Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions

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Abstract

This paper investigates the solvability of discrete Dirichlet boundary value problems by the lower and upper solution method. Here, the second order difference equation with a nonlinear right hand side f is studied and $f(t, u, v)$ can have a superlinear growth both in u and in v. Moreover, the growth conditions on f are one-sided. We compute *a priori* bounds on solutions to the discrete problem and then obtain the existence of at least one solution. It is shown that solutions of the discrete problem will converge to solutions of ordinary differential equations.

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1 Introduction

This paper investigates the following discrete Dirichlet boundary value problem

$$\frac{\nabla \Delta y_k}{h^2} + f(t_k, y_k, \frac{\Delta y_k}{h}) = 0, \qquad k = 1, \ldots, n-1; \tag{1.1}$$

$$y_0 = 0, \quad y_n = 0; \tag{1.2}$$

where: f is a continuous, scalar-valued function; the step size is $h = N/n$ with N a positive constant and $n \geq 2$; the grid points are $t_k = kh$ for $k = 0, \ldots, n$. The differences are given by:

$$\Delta y_k = \begin{cases} y_{k+1} - y_k, & \text{for } k = 0, \ldots, n-1, \\ 0, & \text{for } k = n; \end{cases}$$

$$\nabla \Delta y_k = \begin{cases} y_{k+1} - 2y_k + y_{k-1}, & \text{for } k = 1, \ldots, n-1, \\ 0, & \text{for } k = 0 \text{ or } k = n. \end{cases}$$

This paper addresses two questions of interest regarding the discrete BVP (1.1), (1.2):

- Under what conditions does the discrete BVP (1.1), (1.2) have at least one solution?
- In what sense, if any, will the above solutions to (1.1), (1.2) approximate solutions to the continuous BVP

$$y'' + f(t, y, y') = 0, \qquad t \in [0, N], \tag{1.3}$$

$$y(0) = 0, \qquad y(N) = 0? \tag{1.4}$$

Particular significance in these points lie in the fact that when a BVP is discretized, strange and interesting changes can occur in the solutions. For example, properties such as existence, uniqueness and multiplicity of solutions may not be shared between the "continuous" differential equation and its related "discrete" difference equation [1, p.520]. Moreover, when investigating difference equations, as opposed to differential equations, basic ideas from calculus are not necessarily available to use, such as the intermediate value theorem; the mean value theorem and Rolle's theorem. Thus, new challenges are faced and innovation is required.

In this paper we will significantly extend the ideas from [12], where the solvability of problem (1.1), (1.2) has been proved provided $f(t, u, v)$ has sublinear or linear growth in u and v. Here, using the lower and upper solutions method for our problem, we will prove its solvability, even for nonlinearities $f(t, u, v)$ growing superlinearly in u and v.

The paper is organised as follows.

In Section 2, a new discrete Nagumo condition for (1.1), (1.2) is formulated. The condition is one of the main results of the paper. In short, we gain sufficient conditions, in terms of a general, one-sided growth condition on f, so that all possible solutions to (1.1), (1.2) have an *a priori* bound on their first differences, with this bound being independent of h but dependent on a bound on solutions.

In Section 3 we present the classical method of lower and upper solutions to (1.1), (1.2). We also gain the existence of solutions to a certain "modified" version of (1.1), (1.2) where the boundedness on the right-hand side of the modified difference equation is utilized.

In Section 4 the ideas from the two previous sections are combined and applied to establish new solvability results for (1.1), (1.2). These existence results form another main contribution of the paper.

In Section 5 the *a priori* bound results from Section 2 and 3 are applied to show that solutions to the discrete BVP (1.1), (1.2) will converge to solutions of the continuous BVP (1.3), (1.4). An example is presented to illustrate how the new theory advances existing results from the literature.

For recent and classical results on difference equations and their comparison with differential equations, including existence, uniqueness and spurious solutions, the reader is referred to: [1]-[17].

A solution to equation (1.3) is a twice continuously differentiable function $y = y(t)$ that satisfies (1.3) for all $t \in [0, N]$.

A solution to equation (1.1) is a vector $\mathbf{y} = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1}$ satisfying (1.1) for $k = 1, \ldots, n-1$.

2 A priori estimates

In this section we will prove *a priori* estimates on first differences of vectors, in terms of an *a priori* bound on vectors themselves. The estimates on first differences do not depend on the step-size h or on vectors $\mathbf{y}$.

The following lemma contains one-sided growth conditions which imply the aforementioned estimates on differences.

Lemma 2.1 *Let $\varepsilon \in (0, 2]$, $r \in (0, \infty)$ and $c, K \in [0, \infty)$ be constants. Then there exists a $r^* \in [1, \infty)$ such that for each step-size $h \in (0, \frac{N}{2}]$ and each $\mathbf{y} = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1}$ that satisfy:*

$$\frac{\nabla \Delta y_k}{h^2} \operatorname{sign} y_k \geq -c \left| \frac{\Delta y_k}{h} \right|^{2-\varepsilon} - K, \quad k = 1, \ldots, n-1; \tag{2.1}$$

and

$$\max\{|y_i| : i = 1, \ldots, n-1\} \leq r, \quad y_0 = y_n = 0; \tag{2.2}$$

the estimate

$$\max \left\{ \left| \frac{\Delta y_i}{h} \right| : i = 0, \ldots, n-1 \right\} \leq r^* \tag{2.3}$$

is valid.

Proof Choose an arbitrary $\mathbf{y} = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1}$ that satisfies (2.1), (2.2) and denote

$$\max\left\{\left|\frac{\Delta y_i}{h}\right| : i = 0, \ldots, n-1\right\} = \left|\frac{\Delta y_j}{h}\right| = \rho. \tag{2.4}$$

If $\rho < 1$, then (2.3) holds for $r^* = 1$. Now, assume that $\rho \geq 1$. We will discuss four cases.

Case 1: $\Delta y_j > 0$ and $y_j < 0$.

Then $0 < j \leq n-1$ and we can find $\ell \in [0, j-1]$ such that

$$\Delta y_\ell \leq 0 \qquad \text{and} \qquad \ell < j-1 \Longrightarrow \Delta y_{\ell+1} > 0, \ldots, \Delta y_{j-1} > 0. \tag{2.5}$$

Therefore, by (2.2) and (2.5),

$$\sum_{i=\ell+1}^{j} |\Delta y_i| = \sum_{i=\ell+1}^{j} \Delta y_i = y_{j+1} - y_{\ell+1} \leq 2r. \tag{2.6}$$

Further, if $\ell < j-1$, then we have $y_i < 0, i = \ell+1, \ldots, j-1$, and hence (2.1) yields

$$\frac{1}{h^2}(\Delta y_j - \Delta y_{j-1}) \leq (c+K)\left|\frac{\Delta y_j}{h}\right|^{2-\varepsilon}, \tag{2.7}$$

$$\frac{1}{h^2}(\Delta y_{j-1} - \Delta y_{j-2}) \leq \left\{\begin{array}{ll} (c+K)\left|\frac{\Delta y_{j-1}}{h}\right|^{2-\varepsilon} & \text{if} \quad \frac{|\Delta y_{j-1}|}{h} \geq 1 \\ c+K & \text{if} \quad \frac{|\Delta y_{j-1}|}{h} < 1, \end{array}\right.$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{1}{h^2}(\Delta y_{\ell+1} - \Delta y_\ell) \leq \left\{\begin{array}{ll} (c+K)\left|\frac{\Delta y_{\ell+1}}{h}\right|^{2-\varepsilon} & \text{if} \quad \frac{|\Delta y_{\ell+1}|}{h} \geq 1 \\ c+K & \text{if} \quad \frac{|\Delta y_{\ell+1}|}{h} < 1. \end{array}\right.$$

Hence, by (2.4), (2.5) and (2.6),

$$\rho = \frac{\Delta y_j}{h} \leq \frac{1}{h}(\Delta y_j - \Delta y_\ell) = \frac{1}{h}\sum_{i=\ell+1}^{j}(\Delta y_i - \Delta y_{i-1})$$

$$< (c+K)\left((j-\ell-1)h + \rho^{1-\varepsilon}\sum_{i=\ell+1}^{j}\Delta y_i\right).$$

Therefore,

$$\rho < \left\{\begin{array}{ll} (c+K)\rho^{1-\varepsilon}(N+2r) & \text{if} \quad \varepsilon \in (0,1] \\ (c+K)(N+2r) & \text{if} \quad \varepsilon \in (1,2]. \end{array}\right. \tag{2.8}$$

If $\ell = j-1$, we have by (2.5) and (2.7)

$$\rho = \frac{\Delta y_j}{h} \le \frac{1}{h}(\Delta y_j - \Delta y_{j-1}) < (c+K)\left(h + \rho^{1-\varepsilon}\Delta y_j\right),$$

which implies (2.8) again. So, we get (2.3) if we put

$$r^* = \max\{[(c+K)(N+2r)]^{1/\varepsilon}, (c+K)(N+2r)\} + 1. \tag{2.9}$$

Case 2: $\Delta y_j < 0$ and $y_j > 0$.

Then $0 < j \le n-1$ and we can find $\ell \in [0, j-1]$ such that

$$\Delta y_\ell \ge 0 \qquad \text{and} \qquad \ell < j-1 \Longrightarrow \Delta y_{\ell+1} < 0, \dots, \Delta y_{j-1} < 0. \tag{2.10}$$

So, if $\ell < j-1$, we have $y_i > 0, i = \ell+1, \dots, j-1$. Therefore, by (2.2) and (2.10),

$$\sum_{i=\ell+1}^{j} |\Delta y_i| = \sum_{i=\ell+1}^{j} -\Delta y_i = -y_{j+1} + y_{\ell+1} \le 2r. \tag{2.11}$$

By (2.4), (2.10), (2.11) and similar arguments as in Case 1, we get (2.8). Consequently (2.3) holds if we define r^* by (2.9).

Case 3: $\Delta y_j > 0$ and $y_j \ge 0$.

Then $0 \le j < n-1$ and we can find $\ell \in [j+1, n-1]$ such that

$$\Delta y_\ell \le 0 \qquad \text{and} \qquad \ell > j+1 \Longrightarrow \Delta y_{j+1} > 0, \dots, \Delta y_{\ell-1} > 0. \tag{2.12}$$

Therefore, by (2.2) and (2.12),

$$\sum_{i=j+1}^{\ell} |\Delta y_i| = -\Delta y_\ell + \sum_{i=j+1}^{\ell-1} \Delta y_i \le 2y_\ell - y_{\ell+1} \le 3r. \tag{2.13}$$

Further, if $\ell > j+1$, then we have $y_i > 0, i = j+1, \dots, \ell$, and hence (2.1) yields

$$\frac{1}{h^2}(\Delta y_{j+1} - \Delta y_j) \ge -\begin{cases} (c+K)\left|\frac{\Delta y_{j+1}}{h}\right|^{2-\varepsilon} & \text{if } \frac{|\Delta y_{j+1}|}{h} \ge 1 \\ c+K & \text{if } \frac{|\Delta y_{j+1}|}{h} < 1, \end{cases} \tag{2.14}$$

$$\frac{1}{h^2}(\Delta y_{j+2} - \Delta y_{j+1}) \ge -\begin{cases} (c+K)\left|\frac{\Delta y_{j+2}}{h}\right|^{2-\varepsilon} & \text{if } \frac{|\Delta y_{j+2}|}{h} \ge 1 \\ c+K & \text{if } \frac{|\Delta y_{j+2}|}{h} < 1, \end{cases}$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \end{array}$$

$$\frac{1}{h^2}(\Delta y_\ell - \Delta y_{\ell-1}) \geq -\left\{ \begin{array}{cll} (c+K)\left|\frac{\Delta y_\ell}{h}\right|^{2-\varepsilon} & \text{if} & \frac{|\Delta y_\ell|}{h} \geq 1 \\ c+K & \text{if} & \frac{|\Delta y_\ell|}{h} < 1. \end{array} \right.$$

Hence, by (2.4), (2.12) and (2.13),

$$\begin{aligned} \rho &= \frac{\Delta y_j}{h} \leq \frac{-1}{h}(\Delta y_\ell - \Delta y_j) = \frac{-1}{h}\sum_{i=j+1}^{\ell}(\Delta y_i - \Delta y_{i-1}) \\ &< (c+K)\left((\ell-j-1)h + \rho^{1-\varepsilon}\sum_{i=j+1}^{\ell}|\Delta y_i|\right). \end{aligned}$$

Therefore

$$\rho < \left\{ \begin{array}{cll} (c+K)\rho^{1-\varepsilon}(N+3r) & \text{if} & \varepsilon \in (0,1] \\ (c+K)(N+3r) & \text{if} & \varepsilon \in (1,2]. \end{array} \right. \tag{2.15}$$

If $\ell = j+1$, we have by (2.14),

$$\rho = \frac{\Delta y_j}{h} \leq \frac{-1}{h}(\Delta y_{j+1} - \Delta y_j) \leq (c+K)\left(h + \rho^{1-\varepsilon}\Delta y_{j+1}\right),$$

which implies (2.15) again. We see that (2.3) is true if we put

$$r^* = \max\{[(c+K)(N+3r)]^{1/\varepsilon}, (c+K)(N+3r)\} + 1. \tag{2.16}$$

Case 4: $\Delta y_j < 0$ and $y_j \leq 0$.

Then $0 \leq j < n-1$ and we can find $\ell \in [j+1, n-1]$ such that

$$\Delta y_\ell \geq 0 \qquad \text{and} \qquad \ell > j+1 \Longrightarrow \Delta y_{j+1} < 0, \ldots, \Delta y_{\ell-1} < 0. \tag{2.17}$$

So, if $\ell > j-1$, then we have $y_i < 0, i = j+1, \ldots, \ell$. Therefore, by (2.2) and (2.17),

$$\sum_{i=j+1}^{\ell}|\Delta y_i| = \Delta y_\ell - \sum_{i=j+1}^{\ell-1}\Delta y_i \leq -2y_\ell + y_{\ell+1} \leq 3r. \tag{2.18}$$

By (2.4), (2.17), (2.18) and similar arguments as in Case 3, we get (2.15). Therefore (2.3) is satisfied in all the four cases provided r^* is given by (2.16). $\square$

3 Lower and upper solutions

Lower and upper solutions are important tools in the investigation of solvability of boundary value problems. We now state their definition for problem (1.1), (1.2).

Assume that $\alpha(t)$ and $\beta(t)$ are continuous functions on $[0,N]$. For $n \geq 2$ and $h = \frac{N}{n}$ denote $t_k = kh$, $\alpha_k = \alpha(t_k)$, $\beta_k = \beta(t_k)$, $k = 0,\dots,n$.

We call $(\alpha_0,\dots,\alpha_n) \in \mathbb{R}^{n+1}$ *a lower solution* to problem (1.1), (1.2) if

$$\frac{\nabla\Delta\alpha_k}{h^2} + f(t_k,\alpha_k,\frac{\Delta\alpha_k}{h}) \geq 0, \quad k = 1,\dots,n-1, \tag{3.1}$$

$$\alpha_0 \leq 0, \quad \alpha_n \leq 0. \tag{3.2}$$

We call $(\beta_0,\dots,\beta_n) \in \mathbb{R}^{n+1}$ *an upper solution* of problem (1.1), (1.2) if

$$\frac{\nabla\Delta\beta_k}{h^2} + f(t_k,\beta_k,\frac{\Delta\beta_k}{h}) \leq 0, \quad k = 1,\dots,n-1, \tag{3.3}$$

$$\beta_0 \geq 0, \quad \beta_n \geq 0. \tag{3.4}$$

The next theorem yields solvability of problem (1.1), (1.2) in presence of lower and upper solutions, where we will let

$$r := \max\{\max_{i\in\{0,\dots,n\}}\{|\alpha_i|,|\beta_i|\}\}, \quad r_h := \frac{2r}{h}. \tag{3.5}$$

Theorem 3.1 (Lower and upper solutions method I) *Let $(\alpha_0,\dots,\alpha_n)$ and $(\beta_0,\dots,\beta_n)$ be, respectively, a lower and an upper solution of (1.1), (1.2) with $\alpha_k \leq \beta_k$, $k = 1,\dots,n-1$. Further, for each fixed $t \in [0,N]$ and $u \in [\alpha(t),\beta(t)]$ assume*

$$f(t,u,v) \ \text{ is nondecreasing in } \ v \ \text{ for } v \in [-r_h,r_h], \tag{3.6}$$

where r_h is given by (3.5). Then problem (1.1), (1.2) has at least one solution $\mathbf{y} = (y_0,\dots,y_n)$ satisfying

$$\alpha_k \leq y_k \leq \beta_k, \quad k = 0,\dots,n. \tag{3.7}$$

If, moreover, there exists $M > 0$ such that

$$|f(t,u,v)| \leq M \quad \text{for } t \in [0,N],\ \alpha(t) \leq u \leq \beta(t),\ v \in \mathbb{R}, \tag{3.8}$$

then

$$\left|\frac{\Delta y_k}{h}\right| \leq \frac{NM}{2}, \quad k = 0,\dots,n-1. \tag{3.9}$$

Proof *Step* 1. *Solvability of an auxiliary problem.* For $t \in [0, N]$, $x, z \in \mathbb{R}$, we define functions

$$\sigma(t, z) := \begin{cases} \beta(t+h) & \text{if} \quad z > \beta(t+h), \\ z & \text{if} \quad \alpha(t+h) \le z \le \beta(t+h), \\ \alpha(t+h) & \text{if} \quad z < \alpha(t+h); \end{cases}$$

$$\tilde{f}(t, x, \frac{z-x}{h}) := \begin{cases} f\big(t, \beta(t), \dfrac{\sigma(t,z) - \beta(t)}{h}\big) - \dfrac{x - \beta(t)}{x - \beta(t) + 1} & \text{if} \quad x > \beta(t), \\ f\big(t, x, \dfrac{\sigma(t,z) - x}{h}\big) & \text{if} \quad \alpha(t) \le x \le \beta(t), \\ f(t, \alpha(t), \dfrac{\sigma(t,z) - \alpha(t)}{h}\big) + \dfrac{\alpha(t) - x}{\alpha(t) - x + 1} & \text{if} \quad x < \alpha(t); \end{cases}$$

and we obtain the auxiliary difference equation

$$\frac{\nabla \Delta y_k}{h^2} + \tilde{f}\big(t_k, y_k, \frac{\Delta y_k}{h}\big) = 0, \quad k = 1, \dots, n-1. \tag{3.10}$$

We see that $\tilde{f}$ is continuous on $[0, N] \times \mathbb{R}^2$ and there exists $\tilde{M} > 0$ such that

$$|\tilde{f}(t, u, v)| \le \tilde{M}, \quad \text{for } t \in [0, N],\ u, v \in \mathbb{R}. \tag{3.11}$$

By [12, Corollary 2.3], problem (3.10), (1.2) has at least one solution.

Step 2. *Solvability of problem* (1.1), (1.2). Let $\mathbf{y} = (y_0, \dots, y_n)$ be an arbitrary solution to (3.10), (1.2). We prove that this solution $\mathbf{y}$ of (3.10), (1.2) satisfies (3.7). Put $v_k = y_k - \beta_k$, $k = 0, \dots, n$, and assume that

$$\max\{v_k \colon\ k = 0, \dots, n\} = v_\ell > 0. \tag{3.12}$$

Conditions (1.2) and (3.4) imply $\ell \in \{1, \dots, n-1\}$. Thus we have $v_{\ell+1} \le v_\ell$, $v_{\ell-1} \le v_\ell$, and consequently $\Delta y_\ell \le \Delta \beta_\ell$, $\Delta y_{\ell-1} \ge \Delta \beta_{\ell-1}$. This leads to

$$\nabla \Delta y_\ell \le \nabla \Delta \beta_\ell. \tag{3.13}$$

On the other hand, since f is nondecreasing in its third variable on $[-r_h, r_h]$, we get by (3.10) and (3.12)

$$\frac{1}{h^2}(\nabla \Delta y_\ell - \nabla \Delta \beta_\ell) = -\tilde{f}(t_\ell, y_\ell, \frac{\Delta y_\ell}{h}) - \frac{\nabla \Delta \beta_\ell}{h^2} = -f(t_\ell, \beta_\ell, \frac{\sigma(t_\ell, y_{\ell+1}) - \beta_\ell}{h})$$

$$+ \frac{y_\ell - \beta_\ell}{y_\ell - \beta_\ell + 1} - \frac{\nabla \Delta \beta_\ell}{h^2} \ge -f(t_\ell, \beta_\ell, \frac{\Delta \beta_\ell}{h}) + \frac{v_\ell}{v_\ell + 1} - \frac{\nabla \Delta \beta_\ell}{h^2} \ge \frac{v_\ell}{v_\ell + 1} > 0, \tag{3.14}$$

which contradicts (3.13). So, we have proved $y_k \le \beta_k$, for $k = 0, \dots, n$. The inequality $\alpha_k \le y_k$, for $k = 0, \dots, n$, can be proved similarly. Therefore $\mathbf{y}$ satisfies (3.7) and hence $\mathbf{y}$ is a solution of problem (1.1), (1.2).

Step 3. *Estimates of differences.* Now assume that there exists $M > 0$ satisfying (3.8). By the proof of [12, Theorem 2.1], problem (1.1), (1.2) is equivalent to the summation equation

$$y_k = h \sum_{i=1}^{n-1} G(t_k, s_i) f(s_i, y_i, \frac{\Delta y_i}{h}), \quad k = 0, \dots, n; \tag{3.15}$$

where G is the Green function of the homogeneous problem $\frac{\nabla \Delta y_k}{h^2} = 0$, (1.2) and

$$\sum_{i=1}^{n-1} |\Delta G(t_k, s_i)| \le \frac{N}{2}, \quad k = 0, \dots, n-1.$$

Therefore, by (3.8) and (3.15), we get (3.9). □

Theorem 3.1 is valid for an arbitrary fixed step size $h \in (0, N/2]$. Section 5 deals with convergence results and so our consideration there can be restricted on small steps. To this purpose it will be useful to formulate a modification of Theorem 3.1, which is valid for each sufficiently small step size h and where the monotonicity assumption (3.6) can be omitted.

Theorem 3.2 (Lower and upper solutions method II) *Let* $(\alpha_0, \dots, \alpha_n)$ *and* $(\beta_0, \dots, \beta_n)$ *be, respectively, a lower and an upper solution of* (1.1), (1.2) *with* $\alpha_k \le \beta_k$, $k = 1, \dots, n-1$, *and let there exist* $\rho > 0$ *such that for each* $n \in \mathbb{N}$, $n \ge 2$ *and* $h = N/n$

$$\left| \frac{\Delta \alpha_k}{h} \right| \le \rho, \quad \left| \frac{\Delta \beta_k}{h} \right| \le \rho, \quad k = 0, \dots, n-1. \tag{3.16}$$

Further, assume that there exists $M > 0$ *satisfying* (3.8). *Then there exists* $n^* \ge 2$ *such that for each* $n \in \mathbb{N}$, $n \ge n^*$, *problem* (1.1), (1.2) *has at least one solution* $\mathbf{y} = (y_0, \dots, y_n)$ *satisfying* (3.7) *and* (3.9).

Note that if functions $\alpha(t)$ and $\beta(t)$ have continuous derivatives on $[0, N]$, the condition (3.16) is satisfied.

Proof We argue as in the proof of Theorem 3.1 and get (3.10) – (3.13). By (3.12) we have $y_\ell > \beta_\ell$ which yields $c > 0$ such that $y_\ell = c + \beta_\ell$. In order to prove (3.14) we will show that for a sufficiently large $n \in \mathbb{N}$

$$y_\ell > \beta_\ell \quad \Longrightarrow \quad y_{\ell+1} \ge \beta_{\ell+1}.$$

Using (3.9), (3.12) and (3.16), we have

$$y_{\ell+1} = y_\ell + \Delta y_\ell = c + \beta_\ell + \Delta y_\ell = c + \beta_{\ell+1} - \Delta \beta_\ell + \Delta y_\ell$$
$$\ge c + \beta_{\ell+1} - |\Delta \beta_\ell| - |\Delta y_\ell| \ge c + \beta_{\ell+1} - \rho h - \frac{NM}{2} h \ge \beta_{\ell+1},$$

if $n \ge n^*$ and $n^* = \frac{1}{c}(\rho N + \frac{N^2 M}{2})$, $h = N/n$. Therefore $\sigma(t_\ell, y_{\ell+1}) = \beta_{\ell+1}$ and we get (3.14) without using (3.6). The rest can be proved as for Theorem 3.1. □

4 Solvability

In this section we prove the solvability of problem (1.1), (1.2) for those f that can have a superlinear growth in its second and third variables. The proofs are based on the *a priori* estimates furnished by Lemma **??** and on the lower and upper solutions method of Theorem 3.1 or Theorem 3.2. The first existence result holds for each step size $h \in (0, N/2]$ and follows from Theorem 3.1.

Theorem 4.1 *Let $(\alpha_0, \dots, \alpha_n)$ and $(\beta_0, \dots, \beta_n)$ be, respectively, a lower and an upper solution of (1.1), (1.2) with $\alpha_k \le \beta_k$, for $k = 1, \dots, n-1$. Further, let r and r_h be given by (3.5) and for each fixed $t \in [0, N]$ and $u \in [\alpha(t), \beta(t)]$ let the function $f(t,u,v)$ be nondecreasing in v for $v \in [-r_h, r_h]$. Moreover, assume that there exist $\varepsilon \in (0, 2]$ and $c, K \in [0, \infty)$ such that*

$$(4.1) \qquad f(t,u,v)\ \mathrm{sign}\ u \le c|v|^{2-\varepsilon} + K \quad \textit{for } t \in [0,N],\ \alpha(t) \le u \le \beta(t),\ v \in \mathbb{R}.$$

Then problem (1.1), (1.2) has a solution $\mathbf{y} = (y_0, \dots, y_n)$ satisfying (3.7) and (2.3), where r^* is from Lemma 2.1.

Proof Denote $R_h := \max\{r^*, r_h\}$ and for $t \in [0, N], u, v \in \mathbb{R}$ define

$$\chi(|v|, R_h) := \begin{cases} 1 & \text{if} \quad |v| \le R_h, \\ \dfrac{2R_h - |v|}{R_h} & \text{if} \quad R_h < |v| < 2R_h, \\ 0 & \text{if} \quad |v| \ge 2R_h; \end{cases}$$

and

$$(4.2) \qquad f^*(t,u,v) := \chi(|v|, R_h) f(t,u,v).$$

We will consider the auxiliary difference equation

$$(4.3) \qquad \frac{\nabla \Delta y_k}{h^2} + f^*(t_k, y_k, \frac{\Delta y_k}{h}) = 0,\ k = 1, \dots, n-1.$$

We see that $(\alpha_0, \dots, \alpha_n)$ and $(\beta_0, \dots, \beta_n)$ are lower and upper functions of (4.3), (1.2), respectively. Further f^* is continuous on $[0, N] \times \mathbb{R}^2$ and there exists $M^* > 0$ such that

$$(4.4) \qquad |f^*(t,u,v)| \le M^*, \quad \text{for } t \in [0,N],\ \alpha(t) \le u \le \beta(t),\ v \in \mathbb{R}.$$

Finally, for each fixed $t \in [0, N]$ and $u \in [\alpha(t), \beta(t)]$ the function $f^*(t,u,v)$ is nonincreasing in v for $v \in [-r_h, r_h]$. Therefore, by Theorem 3.1, problem (1.1), (1.2) has at least one solution $\mathbf{y} = (y_0, \dots, y_n)$ satisfying (3.7). Moreover, by (4.1) and (4.3),

$$\frac{\nabla \Delta y_k}{h^2}\ \mathrm{sign}\ y_k = -\chi\left(\frac{|\Delta y_k|}{h}, R_h\right) f(t_k, y_k, \frac{\Delta y_k}{h})\ \mathrm{sign}\ y_k$$

$$\geq \chi\left(\frac{|\Delta y_k|}{h}, R_h\right)\left(-c\left|\frac{\Delta y_k}{h}\right|^{2-\varepsilon} - K\right) \geq -c\left|\frac{\Delta y_k}{h}\right|^{2-\varepsilon} - K, \quad k = 1, \dots, n-1.$$

So, we can use Lemma 2.1 and get the estimate (2.3). Therefore, by (4.2), $\mathbf{y}$ is a solution of problem (1.1), (1.2). □

The second existence result relies upon Theorem 3.2 and hence it holds for small steps only.

Theorem 4.2 *Let* $(\alpha_0, \dots, \alpha_n)$ *and* $(\beta_0, \dots, \beta_n)$ *be, respectively, a lower and an upper solution of* (1.1), (1.2) *with* $\alpha_k \leq \beta_k$, *for* $k = 1, \dots, n-1$, *and let there exist* $\rho > 0$ *such that* (2.4) *holds for each* $n \in \mathbb{N}$, $n \geq 2$ *and* $h = N/n$. *Moreover, assume that there exist* $\varepsilon \in (0, 2]$ *and* $c, K \in [0, \infty)$ *such that* (4.1) *is satisfied. Then there exists* $n^* \geq 2$ *such that for each* $n \in \mathbb{N}$, $n \geq n^*$, *problem* (1.1), (1.2) *has a solution* $\mathbf{y} = (y_0, \dots, y_n)$ *satisfying* (3.7) *and* (2.3), *where* r^* *is from Lemma 2.1.*

Proof We define r and r_h by (3.5) and argue similarly as in the proof of Theorem 4.1 using Theorem 3.2 instead of Theorem 3.1. □

Consider $a, b \in [0, \infty)$ and put $\alpha(t) = -a$, $\beta(t) = b$ for $t \in [0, N]$. Then we see that conditions (3.1) – (3.4) are satisfied, i.e. $(\alpha_0, \dots, \alpha_n)$ and $(\beta_0, \dots, \beta_n)$ are lower and upper solutions of (1.1), (1.2), respectively. Moreover (2.4) is valid for an arbitrary $\rho > 0$. Therefore Theorem 4.2 yields the following corollary.

Corollary 4.3 *Assume that there exist* $a, b \in (0, \infty)$ *such that*

$$f(t, -a, 0) \geq 0, \quad f(t, b, 0) \leq 0 \quad \text{for } t \in [0, N]. \tag{4.5}$$

Moreover, assume that there exist $\varepsilon \in (0, 2]$ *and* $c, K \in [0, \infty)$ *such that*

$$f(t, u, v)\ \text{sign}\ u \leq c|v|^{2-\varepsilon} + K \quad \text{for } t \in [0, N],\ u \in [-a, b],\ v \in \mathbb{R}. \tag{4.6}$$

Then there exists $n^* \geq 2$ *such that for each* $n \geq n^*$, *problem* (1.1), (1.2) *has a solution* $\mathbf{y} = (y_0, \dots, y_n)$ *satisfying*

$$-a \leq y_k \leq b, \quad k = 0, \dots, n,$$

and (2.3), *where* r^* *is from Lemma 2.1.*

5 Convergence of Solutions

In this section the results of Section 2, 3 and 4 are applied to formulate some convergence theorems. The convergence ideas rely on the work of Robert Gaines [3].

We will need the following definition.

Assume that $\alpha(t)$ and $\beta(t)$ are twice continuously differentiable functions on $[0, N]$ and $\mu > 0$.

We call α a lower μ solution to problem (1.3), (1.4) if

$$\alpha'' + f(t,\alpha,\alpha') \geq \mu, \quad t \in [0,N], \tag{5.1}$$

$$\alpha(0) \leq -\mu, \quad \alpha(N) \leq -\mu. \tag{5.2}$$

We call β an upper μ solution of problem (1.3), (1.4) if

$$\beta'' + f(t,\beta,\beta') \leq -\mu, \quad t \in [0,N], \tag{5.3}$$

$$\beta(0) \geq \mu, \quad \beta(N) \geq \mu. \tag{5.4}$$

The following theorem answers the second question from the Introduction concerning the convergence of solutions for the discrete problem.

Theorem 5.1 *Let α and β be lower and upper μ solutions, respectively, to* (1.3), (1.4) *with $\alpha \leq \beta$. Let f be continuous. Moreover, assume that there exist $\varepsilon \in (0,2]$ and $c, K \in [0,\infty)$ such that*

$$f(t,u,v)\ \mathrm{sign}\ u \leq c|v|^{2-\varepsilon} + K \quad \text{for } t \in [0,N],\ \alpha(t) \leq u \leq \beta(t),\ v \in \mathbb{R}. \tag{5.5}$$

Then:

(A) The continuous BVP (1.3), (1.4) *has at least one twice continuously differentiable solution y with $\alpha \leq y \leq \beta$;*

(B) There exists a $\delta(\mu)$ such that for $h < \delta(\mu)$, the discrete BVP (1.1), (1.2) *has at least one solution $\mathbf{y}$ with $\alpha(t_k) \leq y_k \leq \beta(t_k)$, for $k = 0,\ldots,n$;*

(C) Those solutions $\mathbf{y}$ to (1.1), (1.2) *with $\alpha(t_k) \leq y_k \leq \beta(t_k)$, for $k = 0,\ldots,n$ will converge to solutions of* (1.3), (1.4) *in the following sense:*

For any $\varepsilon_1 > 0$ there exists a $h(\varepsilon_1)$ such that if $h \leq h(\varepsilon_1)$ and $\mathbf{y}$ is a solution to (1.1), (1.2), *then there is a solution y to* (1.3), (1.4) *such that*

$$\max_{[0,N]} |y(t,\mathbf{y}) - y(t)| \leq \varepsilon_1,$$
$$\max_{[0,N]} |v(t,\mathbf{y}) - y'(t)| \leq \varepsilon_1,$$

where

$$\begin{aligned} y(t,\mathbf{y}) &:= y_k + (y_{k+1} - y_k)h^{-1}(t - t_k), \quad t_k \leq t \leq t_{k+1}, \\ v(t,\mathbf{y}) &:= \begin{cases} (y_k - y_{k-1})/h + (y_{k+1} - 2y_k + y_{k-1})h^{-2}(t - t_k), & t_k \leq t \leq t_{k+1}, \\ (y_1 - y_0)/h, & 0 \leq t \leq t_1. \end{cases} \end{aligned}$$

Proof The proof of conclusion (A) follows from [8, Theorem 3.3] and so is omitted.

For conclusion (B), it follows from [3, Lemma 5.1] that since α and β are, respectively, lower and upper μ solutions for (1.3), (1.4), there exists a $\delta(\mu)$ such that for $h < \delta(\mu)$ we have

$$(\alpha_0, \ldots, \alpha_n) := (\alpha(t_0), \ldots, \alpha(t_n))$$
$$(\beta_0, \ldots, \beta_n) := (\beta(t_0), \ldots, \beta(t_n))$$

being, respectively, lower and upper solutions for (1.1), (1.2). Further, since α and β have continuous derivatives on $[0, N]$, we see that (2.4) is fulfilled with

$$\rho = \max\{\max_{t\in[0,N]} |\alpha'(t)|, \max_{t\in[0,N]} |\beta'(t)|\}.$$

In conjunction with (5.5) holding, all of the conditions of Theorem 4.2 are satisfied and conclusion (B) follows from there.

Conclusion (C) follows in a straightforward manner from [3, Theorem 2.5] and is omitted for brevity. □

Example Let c_i, $i = 1, 2, 3, 4$, be continuous functions on $[0, N]$, c_1, c_2 be nonpositive and let $c_1(t) + c_2(t) < 0$ for $t \in [0, N]$. Then we can find $a, b \in (0, \infty)$ such that the function

$$f(t, u, v) = c_1(t)u\mathrm{e}^v + c_2(t)u^3 + c_3(t)v\sqrt{|v|} + c_4(t) \tag{5.6}$$

satisfies (4.5). It follows from the fact that

$$\lim_{u\to-\infty} f(t, u, 0) = \infty, \quad \lim_{u\to\infty} f(t, x, 0) = -\infty \text{ uniformly on } [0, N].$$

Further, f satisfies (4.6), because on $[0, N] \times R^2$,

$$f(t, u, v) \operatorname{sign} u = c_1(t)|u|\mathrm{e}^v + c_2(t)|u|u^2 + (c_3(t)v\sqrt{|v|} + c_4(t)) \operatorname{sign} u \le c|v|^{3/2} + K,$$

where $c = \max_{t\in[0,N]}\{|c_3(t)|\}$ and $K = \max_{t\in[0,N]}\{|c_4(t)|\}$. Therefore, by Corollary 4.3, problem (1.1), (1.2) with f by (5.6) and with an arbitrary sufficiently small step size h has a solution $\mathbf{y}$ lying between $-a$ and b. Moreover, if we put $\alpha(t) = -a$ and $\beta(t) = b$ for $t \in [0, N]$, then all assumptions of Theorem 5.1 are satisfied and the corresponding convergence result holds.

Note that f in (5.6) need not fulfill the monotonicity condition (3.6). Therefore we cannot use Theorem 4.1 here. Thus we can assure the solvability of problem (1.1), (1.2) with f by (5.6) only for small steps.

Now, assume moreover that $c_2 < 0$ and $c_3 \ge 0$ on $[0, N]$. Then we can find $a, b \in (0, \infty)$ such that the function

$$f(t, u, v) = c_1(t)(u - |u|)\mathrm{e}^{-uv} + c_2(t)u^3 + c_3(t)v\sqrt{|v|} + c_4(t) \tag{5.7}$$

satisfies both (4.5), (4.6) and (3.6). Thus, we can apply Theorems 4.1, 4.2 and Theorem 5.1 on problem (1.1), (1.2) with f by (5.7). Consequently this problem is solvable for each step size $h \in (0, N/2]$.

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