoted by $\mu(c)$, such that the problem

$$\begin{gathered} (r(x(t))x'(t))' = \mu(c)q(t)f(t,x(t)), \quad t \in (0,c) \\ x(0) = x(c) = 0,\ x(t) > 0 \text{ on } (0,c),\ \max\{x(t) : 0 \le t \le c\} = A \end{gathered} \tag{2.7}$$

has a (unique) solution which we will denote by u_c. In such a way we get the function

$$\mu : (0,T] \to (0,\infty).$$

Lemma 2.4. ([10, Proposition 3.1]) *The function $\mu(c)$ is continuous and nonincreasing on $(0,T]$.*

So, for given $A > 0$ and $c \in (0,T)$ we have the uniquely determined parameter $\mu(c)$ and, by Lemma 2.2 for $\mu = \mu(c)$, $a = c$, $b = T$, there exists exactly one solution of the problem

$$\begin{gathered}(r(x(t))x'(t))' = \mu(c)q(t)f(t,x(t)), \quad t \in (c,T) \\ x(c) = x(T) = 0, \ x(t) < 0 \text{ on } (c,T),\end{gathered} \tag{2.8}$$

which we denote by v_c. Let us define the function $\Delta_A : (0,T) \to (-\infty, 0)$ by the formula

$$\Delta_A(c) = \min\{v_c(t) : c \le t \le T\} \tag{2.9}$$

Lemma 2.5. *The function Δ_A is continuous and nondecreasing on $(0,T)$.*

Proof. Let the functions $r^* : \mathbb{R} \to [r_0, \infty)$, $f^* : [0,T] \times D \to \mathbb{R}$ and $g^* : D \to \mathbb{R}$ be defined by

$$r^*(x) = r(-x), \quad f^*(t,x) = -f(t,-x), \quad g^*(x) = g(-x)$$

and consider the differential equation

$$(r^*(x(t))x'(t))' = \mu q(t) f^*(t,x(t)). \tag{2.10}$$

Then u is a solution of problem (1.1), (2.1) if and only if the function $u^* = -u$ on $[a,b]$ is a solution of problem (2.10), (2.2) (see [10]). Since assumptions (H1) and (H3) are satisfied with r^*, f^* and g^* instead of r, f and g, it follows immediately from Lemma 2.3 in [10]:

(i) if $0 < \mu_1 < \mu_2$, $0 \le a < b \le T$ and u_i is a (unique) solution of problem (1.1), (2.2) with $\mu = \mu_i$, $i = 1,2$, then $u_1(t) > u_2(t)$ for $t \in (a,b)$.

As a direct consequence of Lemma 2.3 and (i) we get that Δ_A is nondecreasing on $(0,T)$.

Suppose that Δ_A is discontinuous on the right at some $c_0 \in (0,T)$, i.e. there is a decreasing sequence $\{c_n\} \subset (c_0, T)$ such that $\lim_{n\to\infty} c_n = c_0$ and

$$\lim_{n\to\infty} \Delta_A(c_n) > \Delta_A(c_0). \tag{2.11}$$

Consider the corresponding sequence $\{\mu(c_n)\}$ of parameter values and the sequence $\{v_{c_n}\}$ of solutions to problem (2.8) for $c = c_n, n \in \mathbb{N} \cup \{0\}$. Then $\Delta_A(c_n) = \min\{v_{c_n}(t) : c_n \le t \le T\}$ and $\Delta_A(c_0) = \min\{v_{c_0}(t) : c_0 \le t \le T\}$.

Using the procedure as in the proof of Proposition 3.1 in [10] we can prove, after evident modifications, that there exists a subsequence $\{v_{c_{k_n}}\}$ of $\{v_{c_n}\}$ such that $\lim_{n\to\infty} v_{c_{k_n}}(t) = v_{c_0}(t)$ locally uniformly on $(c_0, T]$. Hence $\lim_{n\to\infty} \Delta_A(c_n) = \Delta_A(c_0)$, contrary to (2.11).

The left continuity can be proved similarly. $\square$

Lemma 2.6. $\lim_{c\to T^-} \Delta_A(c) = 0$.

Proof. Let $\{c_n\} \subset (0,T)$ be an increasing sequence and let $\lim_{n\to\infty} c_n = T$. By Lemma 2.5, $\{\Delta_A(c_n)\}$ is a nondecreasing sequence, so $\lim_{n\to\infty} \Delta_A(c_n) = \gamma \le 0$. Moreover

$$\mu(c_n) \ge \frac{2\Big(\int_{\Delta_A(c_n)}^0 r(s)\,ds\Big)^2}{(T-c_n)^2 Q \int_{\Delta_A(c_n)}^0 r(s)g(s)\,ds} \quad \text{for } n \in \mathbb{N} \tag{2.12}$$

which follows from Theorem 2.6* in [10]. Here $\{\mu(c_n)\}$ is the corresponding sequence of parameter values to $\{v_{c_n}\}$. If $\gamma < 0$, then from (2.12) we deduce that $\lim_{n\to\infty} \mu(c_n) = \infty$, contrary to Lemma 2.4. Hence $\gamma = 0$ and the lemma is proved. $\square$

Lemma 2.7. $\lim_{c\to 0^+} \Delta_A(c) = -\infty$.

Proof. Let $\{c_n\} \subset (0,T)$ be a decreasing sequence and let $\lim_{n\to\infty} c_n = 0$. Let $\{\mu(c_n)\}$ be the corresponding sequence of parameter values to $\{v_{c_n}\}$. If we put $a = 0$, $b = c_n$ and use Lemma 2.1, we get $\mu(c_n) \ge m_+(0, c_n; A)$ for $n \in \mathbb{N}$, where

$$m_+(0, c_n; A) = \frac{2\Big(\int_0^A r(s)\,ds\Big)^2}{c_n^2 Q \int_0^A r(s)g(s)\,ds}.$$

Since $\lim_{n\to\infty} m_+(0, c_n; A) = \infty$, we yield

$$\lim_{n\to\infty} \mu(c_n) = \infty. \tag{2.13}$$

Now, assume, on the contrary, that the assertion of the lemma is not true. Taking into account Lemma 2.5, there exists $M > 0$ such that

$$0 \ge v_{c_n}(t) \ge -M \quad \text{for } t \in [c_n, T],\ n \in \mathbb{N}. \tag{2.14}$$

Let $c_n \le \frac{T}{2}$ for $n \ge n_0$ with some $n_0 \in \mathbb{N}$. Then $\Delta_A(c_n) \le H^{-1}(\mu(c_n) K_M(\frac{T}{2}, T))$ for $n \ge n_0$ by Lemma 2.2 with $a = c_n$ and $b = T$. Since

$$\lim_{n\to\infty} H^{-1}\Big(\mu(c_n) K_M\Big(\frac{T}{2}, T\Big)\Big) = -\infty$$

by (2.13), we have $\lim_{n\to\infty} \Delta_A(c_n) = -\infty$, contrary to (2.14). $\square$

3 Weak and strong singularity and smoothness of solutions

The weak singularity of f at $x = 0$, which has been defined in Section 1, produces that solutions of problem (1.1), (1.2) cannot increase or decrease in neighbourhoods of their zeros extremely quickly, and so those have finite derivatives at their zeros. This enables to get solutions of problem (1.1), (1.2) in the class $C^1([0,T])$. For other papers working with assumptions of the type (1.4), (1.5) for nonnegative solutions of the Dirichlet problem we can refer to [1], [11] or [12].

The smoothness of solutions can be also obtained by some modifications of conditions (1.4) and (1.5). For example in [4], the authors consider the problem

$$x'' = f(t,x), \quad x(0) = x(1) = 0, \tag{3.1}$$

and assume the validity of second condition in (1.3) together with the inequalities

$$0 < \int_0^1 f(t, c\Phi(t))\, dt < \infty \quad \text{for } c \in (0,\infty), \tag{3.2}$$

where

$$\Phi(t) = \begin{cases} t & \text{for } t \in [0, \frac{1}{2}] \\ 1-t & \text{for } t \in (\frac{1}{2}, 1]. \end{cases} \tag{3.3}$$

By means of (3.2) and (3.3) they get positive (on $(0,1)$) solutions of problem (3.1) in the space $C^1([0,1])$. To the same purpose conditions (3.2) and (3.3) are used in [17]. In [14], the behaviour of f in a right neighbourhood of its singularity $x = 0$ is controlled by the inequalities

$$0 < f(t,x) \le \alpha(t) g(x) \quad \text{for } (t,x) \in (0,1) \times (0,\infty), \tag{3.4}$$

where

$$\int_0^1 t(1-t)\alpha(t)\, dt < \infty, \ \int_0^1 g(kt(1-t))\alpha(t)\, dt < \infty \quad \text{for } k \in (0,\infty). \tag{3.5}$$

It is proved that assumptions (3.4), (3.5) guarantee the smoothness of solutions to (3.1). The same is proved in [9] or in [13] for a special case of (3.4) and (3.5). The common feature of all these conditions is the convergence of auxiliary integrals of a majorant to f.

Motivated by the example $f(t,x) = \dfrac{\operatorname{sign} x}{|x|^\alpha}$, $\alpha \ge 1$, and by the fact that $\displaystyle\int_0 \frac{dx}{x^\alpha} = \infty$, $\displaystyle\int^0 \frac{dx}{|x|^\alpha} = \infty$ if $\alpha \ge 1$, we define other type of singularity of f at $x = 0$.

Let $f : [0,T] \times D \to \mathbb{R}$ and there exist a positive function $p \in C^0(D)$ such that

$$0 < p(x) \le f(t,x) \operatorname{sign} x \quad \text{for } (t,x) \in [0,T] \times D. \tag{3.6}$$

We say that f has the *strong left singularity* at $x = 0$ if

$$\int^0 p(x)\,dx = \infty \tag{3.7}$$

and we say that f has the *strong right singularity* at $x = 0$ if

$$\int_0 p(x)\,dx = \infty. \tag{3.8}$$

The strong left (right) singularity of f at $x = 0$ means that the time for which solution x of problem (1.1), (1.2) stays near 0 is very small, which consequently breaks the boundedness of derivatives of x at its zeros. For example papers [2], [3], [5]–[9], [13]–[16] consider the Dirichlet problem for differential equations with nonlinearities admitting the strong right singularity at $x = 0$ and thus nonnegative solutions having continuous first derivatives only on $(0, 1)$ are guaranteed there.

The fact that the presence of strong left (right) singularity produces the loss of the finiteness of derivatives of solutions at their zeros is demonstrated in the following two lemmas.

Lemma 3.1. *Let assumptions* (H1), (H2) *and*

(H4) $f \in C^0([0,T] \times D)$, $f(t, \cdot)$ *is nonincreasing on* D *for* $t \in [0,T]$, f *satisfies* (3.6) *and has strong right singularity at* $x = 0$

be satisfied. Further assume that $\mu > 0$, $[a, b] \subset [0,T]$, $0 < b - a < T$, $u \in C^0([a, b]) \cap C^1((a, b))$ *satisfies* (2.1), $r(u)u' \in C^1((a, b))$ *and* (1.1) *with* $x = u$ *and this* μ *is fulfilled on* (a, b). *Then*

$$\lim_{t \to a^+} u'(t) = \infty \tag{3.9}$$

provided $a > 0$ *and*

$$\lim_{t \to b^-} u'(t) = -\infty \tag{3.10}$$

provided $b < T$.

Proof. First assume that $a > 0$. By our assumptions, $u \in C^0([a, b]) \cap C^1((a, b))$ satisfies (2.1) and $(r(u(t))u'(t))' = \mu q(t) f(t, u(t))$ for $t \in (a, b)$. Set

$$A = \max\{u(t) : a \le t \le b\} = u(t_0).$$

Then $A > 0$, $t_0 \in (a, b)$ and

$$u'(t_0) = 0. \tag{3.11}$$

Since $\mu q(t) f(t, u(t)) < 0$ for $t \in (a, b)$, $r(u)u'$ is decreasing on (a, b) and, by (H1) and (3.11), $r(u)u' > 0$ on (a, t_0). Let $q_0 = \max\{q(t) : a \le t \le t_0\}$. Then $q_0 < 0$ by (H2), and from the inequality

$$\mu q(t) f(t, u(t)) \le \mu q_0 p(u(t)), \quad t \in (a, t_0]$$

it follows that

$$(r(u(t))u'(t))'r(u(t))u'(t) \le \mu q_0 p(u(t))r(u(t))u'(t) \le \mu q_0 r_0 p(u(t))u'(t)$$

for $t \in (a, t_0]$. Choose $\varepsilon \in (0, t_0 - a)$. Integrating the inequality

$$(r(u(t))u'(t))'r(u(t))u'(t) \le \mu q_0 r_0 p(u(t))u'(t), \quad t \in (a, t_0]$$

from $a + \varepsilon$ to t_0, we get

$$(r(u(a+\varepsilon))u'(a+\varepsilon))^2 \ge 2\mu q_0 r_0 \int_{t_0}^{a+\varepsilon} p(u(t))u'(t)\,dt = 2\mu q_0 r_0 \int_A^{u(a+\varepsilon)} p(s)\,ds.$$

Consequently,

$$\lim_{\varepsilon\to 0^+} (r(u(a+\varepsilon))u'(a+\varepsilon))^2 = r^2(0) \lim_{\varepsilon\to 0^+} (u'(a+\varepsilon))^2 \ge 2\mu|q_0|r_0 \int_0^A p(s)\,ds = \infty$$

which gives (3.9).

If $b < T$ then (3.10) can be proved similarly. □

Remark 3.2. If the function q in (1.1) satisfies (H2) with the inequality $q(t) \le q_0 < 0$ for $t \in [0, T]$, then the assertions of Lemma 3.1 hold for each $0 \le a < b \le T$.

Theorem 3.3. *Let assumptions* (H1), (H2) *and* (H4) *be satisfied. Then problem* (1.1), (1.2) *has no solution.*

Proof. Assume that u is a solution of problem (1.1), (1.2). Then $u \in C^1([0, T])$ and there exists precisely one $t_0 \in (0, T)$ such that $u(t_0) = 0$ and either $u > 0$ on $(0, t_0)$, $u < 0$ on (t_0, T) or $u < 0$ on $(0, t_0)$, $u > 0$ on (t_0, T); say $u < 0$ on $(0, t_0)$, $u > 0$ on (t_0, T). Then, by Lemma 3.1 with $a = t_0$ and $b = T$, we have $\lim_{t\to t_0^+} u'(t) = \infty$, contrary to $u \in C^1([0, T])$. □

Since the proofs of next Lemma 3.4 and Theorem 3.5 are similar to those of Lemma 3.1 and Theorem 3.3, they will be omitted.

Lemma 3.4. *Let assumptions of Lemma* 3.1 *be satisfied with the difference that now u fulfils* (2.2) *instead of* (2.1) *and* (H4) *is replaced with*

(H5) $f \in C^0([0, T] \times D)$, *$f(t, \cdot)$ is nonincreasing on D for $t \in [0, T]$, f satisfies* (3.6) *and has strong left singularity at $x = 0$.*

Then $\lim_{t\to a^+} u'(t) = -\infty$ *provided $a > 0$ and* $\lim_{t\to b^-} u'(t) = \infty$ *provided $b < T$.*

Theorem 3.5. *Suppose that assumptions* (H1), (H2) *and* (H5) *hold. Then problem* (1.1), (1.2) *has no solution.*

Remark 3.6. Observe that the nonexistence of a solution of problem (1.1), (1.2) follows from the fact that any solution of this problem vanishes just at one point of $(0, T)$ and that the nonlinearity of (1.1) has strong left (or right) singularity at $x = 0$. Even in the case that the condition $x \in C^1([0, T])$ in our definition of the solution x of problem (1.1), (1.2) is replaced with the weaker assumption $x \in C^0([0, T]) \cap C^1((0, T))$, the assertions of Theorems 3.3 and 3.5 are true.

Now, we generalize the notion of a solution of problem (1.1), (1.2). We say that $x \in C^0([0,T])$ is a *w-solution of problem* (1.1), (1.2), if x has precisely one zero $t_0 \in (0,T)$, $x \in C^1([0,T] \setminus \{t_0\})$, there exist finite $\lim_{t\to t_0^-} x'(t)$, $\lim_{t\to t_0^+} x'(t)$, $r(x)x' \in C^1((0,T)\setminus\{t_0\})$, x fulfils (1.2) and finally there exists $\mu_0 > 0$ such that (1.1) with $\mu = \mu_0$ is satisfied on $(0,T)\setminus\{t_0\}$. Multiplicity results for w-solutions of problem (1.1), (1.2) are presented in the following Theorems 3.7, 3.8 and 3.10. Note, that the first two theorems give exact multiplicity results while the third one affords a lower bound of a number of w-solutions.

Theorem 3.7. *Suppose that assumptions* (H1)-(H3) *hold. Let* $A > 0$. *Then for each* $t_0 \in (0,T)$ *problem* (1.1), (1.2) *has just two w-solutions vanishing at* t_0 *and having their maximum value on* $[0,T]$ *equal to* A.

Proof. Choose $t_0 \in (0,T)$ and put

$$Q_0 = \sup\{|q(t)| : 0 \le t \le t_0\}, \quad Q_T = \sup\{|q(t)| : t_0 \le t \le T\}$$

and, according to (2.4), denote

$$m_+(0,t_0;A) = \frac{2\Big(\int_0^A r(s)\,ds\Big)^2}{t_0^2 Q_0 \int_0^A r(s)g(s)\,ds},$$

$$m_+(t_0,T;A) = \frac{2\Big(\int_0^A r(s)\,ds\Big)^2}{(T-t_0)^2 Q_T \int_0^A r(s)g(s)\,ds}.$$

By Lemma 2.1, for $a = 0$, $b = t_0$, there exists just one value $\mu_0 \in [m_+(0,t_0;A), \infty)$ of the parameter μ such that problem (1.1), (2.1) with $\mu = \mu_0$ has a (unique) solution u_1 satisfying

$$\max\{u_1(t) : 0 \le t \le t_0\} = A. \tag{3.12}$$

Further, by Lemma 2.2, for $a = t_0$, $b = T$, problem (1.1), (2.2) with $\mu = \mu_0$ has a (unique) solution v_1. Now, we can simply put

$$x_1(t) = \begin{cases} u_1(t) & \text{for } t \in [0,t_0] \\ v_1(t) & \text{for } t \in (t_0,T]. \end{cases}$$

The function x_1 need not have a continuous first derivative at t_0 but there exist finite $\lim_{t\to t_0^-} x_1'(t)$, $\lim_{t\to t_0^+} x_1'(t)$, and so x_1 is a w-solution of problem (1.1), (1.2). Since $v_1 \le 0$ on $[t_0,T]$, we get from (3.12) that $\max\{x_1(t) : 0 \le t \le T\} = A$.

To get the second w-solution of problem (1.1), (1.2) with a zero at t_0 and with the maximum value A, we can use Lemma 2.1 for $a = t_0$, $b = T$. Then there exists just one value $\mu_T \in [m_+(t_0,T;A), \infty)$ of the parameter μ such that problem (1.1), (2.1) with $\mu = \mu_T$ has a (unique) solution u_2 satisfying $\max\{u_2(t) : t_0 \le$

$t \leq T\} = A$. Finally, by Lemma 2.2 for $a = 0$, $b = t_0$, problem (1.1), (2.2) with $\mu = \mu_T$ has a (unique) solution v_2 and therefore the function

$$x_2(t) = \begin{cases} v_2(t) & \text{for } t \in [0, t_0] \\ u_2(t) & \text{for } t \in (t_0, T] \end{cases}$$

is our second w-solution of problem (1.1), (1.2) with $\max\{x_2(t) : 0 \leq t \leq T\} = A$. □

In a similar way we can prove the next theorem.

Theorem 3.8. *Let $B < 0$ and let assumptions* (H1)-(H3) *hold. Then for each $t_0 \in (0, T)$ problem* (1.1), (1.2) *has just two w-solutions vanishing at t_0 and having their minimum value on $[0, T]$ equal to B.*

Example 3.9. By Theorems 3.7 and 3.8, for $A > 0$, $B < 0$ and $t_0 \in (0, T)$ given, there exist just two w–solutions u_i and just two w-solutions v_i of problem (1.7), (1.2) such that $u_i(t_0) = v_i(t_0) = 0$ and $\max\{u_i(t) : 0 \leq t \leq T\} = A$, $\min\{v_i(t) : 0 \leq t \leq T\} = B$, $i = 1, 2$.

Theorem 3.10. *Suppose that* (H1)-(H3) *hold. Then for each $A > 0$, $B < 0$ there exist at least two w–solutions of problem* (1.1), (1.2) *having their maximum value on $[0, T]$ equal to A and their minimum value on $[0, T]$ equal to B.*

Proof. Fix $A > 0$, $B < 0$ and consider the function $\Delta_A : (0, T) \to (-\infty, 0)$ which is defined by (2.9). By Lemmas 2.5-2.7, Δ_A is continuous and nondecreasing on $(0, T)$, $\lim_{c \to T^-} \Delta_A(c) = 0$ and $\lim_{c \to 0^+} \Delta_A(c) = -\infty$. Hence there exists at least one solution of the equation $\Delta_A(c) = B$ in $(0, T)$, say c_*. Now, taking the notation u_c for the unique solution of problem (2.7) and v_c for the unique solution of problem (2.8), we see that

$$x_1(t) = \begin{cases} u_{c_*}(t) & \text{for } t \in [0, c_*] \\ v_{c_*}(t) & \text{for } t \in (c_*, T] \end{cases}$$

is a w-solution x of problem (1.1), (1.2) such that

$$\max\{x_1(t) : 0 \leq t \leq T\} = \max\{u_{c_*}(t) : 0 \leq t \leq c_*\} = A,$$

$$\min\{x_1(t) : 0 \leq t \leq T\} = \min\{v_{c_*}(t) : c_* \leq t \leq T\} = B.$$

To get the second w-solution we consider the differential equation (2.10) instead of (1.1) and use relations between solutions of both the differential equations given in the proof of Lemma 2.5. Then there is a $c_0 \in (0, T)$ such that

$$x_2(t) = \begin{cases} v_{c_0}(t) & \text{for } t \in [0, c_0] \\ u_{c_0}(t) & \text{for } t \in (c_0, T] \end{cases}$$

is a w-solution x of problem (1.1), (1.2) such that

$$\max\{x_2(t) : 0 \leq t \leq T\} = \max\{u_{c_0}(t) : c_0 \leq t \leq T\} = A,$$

$$\min\{x_2(t) : 0 \le t \le T\} = \min\{v_{c_0}(t) : 0 \le t \le c_0\} = B.$$

□

Remark 3.11. Summarizing our results for solutions and w-solutions of problem (1.1), (1.2) which are given in Theorems 1.1–1.4 and Theorems 3.3 and 3.5 for solutions and in Theorems 3.7, 3.8 and 3.10 for w-solutions we get:

1) If f has the one-sided strong singularity at $x = 0$, then problem (1.1), (1.2) has no solution and no w-solution;

2) If f has the week singularity at $x = 0$ then in the case of solutions we yield:

   (i) for each $A > 0$, (resp. $B < 0$) there exists at least one solution having its maximum value on $[0, T]$ which is $\le A$ (resp. minimum value on $[0, T]$ which is $\ge B$)

   (ii) zeros of solutions on $(0, T)$ are not precisely localized

   and in the case of w-solutions we receive:

   (j) for each $A > 0$, $B < 0$ and $t_0 \in (0, T)$ there exist precisely two w-solutions having zero at $t = t_0$ with the maximum value on $[0, T]$ equals A and there exist precisely two w-solutions having zero at $t = t_0$ with the minimum value on $[0, T]$ equals B

   (jj) for each $A > 0$, $B < 0$ there exist at least two w-solutions with the maximum value on $[0, T]$ equals A and with the minimum value on $[0, T]$ equals B. Zeros of these w-solutions on $[0, T]$ are not precisely localized.

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