Singular discrete problem arising in the theory of shallow membrane caps

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Abstract. Assume that $T \in (0, \infty)$, $n \in \mathbb{N}$, $n \geq 2$ and $h = \frac{T}{n}$. We use the lower and upper functions method to prove the existence of a positive solution of the singular discrete problem

$$\frac{1}{h^2}\Delta(t_k^3 \Delta u_{k-1}) + t_k^3 \left(\frac{1}{8u_k^2} - \frac{a_0}{u_k} - b_0 t_k^{2\gamma-4} \right) = 0, \quad k = 1, \ldots, n-1,$$

$$\Delta u_0 = 0, \quad u_n = 0,$$

where $a_0 \geq 0$, $b_0 > 0$, $\gamma > 1$. We prove that for $n \to \infty$ the sequence of solutions of the above discrete problems converges to a solution y of the corresponding continuous boundary value problem

$$(t^3 y')' + t^3 \left(\frac{1}{8y^2} - \frac{a_0}{y} - b_0 t^{2\gamma-4} \right) = 0,$$

$$\lim_{t \to 0+} t^3 y'(t) = 0, \quad y(T) = 0.$$

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1 Introduction

Let $T \in (0, \infty)$, $n \in \mathbb{N}$, $n \geq 2$. Put $h = \frac{T}{n}$ and for $k = 0, \ldots, n$ define $t_k = hk$.

We will investigate the solvability of the singular mixed discrete boundary value problem

$$\frac{1}{h^2}\Delta(t_k^3 \Delta u_{k-1}) + t_k^3 \left(\frac{1}{8u_k^2} - \frac{a_0}{u_k} - b_0 t_k^{2\gamma-4} \right) = 0, \quad k = 1, \ldots, n-1, \tag{1.1}$$

$$\Delta u_0 = 0, \quad u_n = 0, \tag{1.2}$$

where $a_0 \geq 0$, $b_0 > 0$, $\gamma > 1$. Here Δ denotes the forward difference operator, i.e.

$$\Delta u_{k-1} = u_k - u_{k-1}.$$

In particular, we are interested in the existence of a positive solution of problem (1.1), (1.2), which arises in the theory of shallow membrane caps [9], [13]. The continuous version of problem (1.1), (1.2) has the form

$$(t^3 y')' + t^3 \left(\frac{1}{8y^2} - \frac{a_0}{y} - b_0 t^{2\gamma-4} \right) = 0, \tag{1.3}$$

$$\lim_{t \to 0+} t^3 y'(t) = 0, \quad y(T) = 0, \tag{1.4}$$

and it was investigated for example in [14] and [21].

Solvability of discrete second order boundary value problems is studied in the monographs [1], [3], [4], [15] and in many papers, e.g. [5]–[7], [8], [11], [17]–[19], [23]. It is of interest to note that singular problems for differential equations have been intensively studied in literature. For the second order singular differential equations we can refer to the monographs [16], [20], [24]. However there are only few results for its discrete analogue, see [2], [4], [22]. Here we prove that for each n there exists a positive solution of (1.1), (1.2) and we show that these solutions converge (for $n \to \infty$) to a solution of the continuous problem (1.3), (1.4). Similar results about discrete approximation of regular problems can be found in [10], [12], [25], [26], [27]. Note that equation (1.3) is singular and that it becomes undefined when $t = 0$ and $y = 0$. We can see it if we transform (1.3) onto the system

$$x_1' = f_1(t, x_1, x_2) = \frac{x_2}{t^3},$$

$$x_2' = f_2(t, x_1, x_2) = -t^3 \left(\frac{1}{8x_1^2} - \frac{a_0}{x_1} - b_0 t^{2\gamma-4} \right),$$

using the substitution $x_1 = y$, $x_2 = t^3 y'$. Then f_1 is not integrable in t on a right neighbourhood of 0 (f_1 has a singularity at $t = 0$) and f_2 is not continuous in x_1 (f_2 has a singularity at $x_1 = 0$). Our main tools are the lower and upper functions method and the approximation principle which are proved here for the more general singular difference equation

$$\frac{1}{h^2}\Delta(p(t_k)\Delta u_{k-1}) + f(t_k, u_k) = 0, \quad k = 1, \dots, n-1, \tag{1.5}$$

and the corresponding singular differential equation

$$(p(t)y'(t))' + f(t, y(t)) = 0. \tag{1.6}$$

Equation (1.5) is studied with the boundary conditions (1.2) while equation (1.6) is investigated with the boundary conditions

$$\lim_{t\to 0+} p(t)y'(t) = 0, \quad y(T) = 0. \tag{1.7}$$

Now we characterize more precisely singular problems which are considered in the paper.

The continuous problem (1.6), (1.7) is supposed to be singular at $x = 0$ and it may also be singular at $t = 0$. These assumptions have the form

$$p \in C[0, T] \quad \text{is positive on } (0, T], \tag{1.8}$$

and

$$f \in C([0, T] \times (0, \infty)), \quad \limsup_{x\to 0+} |f(t, x)| = \infty \quad \text{for } t \in (0, T]. \tag{1.9}$$

By (1.9), $f(t, x)$ is not defined at $x = 0$ and hence problem (1.6), (1.7) is singular at $x = 0$. Further, we see that (1.8) yields that $p(0) = 0$ may occur and consequently $\frac{1}{p(t)}$ becomes undefined at $t = 0$. Moreover, the following integral

$$\int_0^T \frac{\mathrm{d}t}{p(t)}$$

can be divergent. This is the case that problem (1.6), (1.7) is also singular at $t = 0$.

The discrete problem (1.5), (1.2) is supposed to be singular at $x = 0$, i.e. the nonlinearity f in (1.5) satisfies (1.9). We also assume that p in (1.5) fulfils (1.8). But since only the values $p(t_k)$, $k = 1, \dots, n$, are relevant in (1.5), the condition $p(0) = 0$ does not imply that problem (1.5), (1.2) is singular at $t = 0$.

Definition 1.1 A vector $(u_0, \dots, u_n) \in \mathbb{R}^{n+1}$ satisfying (1.5), (1.2) and $u_k > 0$ for $k = 0, \dots, n-1$ is called *a positive solution* of the discrete problem (1.5), (1.2).

Definition 1.2 A function $y \in C[0, T] \cap C^2(0, T)$ with $y > 0$ on $(0, T)$, which satisfies (1.6) for $t \in (0, T)$ and fulfils (1.7), is called *a positive solution* of the continuous problem (1.6), (1.7).

2 The Green function

Let $g\colon [0,T] \to \mathbb{R}$ be continuous and let (1.8) be satisfied. Consider the linear difference equation

$$\frac{1}{h^2}\Delta(p(t_k)\Delta u_{k-1}) + g(t_k) = 0, \quad k = 1,\dots,n-1, \tag{2.1}$$

and the corresponding homogeneous equation

$$\frac{1}{h^2}\Delta(p(t_k)\Delta u_{k-1}) = 0, \quad k = 1,\dots,n-1. \tag{2.2}$$

Since problem (2.2), (1.2) has just the trivial solution, there exists the Green function G of this problem. Define

$$P(t_k) = \sum_{i=1}^{k} \frac{1}{p(t_i)}, \quad k = 1,\dots,n.$$

Then the Green function G can be written in the form

$$G(t_k,s_i) = h^2 \begin{cases} P(t_k) - P(T) & \text{for } 0 < s_i \le t_k \le T, \\ P(s_i) - P(T) & \text{for } 0 \le t_k < s_i \le T, \end{cases} \tag{2.3}$$

where $s_i = hi$, $t_k = hk$, $k = 0,\dots,n$, $i = 1,\dots,n$. Indeed, we can check that

$$G(T,s_i) = 0, \quad \Delta G(t_0,s_i) = 0, \quad i = 1,\dots,n, \tag{2.4}$$

and

$$\frac{1}{h^2}\Delta(p(t_k)\Delta G(t_{k-1},s_i)) = \delta_{ik}, \quad i,k = 1,\dots,n-1, \tag{2.5}$$

hold. Here

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \quad (\text{i.e. } s_i = t_k), \\ 0 & \text{if } i \ne k \quad (\text{i.e. } s_i \ne t_k). \end{cases}$$

Moreover, if we denote

$$M_p = \max\left\{\frac{1}{p(t_k)} \colon k = 1,\dots,n\right\},$$

we have

$$-hTM_p < G(t_k,s_i) < 0, \quad i = 1,\dots,n-1, \quad k = 0,\dots,n-1. \tag{2.6}$$

Lemma 2.1 *Assume that condition* (1.8) *holds. Then problem* (2.1), (1.2) *has a unique solution* $(u_0,\dots,u_n) \in \mathbb{R}^{n+1}$. *The solution* $(u_0,\dots,u_n)$ *has the form*

$$u_k = -\sum_{i=1}^{n-1} G(t_k,s_i)g(s_i), \quad k = 0,\dots,n. \tag{2.7}$$

Proof. Since the homogeneous problem (2.2), (1.2) has just the trivial solution, the nonhomogeneous problem (2.1), (1.2) has a unique solution. Let us show that this solution is given by (2.7). By virtue of (2.4) we get

$$u_n = -\sum_{i=1}^{n-1} G(T, s_i)g(s_i) = 0,$$

and

$$\begin{aligned}\Delta u_0 = u_1 - u_0 &= -\sum_{i=1}^{n-1} G(t_1, s_i)g(s_i) + \sum_{i=1}^{n-1} G(t_0, s_i)g(s_i) \\ &= -\sum_{i=1}^{n-1} \Delta G(t_0, s_i)g(s_i) = 0.\end{aligned}$$

Hence $(u_0, \dots, u_n)$ satisfies conditon (1.2).

Further, using equality (2.5), we obtain

$$\begin{aligned}\frac{1}{h^2}\Delta(p(t_k)\Delta u_{k-1}) &= \frac{1}{h^2}\Delta\left(p(t_k)\Delta\left(-\sum_{i=1}^{n-1} G(t_{k-1}, s_i)g(s_i)\right)\right) \\ = -\sum_{i=1}^{n-1}\frac{1}{h^2}\Delta(p(t_k)\Delta G(t_{k-1}, s_i)g(s_i)) &= -\sum_{i=1}^{n-1}\delta_{ik}g(s_i) \\ &= -g(t_k), \quad k = 1, \dots, n-1.\end{aligned}$$

Therefore $(u_0, \dots, u_n)$ satisfies equation (2.1). □

3 Lower and upper functions method

The lower and upper functions method for regular discrete problems can be found e.g. in [5], [12], [23]. In this section we extend this method for singular discrete problem (1.5), (1.2).

Definition 3.1 The vector $(\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1}$ is called a *lower function* of problem (1.5), (1.2) if

$$\frac{1}{h^2}\Delta(p(t_k)\Delta\alpha_{k-1}) + f(t_k, \alpha_k) \geq 0, \quad k = 1, \dots, n-1, \tag{3.1}$$

$$\Delta\alpha_0 \geq 0, \quad \alpha_n \leq 0. \tag{3.2}$$

Definition 3.2 The vector $(\beta_0, \dots, \beta_n) \in \mathbb{R}^{n+1}$ is called an *upper function* of problem (1.5), (1.2) if

$$\frac{1}{h^2}\Delta(p(t_k)\Delta\beta_{k-1}) + f(t_k, \beta_k) \leq 0, \quad k = 1, \dots, n-1, \tag{3.3}$$

$$\Delta\beta_0 \leq 0, \quad \beta_n \geq 0. \tag{3.4}$$

The lower and upper functions method guarantees the existence of a solution of the singular problem (1.5), (1.2) under the assumption that there exists a well ordered couple of lower and upper functions. This is expressed in the next theorem.

Theorem 3.3 *Assume that conditons* (1.8) *and* (1.9) *hold. Let* $(\alpha_0,\dots,\alpha_n)$ *and* $(\beta_0,\dots,\beta_n)$ *be, respectively, a lower and an upper function of problem* (1.5), (1.2) *with* $0<\alpha_k\le\beta_k$, $k=1,\dots,n-1$. *Then problem* (1.5), (1.2) *has a positive solution* $(u_0,\dots,u_n)$ *satisfying*

$$\alpha_k\le u_k\le\beta_k,\quad k=0,\dots,n. \tag{3.5}$$

Proof. For $k\in\{1,\dots,n-1\}$, $x\in\mathbb{R}$ define a function

$$\tilde f(t_k,x)=\begin{cases} f(t_k,\beta_k)-\frac{x-\beta_k}{x-\beta_k+1} & \text{if } x>\beta_k,\\ f(t_k,x) & \text{if } \alpha_k\le x\le\beta_k,\\ f(t_k,\alpha_k)+\frac{\alpha_k-x}{\alpha_k-x+1} & \text{if } x<\alpha_k.\end{cases}$$

We see that $\tilde f(t_k,\cdot)\colon\mathbb{R}\to\mathbb{R}$ is continuous for $k=1,\dots,n-1$ and there exists $M>0$ such that

$$|\tilde f(t_k,x)|\le M\quad\text{for } k=1,\dots,n-1,\ x\in\mathbb{R}.$$

Consider the auxiliary regular difference equation

$$\frac{1}{h^2}\Delta(p(t_k)\Delta u_{k-1})+\tilde f(t_k,u_k)=0,\quad k=1,\dots,n-1. \tag{3.6}$$

Denote

$$E=\{\mathbf{v}=(v_0,\dots,v_n)\in\mathbb{R}^{n+1}\colon \Delta v_0=0,\ v_n=0\},$$

and define $\|\mathbf{v}\|=\max\{|v_k|\colon k=1,\dots,n-1\}$. Then E is a Banach space with $\dim E=n-1$. Let the function G be given by formula (2.3). Define an operator $\mathcal{F}\colon E\to E$ by

$$(\mathcal{F}\mathbf{v})_k=-\sum_{i=1}^{n-1}G(t_k,s_i)\tilde f(s_i,v_i),\quad k=0,\dots,n. \tag{3.7}$$

Estimate (2.6) implies

$$|(\mathcal{F}\mathbf{v})_k|<nhTM_pM=T^2M_pM,\quad k=0,\dots,n.$$

Therefore if we denote $r^*=T^2M_pM$ and consider the closed ball $\overline{\mathcal{K}(r^*)}=\{\mathbf{v}\in E\colon\|\mathbf{v}\|\le r^*\}$, we see that $\mathcal{F}$ maps $\overline{\mathcal{K}(r^*)}$ into itself. Since $\mathcal{F}$ is continuous, the Brouwer fixed point theorem yields a fixed point $\mathbf{u}\in\overline{\mathcal{K}(r^*)}$ of the operator $\mathcal{F}$. So, we have $\mathbf{u}=\mathcal{F}\mathbf{u}$ and consequently, by (3.7),

$$u_k=-\sum_{i=1}^{n-1}G(t_k,s_i)\tilde f(s_i,u_i),\quad k=0,\dots,n.$$

So, according to Lemma 2.1, the vector $\mathbf{u} = (u_0, \dots, u_n)$ is a solution of problem (3.6), (1.2).

Now we will prove estimate (3.5). Let us put $z_k = \alpha_k - u_k$ and assume

$$\max\{z_k : k = 0, \dots, n\} = z_\ell > 0. \tag{3.8}$$

Then $\ell \in \{1, \dots, n-1\}$ because $z_n = \alpha_n - u_n \le 0$ and $\Delta z_0 = z_1 - z_0 = \alpha_1 - \alpha_0 - (u_1 - u_0) \ge 0$. Consequently $z_1 \ge z_0$. Since z_ℓ is maximal, we have

$$z_{\ell-1} \le z_\ell, \quad z_\ell \ge z_{\ell+1}. \tag{3.9}$$

The first inequality in (3.9) implies $\alpha_{\ell-1} - u_{\ell-1} \le \alpha_\ell - u_\ell$ and

$$p_\ell \Delta u_{\ell-1} \le p_\ell \Delta \alpha_{\ell-1}. \tag{3.10}$$

Similarly the second inequality in (3.9) gives

$$p_{\ell+1} \Delta u_\ell \ge p_{\ell+1} \Delta \alpha_\ell. \tag{3.11}$$

Inequalities (3.10) and (3.11) lead to

$$\Delta(p_\ell \Delta u_{\ell-1}) \ge \Delta(p_\ell \Delta \alpha_{\ell-1}). \tag{3.12}$$

On the other hand, by Definitions 1.1 and 3.1, we obtain

$$\frac{1}{h^2}(\Delta(p_\ell \Delta \alpha_{\ell-1}) - \Delta(p_\ell \Delta u_{\ell-1})) = \frac{1}{h^2}\Delta(p_\ell \Delta \alpha_{\ell-1}) + \tilde{f}(t_\ell, u_\ell)$$

$$= \frac{1}{h^2}\Delta(p_\ell \Delta \alpha_{\ell-1}) + f(t_\ell, \alpha_\ell) + \frac{\alpha_\ell - u_\ell}{\alpha_\ell - u_\ell + 1} \ge \frac{z_\ell}{z_\ell + 1} > 0,$$

contrary to (3.12). So, we have proved $\alpha_k \le u_k$, $k = 0, \dots, n$. The estimate $u_k \le \beta_k$, $k = 0, \dots, n$ can be proved similarly. Therefore $(u_0, \dots, u_n)$ satisfies (3.5) and hence $(u_0, \dots, u_n)$ is also a solution of problem (1.5), (1.2). □

4 Approximation principle

The main result of this section is contained in Theorem 4.1 which characterizes a connection between positive solutions of the singular difference problem (1.5), (1.2) and positive solutions of the corresponding singular differential problem (1.6), (1.7).

Theorem 4.1 *Let* (1.8) *and* (1.9) *hold. Assume that for each sufficiently large* $n \in \mathbb{N}$ *the singular problem* (1.5), (1.2) *has a solution* $(u_0, \dots, u_n)$ *and let there exist functions* $\alpha, \beta \in C[0, T]$ *satisfying*

$$\beta(T) = 0, \quad 0 < \alpha(t_k) \le u_k \le \beta(t_k), \quad k = 1, \dots, n-1. \tag{4.1}$$

Let $y^{[n]} \in C[0,T]$ *be a piece-wise linear function with*

$$y^{[n]}(t_k) = u_k, \quad k = 0, \dots, n,$$

and let

$$|f(t,x)| \le g(t,x) \quad \textit{for } t \in (0,T),\ \alpha(t) \le x \le \beta(t), \tag{4.2}$$

where $g \in C([0,T] \times (0,\infty))$ *is nonincreasing in its second variable with*

$$\int_0^{T/2} g(t,\alpha(t))\mathrm{d}t < \infty, \quad \int_0^{T/2} \frac{1}{p(t)} \int_0^t g(\tau,\alpha(\tau))\mathrm{d}\tau\mathrm{d}t < \infty. \tag{4.3}$$

Then the following approximation principle is valid:

There exists a subsequence $\{y^{[m]}\} \subset \{y^{[n]}\}$ *such that*

$$\lim_{m\to\infty} y^{[m]}(t) = y(t) \quad \textit{locally uniformly on } (0,T),$$

and $y \in C[0,T] \cap C^2(0,T)$ *is a positive solution of the singular problem* (1.6), (1.7).

First we will prove some lemmas about functions $y^{[n]}$ and $z^{[n]}$ which are defined by

$$y^{[n]}(t) = u_k + \frac{\Delta u_k}{h}(t - t_k), \quad t \in [t_k, t_{k+1}],\ k = 0, \dots, n-1, \tag{4.4}$$

and

$$\begin{cases} z^{[n]}(t) = v_k + \dfrac{\Delta v_k}{h}(t - t_k), \quad t \in [t_k, t_{k+1}],\ k = 0, \dots, n-2, \\ z^{[n]}(t) = v_{n-1}, \quad t \in [t_{n-1}, t_n], \end{cases} \tag{4.5}$$

where

$$v_k = \frac{1}{h} p(t_{k+1}) \Delta u_k, \quad k = 0, \dots, n-1, \tag{4.6}$$

and p satisfies (1.8). Clearly

$$y^{[n]}, z^{[n]} \in C[0,T], \quad y^{[n]}(t_k) = u_k, \quad z^{[n]}(t_k) = v_k, \quad k = 0, \dots, n-1.$$

Lemma 4.2 *Let the assumptions of Theorem* 4.1 *be fulfilled. Then for each* $b \in (0,T)$ *the sequences* $\{y^{[n]}\}$ *and* $\{z^{[n]}\}$ *are bounded on* $[0,b]$.

Proof. Inserting (4.6) into equation (1.5) we get

$$\frac{\Delta v_{k-1}}{h} = -f(t_k, u_k), \quad k = 1, \dots, n-1. \tag{4.7}$$

Since $\Delta u_0 = v_0 = 0$, equations (4.6) and (4.7) can be written in the form

$$u_{k+1} = u_0 + h \sum_{i=1}^{k} \frac{v_i}{p(t_{i+1})}, \quad k = 1, \dots, n-1, \tag{4.8}$$

and

$$v_k = -h \sum_{i=1}^{k} f(t_i, u_i), \quad k = 1, \dots, n-1. \tag{4.9}$$

Choose arbitrary $b, b^* \in (0, T)$, $b < b^*$. Then for each suffiently large $n \in \mathbb{N}$ there is $b_n \in \{1, \dots, n\}$ such that

$$t_{b_n}, t_{b_n+1} \in (b, b^*), \quad \lim_{n\to\infty} t_{b_n} = \lim_{n\to\infty} t_{b_n+1} = b. \tag{4.10}$$

Due to (4.1),

$$\max\{|u_k|: k = 1, \dots, b_n + 1\} \le \max\{\beta(t): t \in [0, b^*]\} =: B. \tag{4.11}$$

The first inequality in (4.3) implies that

$$M := \int_0^{b^*} g(t, \alpha(t))\mathrm{d}t < \infty. \tag{4.12}$$

Therefore there exists $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$,

$$h \sum_{i=1}^{b_n+1} g(t_i, \alpha_i) \le M + 1.$$

By (4.1), (4.2) and (4.9) we have for $k = 1, \dots, b_n + 1$,

$$|v_k| \le h \sum_{i=1}^{k} |f(t_i, u_i)| \le h \sum_{i=1}^{k} g(t_i, u_i)$$

$$\le h \sum_{i=1}^{k} g(t_i, \alpha_i) \le h \sum_{i=1}^{b_n+1} g(t_i, \alpha_i) \le M + 1,$$

and so

$$\max\{|v_k|: k = 1, \dots, b_n + 1\} \le M + 1. \tag{4.13}$$

Consequently, using (4.11) and $\Delta u_0 = 0$, we get

$$\max\{|y^{[n]}(t)|: t \in [0, b]\} \le \max\{|u_k| + |u_{k+1} - u_k|: k = 1, \dots, b_n\} \le 3B,$$

and using (4.13) and $v_0 = 0$, we get

$$\max\{|z^{[n]}(t)|: t \in [0, b]\} \le \max\{|v_k| + |v_{k+1} - v_k|: k = 1, \dots, b_n\} \le 3(M + 1).$$

We have proved that the sequences $\{y^{[n]}\}$ and $\{z^{[n]}\}$ are bounded on $[0, b]$. $\square$

Lemma 4.3 *Let the assumptions of Theorem* 4.1 *be fulfilled. Then for each* $a, b \in (0, T)$, $a < b$, *the sequence* $\{y^{[n]}\}$ *is equicontinuous on* $[a, b]$ *and the sequence* $\{z^{[n]}\}$ *is equicontinuous on* $[0, b]$.

Proof. Choose arbitrary $b, b^* \in (0,T)$, $b < b^*$ and $\tau_1, \tau_2 \in [0,b]$, $\tau_1 < \tau_2$. By (4.10) we find $k, \ell \in \{1, \dots, b_n\}$, $k \le \ell$, such that $\tau_1 \in [t_{k-1}, t_k)$, $\tau_2 \in (t_{\ell-1}, t_\ell)$ and, due to (4.5), (4.7) and (4.2),

$$
\begin{aligned}
&|z^{[n]}(\tau_2) - z^{[n]}(\tau_1)| \\
&\le \sum_{i=k+1}^{\ell-1} \left|\frac{\Delta v_{i-1}}{h}\right| (t_i - t_{i-1}) + \left|\frac{\Delta v_{k-1}}{h}\right| (t_k - \tau_1) + \left|\frac{\Delta v_{\ell-1}}{h}\right| (\tau_2 - t_{\ell-1}) \\
&= \sum_{i=k+1}^{\ell-1} |f(t_i, u_i)|(t_i - t_{i-1}) + |f(t_k, u_k)|(t_k - \tau_1) + |f(t_\ell, u_\ell)|(\tau_2 - t_{\ell-1}) \\
&\le \sum_{i=k+1}^{\ell-1} g(t_i, \alpha_i)(t_i - t_{i-1}) + g(t_k, \alpha_k)(t_k - \tau_1) + g(t_\ell, \alpha_\ell)(\tau_2 - t_{\ell-1}).
\end{aligned}
$$

If $k+1 > \ell - 1$, we put $\sum_{i=k+1}^{\ell-1} = 0$. By virtue of (4.12), for each $\varepsilon > 0$ there exists $n_\varepsilon \ge n_0$ such that for each $n \ge n_\varepsilon$,

$$
\begin{aligned}
&\sum_{i=k+1}^{\ell-1} g(t_i, \alpha_i)(t_i - t_{i-1}) + g(t_k, \alpha_k)(t_k - \tau_1) + g(t_\ell, \alpha_\ell)(\tau_2 - t_{\ell-1}) \\
&\qquad \le \int_{\tau_1}^{\tau_2} g(t, \alpha(t))\mathrm{d}t + \varepsilon.
\end{aligned}
$$

Moreover there exists $\delta > 0$ such that if $\tau_2 - \tau_1 < \delta$, then $|z^{[n]}(\tau_2) - z^{[n]}(\tau_1)| < \varepsilon$ for $n = n_0, \dots, n_\varepsilon$, and

$$
\int_{\tau_1}^{\tau_2} g(t, \alpha(t))\mathrm{d}t < \varepsilon.
$$

We have proved that the sequence $\{z^{[n]}\}$ is equicontinuous on $[0,b]$.

Choose arbitrary $a, a^* \in (0,b)$, $a^* < a$. Then for each sufficiently large $n \in \mathbb{N}$ there is $a_n \in \{1, \dots, n\}$ such that

$$
t_{a_n-1}, t_{a_n} \in (a^*, a), \quad \lim_{n\to\infty} t_{a_n-1} = \lim_{n\to\infty} t_{a_n} = a. \tag{4.14}
$$

By assumption (1.8),

$$
Q := \min\{p(t) \colon t \in [a^*, b^*]\} > 0. \tag{4.15}
$$

Choose $\tau_1, \tau_2 \in [a,b]$, $\tau_1 < \tau_2$. By (4.10) and (4.14) we find $k, \ell \in \{a_n, \dots, b_n\}$, $k \le \ell$, such that $\tau_1 \in (t_{k-1}, t_k)$, $\tau_2 \in (t_{\ell-1}, t_\ell)$ and, due to (4.4), (4.6), (4.13) and (4.15),

$$
\begin{aligned}
&|y^{[n]}(\tau_2) - y^{[n]}(\tau_1)| \\
&\le \sum_{i=k+1}^{\ell-1} \left|\frac{v_{i-1}}{p(t_i)}\right| (t_i - t_{i-1}) + \left|\frac{v_{k-1}}{p(t_k)}\right| (t_k - \tau_1) + \left|\frac{v_{\ell-1}}{p(t_\ell)}\right| (\tau_2 - t_{\ell-1}) \\
&< \frac{M+1}{Q}(\tau_2 - \tau_1).
\end{aligned}
$$

We have proved that the sequence $\{y^{[n]}\}$ is equicontinuous on $[a,b]$. □

Lemma 4.4 *Let the assumptions of Theorem* 4.1 *be fulfilled. Then there exist subsequences* $\{y^{[m]}\} \subset \{y^{[n]}\}$ *and* $\{z^{[m]}\} \subset \{z^{[n]}\}$ *satisfying*

$$\lim_{m\to\infty} y^{[m]}(t) = y(t) \quad \textit{locally uniformly on } (0,T), \tag{4.16}$$

and

$$\lim_{m\to\infty} z^{[m]}(t) = z(t) \quad \textit{locally uniformly on } [0,T). \tag{4.17}$$

Moreover

$$\alpha(t) \le y(t) \le \beta(t) \quad \textit{for } t \in (0,T). \tag{4.18}$$

Proof. Let $a, b \in (0,T)$, $a < b$. By Lemma 4.2, Lemma 4.3 and the Arzelà-Ascoli theorem we can choose subsequences $\{y^{[m]}\}$, $\{z^{[m]}\}$ such that

$$\lim_{m\to\infty} y^{[m]}(t) = y(t) \quad \text{uniformly on } [a,b],$$

and

$$\lim_{m\to\infty} z^{[m]}(t) = z(t) \quad \text{uniformly on } [0,b].$$

Since $a, b \in (0,T)$ are arbitrary, we use the diagonalization theorem (see e.g. [24]) and get that these subsequences can be chosen in such a way that they fulfil (4.16) and (4.17). Moreover, (4.16), (4.17) and $v_0 = 0$ imply that

$$y \in C(0,T), \quad z \in C[0,T), \quad z(0) = 0. \tag{4.19}$$

Now choose $b \in (0,T)$ and assume that the sequence $\{t_{b_m}\}$ fulfils (4.10). By (4.1) we have for $m \in \mathbb{N}$,

$$\alpha(t_{b_m}) \le y^{[m]}(t_{b_m}) \le \beta(t_{b_m}),$$

and letting $m \to \infty$ we obtain

$$\alpha(b) \le y(b) \le \beta(b).$$

Having in mind that $b \in (0,T)$ is arbitrary, estimate (4.18) is valid. □

Proof of Theorem 4.1. Consider the sequences $\{y^{[n]}\}$ and $\{z^{[n]}\}$ given by (4.4) and (4.5). According to (4.8) and (4.9) these sequences fulfil the equations

$$y^{[n]}(t_{k+1}) = y^{[n]}(0) + h\sum_{i=1}^{k} \frac{z^{[n]}(t_i)}{p(t_{i+1})}, \quad k = 1,\ldots,n-1, \tag{4.20}$$

and

$$z^{[n]}(t_k) = -h\sum_{i=1}^{k} f(t_i, y^{[n]}(t_i)), \quad k = 1,\ldots,n-1. \tag{4.21}$$

Let $\{y^{[m]}\}$ and $\{z^{[m]}\}$ be the subsequences of Lemma 4.4. Assume that $0 < a^* < a < b < b^* < T$ and that (4.10) and (4.14) hold. By (4.16) and (4.17) we have

$$\lim_{m\to\infty} y^{[m]}(t_{b_m+1}) = y(b), \quad \lim_{m\to\infty} y^{[m]}(t_{a_m}) = y(a), \tag{4.22}$$

$$\lim_{m\to\infty} z^{[m]}(t_{b_m}) = z(b), \quad \lim_{m\to\infty} z^{[m]}(t_{a_m-1}) = z(a). \tag{4.23}$$

Denote

$$\varrho_m = \max\{|z^{[m]}(t_i) - z(t_{i+1})| \colon i = a_m, \dots, b_m\},$$

and

$$\sigma_m = \max\{|f(t, y^{[m]}(t)) - f(t, y(t))| \colon t = [a^*, b^*]\}.$$

Then by (4.16), (4.17) and (4.19)

$$\lim_{m\to\infty} \varrho_m = 0, \quad \lim_{m\to\infty} \sigma_m = 0.$$

Consequently, having $h = \frac{T}{m}$ and (4.15), we conclude

$$\lim_{m\to\infty} \frac{T}{m} \sum_{i=a_m}^{b_m} \left| \frac{z^{[m]}(t_i)}{p(t_{i+1})} - \frac{z(t_{i+1})}{p(t_{i+1})} \right|$$

$$\le \lim_{m\to\infty} \frac{T(b_m - a_m)}{mQ} \varrho_m \le \frac{T}{Q} \lim_{m\to\infty} \varrho_m = 0, \tag{4.24}$$

and similarly

$$\lim_{m\to\infty} \frac{T}{m} \sum_{i=a_m}^{b_m} |f(t_i, y^{[m]}(t_i)) - f(t_i, y(t_i))|$$

$$\le \lim_{m\to\infty} \frac{T(b_m - a_m)}{m} \sigma_m \le T \lim_{m\to\infty} \sigma_m = 0. \tag{4.25}$$

Since $f(t, y(t))$ is continuous on $(0, T)$, we have

$$\lim_{m\to\infty} \frac{T}{m} \sum_{i=a_m}^{b_m} f(t_i, y(t_i)) = \int_a^b f(\tau, y(\tau)) \mathrm{d}\tau, \tag{4.26}$$

and, by (1.8) and (4.19),

$$\lim_{m\to\infty} \frac{T}{m} \sum_{i=a_m}^{b_m} \frac{z(t_{i+1})}{p(t_{i+1})} = \int_a^b \frac{z(\tau)}{p(\tau)} \mathrm{d}\tau. \tag{4.27}$$

Equation (4.20) yields

$$y^{[m]}(t_{b_m+1}) = y^{[m]}(t_{a_m}) + \frac{T}{m} \sum_{i=a_m}^{b_m} \frac{z^{[m]}(t_i)}{p(t_{i+1})}.$$

Letting $m \to \infty$ and using (4.22), (4.24) and (4.27), we get

$$y(b) = y(a) + \int_a^b \frac{z(\tau)}{p(\tau)} \mathrm{d}\tau.$$

Equation (4.21) yields

$$z^{[m]}(t_{b_m}) = z^{[m]}(t_{a_m-1}) - \frac{T}{m} \sum_{i=a_m}^{b_m} f(t_i, y^{[m]}(t_i)).$$

By (4.23), (4.25) and (4.26), we get for $m \to \infty$,

$$z(b) = z(a) - \int_a^b f(\tau, y(\tau)) \mathrm{d}\tau.$$

Since the interval $[a, b] \subset (0, T)$ is arbitrary, we have

$$y(t) = y\left(\frac{T}{2}\right) + \int_{T/2}^t \frac{z(\tau)}{p(\tau)} \mathrm{d}\tau, \quad t \in (0, T),$$

and

$$z(t) = z\left(\frac{T}{2}\right) - \int_{T/2}^t f(\tau, y(\tau)) \mathrm{d}\tau, \quad t \in (0, T).$$

The first equation gives

$$y'(t) = \frac{z(t)}{p(t)} \quad \text{for } t \in (0, T),$$

and then the second equation can be written in the form

$$p(t)y'(t) = p\left(\frac{T}{2}\right) y'\left(\frac{T}{2}\right) - \int_{T/2}^t f(\tau, y(\tau)) \mathrm{d}\tau.$$

By (4.19) we get

$$\lim_{t \to 0+} p(t)y'(t) = 0 = p\left(\frac{T}{2}\right) y'\left(\frac{T}{2}\right) - \int_{T/2}^0 f(\tau, y(\tau)) \mathrm{d}\tau,$$

and hence

$$p(t)y'(t) = -\int_0^t f(\tau, y(\tau)) \mathrm{d}\tau \quad \text{for } t \in [0, T). \tag{4.28}$$

We have proved that $y \in C^2(0, T)$ fulfils equation (1.6) for $t \in (0, T)$ and satisfies the first condition in (1.7).

Now we describe a behaviour of y at the singular points $t = 0$ and $t = T$. Integrating equation (4.28) we obtain

$$y(t) = y\left(\frac{T}{2}\right) + \int_t^{T/2} \frac{1}{p(\tau)} \int_0^\tau f(s, y(s)) \mathrm{d}s \mathrm{d}\tau = y\left(\frac{T}{2}\right) + \int_t^{T/2} h(\tau) \mathrm{d}\tau.$$

Due to (4.2) and (4.3), we see that h is integrable on $\left[0, \frac{T}{2}\right]$, and so $y \in C[0,T)$. Since $\alpha, \beta \in C[0,T]$ and $\alpha(T) = \beta(T) = 0$, we get by (4.18),

$$\lim_{t \to T-} y(t) = 0.$$

Therefore if we put $y(T) = 0$, we have that $y \in C[0,T]$ satisfies the second condition in (1.7). Finally, by (4.1) and (4.18), $y(t) > 0$ for $t \in (0,T)$. We have proved that y is a positive solution of problem (1.6), (1.7). □

5 Membrane

In this section we investigate the difference problem (1.1), (1.2) and the corresponding differential problem (1.3), (1.4) by means of Theorem 3.3 and Theorem 4.1. To this aim we find functions α and β satisfying the conditions of these theorems.

Choose $\nu, c \in (0, \infty)$ and define

$$\alpha(t) = \begin{cases} \nu(t+\nu)(T-t), & \gamma \geq 2, \\ \nu t^{2-\gamma}(T-t), & \gamma \in (1,2), \end{cases} \quad t \in [0,T], \tag{5.1}$$

and

$$\beta(t) = c\sqrt{T^2 - t^2}, \quad t \in [0,T]. \tag{5.2}$$

For $n \in \mathbb{N}$ denote

$$h = \frac{T}{n}, \quad t_k = hk, \quad \alpha_k = \alpha(t_k), \quad \beta_k = \beta(t_k), \quad k = 0, \ldots, n. \tag{5.3}$$

Lemma 5.1 *For each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, the inequality*

$$\left| \frac{1}{h^2} \Delta(t_k^3 \Delta \alpha_{k-1}) - (t^3 \alpha'(t))' \right| < \varepsilon, \quad t \in [t_{k-1}, t_{k+1}], \ k = 1, \ldots, n-1,$$

is valid.

Proof. Let $k \in \{1, \ldots, n-1\}$. By the Taylor theorem there are $\xi_k \in (t_k, t_{k+1})$ and $\eta_k \in (t_{k-1}, t_k)$ such that

$$t_{k+1}^3 \alpha(t_{k+1}) = t_{k+1}^3 \alpha(t_k) + h t_{k+1}^3 \alpha'(t_k) + \frac{h^2}{2} t_{k+1}^3 \alpha''(\xi_k),$$

$$t_k^3 \alpha(t_{k-1}) = t_k^3 \alpha(t_k) - h t_k^3 \alpha'(t_k) + \frac{h^2}{2} t_k^3 \alpha''(\eta_k).$$

Adding these two equations, we have

$$t_{k+1}^3 \alpha(t_{k+1}) + t_k^3 \alpha(t_{k-1})$$

$$= (t_{k+1}^3 + t_k^3)\alpha(t_k) + h(t_{k+1}^3 - t_k^3)\alpha'(t_k) + \frac{h^2}{2}(t_{k+1}^3\alpha''(\xi_k) + t_k^3\alpha''(\eta_k)).$$

Further,

$$\frac{1}{h^2}\Delta(t_k^3\Delta\alpha_{k-1}) = \frac{1}{h^2}(t_{k+1}^3\alpha_{k+1} - (t_{k+1}^3 + t_k^3)\alpha_k + t_k^3\alpha_{k-1}),$$

and hence

$$\frac{1}{h^2}\Delta(t_k^3\Delta\alpha_{k-1}) = \frac{1}{h}(t_{k+1}^3 - t_k^3)\alpha'(t_k) + \frac{1}{2}(t_{k+1}^3\alpha''(\xi_k) + t_k^3\alpha''(\eta_k)).$$

Finally,

$$\begin{aligned} &\left|\frac{1}{h^2}\Delta(t_k^3\Delta\alpha_{k-1}) - (t^3\alpha'(t))'\right| \\ &\le |\alpha'(t_k)(t_{k+1}^2 + t_{k+1}t_k + t_k^2) - 3t^2\alpha'(t)| \\ &+\frac{1}{2}|t_{k+1}^3\alpha''(\xi_k) - t^3\alpha''(t)| + \frac{1}{2}|t_k^3\alpha''(\eta_k) - t^3\alpha''(t)| < \varepsilon, \end{aligned}$$

for n sufficiently large, since $t^2\alpha'(t)$ and $t^3\alpha''(t)$ are uniformly continuous on $[0,T]$. □

Lemma 5.2 *Let α and β be given* (5.1) *and* (5.2) *and let* (5.3) *hold. Then there exist $\nu^*, c^* \in (0,\infty)$ such that for each $\nu \in (0,\nu^*]$, each $c \ge c^*$ and for each $n \in \mathbb{N}$, $n \ge 2$, the vectors $(\alpha_0,\dots,\alpha_n)$ and $(\beta_0,\dots,\beta_n)$ are lower and upper functions of problem* (1.1), (1.2) *and*

$$0 < \alpha_k \le \beta_k \quad \textit{for } k = 1,\dots,n-1. \tag{5.4}$$

Proof. We will show that we can find $\nu^*, c^* \in (0,\infty)$ such that for each $\nu \in (0,\nu^*]$, each $c \ge c^*$ and for each $n \ge 2$, the vectors $(\alpha_0,\dots,\alpha_n)$ and $(\beta_0,\dots,\beta_n)$ fulfil conditions (3.1)–(3.4), where

$$p(t_k) = t_k^3, \quad f(t_k,x_k) = t_k^3\left(\frac{1}{8x_k^2} - \frac{a_0}{x_k} - b_0t_k^{2\gamma-4}\right), \quad k = 1,\dots,n-1.$$

We see that $\alpha_n = \alpha(T) = 0$, $\beta_n = \beta(T) = 0$,

$$\Delta\alpha_0 = \begin{cases} \nu t_1(T - t_1 - \nu) & \text{if } \gamma \ge 2, \\ \nu t_1^{2-\gamma}(T - t_1) & \text{if } \gamma \in (1,2), \end{cases}$$

$\Delta\beta_0 = c\sqrt{T^2 - t_1^2} - c\sqrt{T^2}$. Therefore for each $c > 0$ and each sufficiently small $\nu > 0$, conditions (3.2) and (3.4) hold. Having in mind that $t_k = hk$ we get

$$\Delta\beta_{k-1} = c\sqrt{T^2 - t_k^2} - c\sqrt{T^2 - t_{k-1}^2} = \frac{-c(t_k^2 - t_{k-1}^2)}{\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2}}$$

$$= \frac{-c(t_k + t_{k-1})h}{\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2}}, \quad k = 1, \ldots, n-1.$$

By inserting $(\beta_0, \ldots, \beta_n)$ into equation (1.1) we obtain

$$\frac{1}{h^2}\Delta(t_k^3 \Delta\beta_{k-1}) + t_k^3 \left(\frac{1}{8c^2(T^2 - t_k^2)} - \frac{a_0}{c\sqrt{T^2 - t_k^2}} - b_0 t_k^{2\gamma-4} \right)$$

$$< \frac{-ct_{k+1}^3}{h} \cdot \frac{t_{k+1} + t_k}{\sqrt{T^2 - t_{k+1}^2} + \sqrt{T^2 - t_k^2}} + \frac{ct_k^3}{h} \cdot \frac{t_k + t_{k-1}}{\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2}}$$

$$+ \frac{t_k^3}{8c^2(T^2 - t_k^2)} = \psi(t_k, c).$$

Let $t_k \leq \frac{T}{2}$. Then we can find $c^* > 0$ such that

$$\psi(t_k, c) < \frac{-2ct_k(t_{k+1}^3 - t_k^3)}{2h\sqrt{T^2 - t_k^2}} + \frac{t_k^3}{8c^2(T^2 - t_k^2)}$$

$$\leq \frac{-ct_k(t_{k+1}^2 + t_{k+1}t_k + t_k^2)}{T} + \frac{t_k^3}{8c^2\left(T^2 - \frac{T^2}{4}\right)}$$

$$\leq \frac{-ct_k^3}{T}\left(3 - \frac{1}{6Tc^3}\right) < 0,$$

for each $c \geq c^*$ and each $n \geq 2$.

Now, let $t_k \geq \frac{T}{2}$. Then we can choose c^* in such a way that

$$\psi(t_k, c) < \frac{-2ct_k^4}{h} \cdot \frac{\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2} - \left(\sqrt{T^2 - t_{k+1}^2} + \sqrt{T^2 - t_k^2}\right)}{\left(\sqrt{T^2 - t_{k+1}^2} + \sqrt{T^2 - t_k^2}\right)\left(\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2}\right)}$$

$$+ \frac{t_k^3}{8c^2(T^2 - t_k^2)} < \frac{-2ct_k^4}{h} \cdot \frac{(T^2 - t_{k-1}^2) - (T^2 - t_{k+1}^2)}{8(T^2 - t_{k-1}^2)^{\frac{3}{2}}} + \frac{t_k^3}{8c^2(T^2 - t_k^2)}$$

$$= \frac{-2ct_k^4(t_{k+1} + t_{k-1})}{8(T^2 - t_{k-1}^2)^{\frac{3}{2}}} + \frac{t_k^3}{8c^2(T^2 - t_{k-1}^2)} \cdot \frac{T^2 - t_{k-1}^2}{T^2 - t_k^2}$$

$$< \frac{-2cT\left(\frac{T}{2}\right)^4}{8(T^2 - t_{k-1}^2)^{\frac{3}{2}}} + \frac{2T^3}{8c^2(T^2 - t_{k-1}^2)}$$

$$= \frac{-cT^3}{4(T^2 - t_{k-1}^2)^{\frac{3}{2}}}\left(\frac{T^2}{16} - \frac{\sqrt{T^2 - t_{k-1}^2}}{c^3}\right) < 0$$

holds for each $c \geq c^*$ and each $n \geq 2$. We have proved that for each $c \geq c^*$ and each $n \geq 2$, the vector $(\beta_0, \ldots, \beta_n)$ is an upper function of problem (1.1), (1.2).

Acoording to (5.1), we have for $t \in [0, T]$,

$$(t^3\alpha'(t))' = \begin{cases} \nu t^2(3T - 8t - 3\nu) & \text{if } \gamma \geq 2, \\ \nu t^{3-\gamma}(T(2-\gamma)(4-\gamma) - (3-\gamma)(5-\gamma)t) & \text{if } \gamma \in (1, 2). \end{cases}$$

Therefore there exist $b, b^* \in (0, T)$, $b < b^*$, such that

$$(t^3\alpha'(t))' \geq 0 \quad \text{for } t \in [0, b^*].$$

By Lemma 5.1 there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$,

$$\frac{1}{h^2}\Delta(t_k^3 \Delta\alpha_{k-1}) \geq 0, \quad k = 1, \dots, b_n, \tag{5.5}$$

where $t_{b_n} \in (b, b^*)$ and $\lim_{n\to\infty} t_{b_n} = b$. Moreover, Lemma 5.1 implies that there exists $M^* > 0$ such that for each $n \geq 2$,

$$\left|\frac{1}{h^2}\Delta(t_k^3 \Delta\alpha_{k-1})\right| \leq M^*, \quad k = 1, \dots, n-1. \tag{5.6}$$

By inserting $(\alpha_0, \dots, \alpha_n)$ into f we obtain

$$\begin{aligned} f(t_k, \alpha_k) &= t_k^3 \left(\frac{1}{8\nu^2(t_k+\nu)^2(T-t_k)^2} - \frac{a_0}{\nu(t_k+\nu)(T-t_k)} - b_0 t_k^{2\gamma-4}\right) \\ &\geq \frac{t_k^3}{\nu^2(t_k+\nu)^2(T-t_k)^2} \cdot \psi_1(\nu), \end{aligned}$$

where $\psi_1(\nu) = \frac{1}{8} - a_0\nu(T+\nu)T - b_0\nu^2(T+\nu)^2T^{2\gamma-2}$ if $\gamma \geq 2$, and

$$\begin{aligned} f(t_k, \alpha_k) &= t_k^3 \left(\frac{1}{8\nu^2 t_k^{4-2\gamma}(T-t_k)^2} - \frac{a_0}{\nu t_k^{2-\gamma}(T-t_k)} - b_0 t_k^{2\gamma-4}\right) \\ &\geq \frac{t_k^{2\gamma-1}}{\nu^2(T-t_k)^2} \cdot \psi_2(\nu), \end{aligned}$$

where $\psi_2(\nu) = \frac{1}{8} - a_0\nu T^{3-\gamma} - b_0\nu^2T^2$ if $\gamma \in (1, 2)$. We see that $\psi_1(\nu) > 0$ and $\psi_2(\nu) > 0$ for ν sufficiently small.

Assume that $n \geq n_0$. Let $k \in \{b_n, \dots, n\}$. Then

$$\lim_{\nu\to 0} \frac{t_k^3\psi_1(\nu)}{\nu^2(t_k+\nu)^2(T-t_k)^2} = \infty, \quad \lim_{\nu\to 0} \frac{t_k^{2\gamma-1}\psi_2(\nu)}{\nu^2(T-t_k)^2} = \infty, \tag{5.7}$$

and, by (5.6), there exists $\nu^* > 0$ such that the inequality

$$\frac{1}{h^2}\Delta(t_k^3\Delta\alpha_{k-1}) + t_k^3 f(t_k, \alpha_k) > 0 \tag{5.8}$$

is valid for $\nu \in (0, \nu^*)$. Let $k \in \{1, \dots, b_n\}$. Then

$$\frac{t_k^3 \psi_1(\nu)}{\nu^2 (t_k + \nu)^2 (T - t_k)^2} > 0, \quad \frac{t_k^{2\gamma-1} \psi_2(\nu)}{\nu^2 (T - t_k)^2} > 0,$$

and, by (5.5), inequality (5.8) is valid for $\nu \in (0, \nu^*)$, as well.

Now, assume that $n \le n_0$. Then (5.7) is satisfied for $k = 1, \dots, n$ and $\nu \in (0, \nu^*)$. Due to (5.6) we get that the vector $(\alpha_0, \dots, \alpha_n)$ fulfils inequality (5.8).

We have proved that for each $\nu \in (0, \nu^*)$ and each $n \ge 2$ the vector $(\alpha_0, \dots, \alpha_n)$ is a lower function of problem (1.1), (1.2). Since we can choose ν^* and c^* such that (5.4) holds, the lemma is proved. $\square$

Now, we are ready to prove the main result.

Theorem 5.3 *Let $a_0 \ge 0$, $b_0 > 0$, $\gamma > 1$. Then for each $n \ge 2$, the difference problem* (1.1), (1.2) *has a positive solution $(u_0, \dots, u_n)$. Let $y^{[n]} \in C[0,T]$ be a piece-wise linear function with $y^{[n]}(t_k) = u_k$, $k = 0, \dots, n$. Then there exists a subsequence $\{y^{[m]}\} \subset \{y^{[n]}\}$ such that*

$$\lim_{m\to\infty} y^{[m]}(t) = y(t) \quad \textit{locally uniformly on } (0,T),$$

and $y \in C[0,T] \cap C^2(0,T)$ is a positive solution of the differential problem (1.3), (1.4).

Proof. Choose $n \in \mathbb{N}$, $n \ge 2$. Lemma 5.2 yields the vectors $(\alpha_0, \dots, \alpha_n)$ and $(\beta_0, \dots, \beta_n)$ which are lower and upper functions of problem (1.1), (1.2) and satisfy (5.4). For $t \in [0,T]$ and $x \in (0, \infty)$ put

$$p(t) = t^3 \quad \text{and} \quad f(t,x) = t^3 \left(\frac{1}{8x^2} - \frac{a_0}{x} - b_0 t^{2\gamma-4} \right).$$

Since conditions (1.8) and (1.9) are fulfilled, Theorem 3.3 guarantees the existence of a positive solution $(u_0, \dots, u_n)$ of problem (1.1), (1.2) satisfying (3.5). Moreover, by (5.1), (5.2) and (5.3), the functions $\alpha, \beta \in C[0,T]$ fulfil (4.1). Let us put

$$g(t,x) = t^3 \left(\frac{1}{8x^2} + \frac{a_0}{x} \right) + b_0 t^{2\gamma-1}.$$

Then $g \in C([0,T] \times (0,\infty))$ is nonincreasing in x and we have for $t \in \left[0, \frac{T}{2}\right]$

$$g(t, \alpha(t)) \le \begin{cases} At^3 + b_0 t^{2\gamma-1} & \text{if } \gamma \ge 2, \\ Bt^{2\gamma-1} & \text{if } \gamma \in (1,2), \end{cases}$$

where

$$A = \frac{1}{2\nu^4 T^2} + \frac{2a_0}{\nu^2 T}, \quad B = \frac{1}{2\nu^2 T^2} + \frac{2a_0}{\nu T} \left(\frac{T}{2} \right)^{2-\gamma} + b_0.$$

Hence (4.3) is fulfilled and the assertion of Theorem 5.3 is true due to Theorem 4.1. $\square$

6 Conclusion and discussion

The classical approach to singular problems, which can be found in literature, is based on the regularization method, where a singular differential equation is approximated by a sequence of auxiliary regular differential equations.

In this paper we have shown a new approach, which we have demonstrated on the second order singular differential equation (1.6) and the mixed boundary conditions (1.7). Particularly, we have found the sequence of corresponding difference equations (1.5), where $h = \frac{T}{n}$ and $n \in \mathbb{N}$. By Theorem 3.3, we have got the solvability of the discrete problems (1.5), (1.2) for each sufficiently large $n \in \mathbb{N}$. To this goal it was sufficient to find well ordered lower and upper functions. By Theorem 4.1 we have decided whether a sequence of solutions of problems (1.5), (1.2) converges for $n \to \infty$ to a solution of the singular differential problem (1.6), (1.7).

The combination of Theorem 3.3 and Theorem 4.1 can be used in the investigation of various singular differential and difference problems. We have shown such application in Section 5. Moreover, these theorems can be extended and modified for other types of equations, for example for equations with one-dimensional p-Laplacian or for equations whose nonlinearity f depends also on the first difference. Such equations have been studied for example in [22]. Other types of boundary conditions (Dirichlet, periodic, Neumann, ...) can be considered as well.

Now, assume that the points $0 = t_0 < t_1 < \cdots < t_n = T$ need not be equidistant. Denote $h_k = t_k - t_{k-1}$, $k = 1, \ldots, n$, and construct for example the following discretization of equation (1.3)

$$\frac{1}{h_{k+1}} \Delta \left(\frac{t_k^3}{h_k} \Delta u_{k-1} \right) + t_k^3 \left(\frac{1}{8u_k^2} - \frac{a_0}{u_k} - b_0 t_k^{2\gamma-4} \right) = 0, \quad k = 1, \ldots, n-1,$$

and the corresponding discretization of equation (1.6)

$$\frac{1}{h_{k+1}} \Delta \left(\frac{p(t_k)}{h_k} \Delta u_{k-1} \right) + f(t_k, u_k) = 0, \quad k = 1, \ldots, n-1.$$

If we put

$$P(t_k) := \sum_{i=1}^{k} \frac{h_i}{p(t_i)}, \quad k = 1, \ldots, n, \quad h_{n+1} := h_n,$$

then we can check that

$$G(t_k, s_i) = h_{i+1} \begin{cases} P(t_k) - P(T) & \text{for } 0 < s_i \le t_k \le T, \\ P(s_i) - P(T) & \text{for } 0 \le t_k < s_i \le T, \end{cases}$$

is the Green function of the problem

$$\frac{1}{h_{k+1}} \Delta \left(\frac{p(t_k)}{h_k} \Delta u_{k-1} \right) = 0, \quad k = 1, \ldots, n-1, \quad \Delta u_0 = 0, \quad u_n = 0.$$

Moreover, Lemma 2.1 is true and the lower and upper functions method can be extended to this discretization. Further, define

$$y^{[n]}(t) = u_k + \frac{\Delta u_k}{h_{k+1}}(t - t_k), \quad t \in [t_k, t_{k+1}], \ k = 0, \dots, n-1,$$

and

$$\begin{cases} z^{[n]}(t) = 0, & t \in [t_0, t_1], \\ z^{[n]}(t) = v_k + \dfrac{\Delta v_k}{h_{k+1}}(t - t_k), & t \in [t_k, t_{k+1}], \ k = 1, \dots, n-1, \end{cases}$$

where

$$v_k = \frac{p(t_k)}{h_k}\Delta u_{k-1}, \quad k = 1, \dots, n.$$

Denote

$$\mu_n = \max\{h_k \colon 1 \le k \le n\}, \quad \nu_n = \min\{h_k \colon 1 \le k \le n\},$$

and assume

$$\lim_{n \to \infty} \mu_n = 0, \quad \lim_{n \to \infty} \frac{\mu_n}{\nu_n} \in (0, \infty).$$

Then we can prove the same convergence result as in Theorem 5.3.

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