On three solutions of the second order periodic boundary value problem *

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Abstract: We consider the periodic boundary value problem $x''+a(t)x'+b(t)x = f(t,x,x')$, $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$, where a, b are Lebesgue integrable functions and f fulfils the Carathéodory conditions. We extend results about the Leray-Schauder topological degree from [5] and [6] and present conditions implying nonzero values of the degree on sets defined by lower and upper functions. Using such results we prove the existence of at least three different solutions to the above problem.
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1 Introduction

We will study the periodic boundary value problem

$$x'' + a(t)x' + b(t)x = f(t, x, x'), \tag{1.1}$$

$$x(0) = x(2\pi),\ x'(0) = x'(2\pi), \tag{1.2}$$

where a, b are Lebesgue integrable functions on $J = [0, 2\pi]$ and f fulfils the Carathéodory conditions on $J \times \mathbb{R}^2$.

Having values of the Leray-Schauder topological degree of an operator which corresponds to problem (1.1), (1.2) and which is defined on proper sets, we can decide whether there are solutions of (1.1), (1.2) lying in these sets. In [5] and

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[6], where the special case of equation (1.1) (with $a = b = 0$ on J and with f having an one-sided Lebesgue integrable bound) was considered, such sets were found by means of lower and upper functions of problem (1.1), (1.2).

Here we extend results about the degreee of [5], [6] to equation (1.1) with nonzero a, b. Moreover we present theorems which guarantee the existence of at least three solutions to (1.1), (1.2).

Throughout the paper we keep the following notations. $\mathrm{L}(J)$ is the Banach space of Lebesgue integrable functions on J equipped with the norm $||x||_1 = \int_0^{2\pi} |x(t)|dt$ and $\mathrm{L}_\infty(J)$ denotes the Banach space of essentially bounded on J functions with the norm $||x||_\infty = \text{ess sup } \{|x(t)| : t \in J\}$. For $k \in \mathbb{N} \cup \{0\}$, $\mathrm{C}^k(J)$ and $\mathrm{AC}^k(J)$ are the Banach spaces of functions having continuous $k-$th derivatives on J and of functions having absolutely continuous $k-$th derivatives on J, respectively. As usual, the corresponding norms are defined by $||x||_{\mathrm{C}^k} = \sum_{i=0}^k \max\{|x^{(i)}(t)| : t \in J\}$ and $||x||_{\mathrm{AC}^k} = ||x||_{\mathrm{C}^k} + ||x^{(k+1)}||_1$. The symbols $\mathrm{C}(J)$ or $\mathrm{AC}(J)$ are used instead of $\mathrm{C}^0(J)$ or $\mathrm{AC}^0(J)$. $\mathrm{Car}(J \times \mathbb{R}^2)$ is the set of functions $f : J \times \mathbb{R}^2 \to \mathbb{R}$ satisfying the *Carathéodory conditions* on $J \times \mathbb{R}^2$, i.e. (i) for each $(x, y) \in \mathbb{R}^2$ the function $f(\cdot, x, y) : J \to \mathbb{R}$ is measurable, (ii) for a.e. $t \in J$ the function $f(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous, (iii) $\sup_{(x,y) \in K} |f(t, x, y)| \in \mathrm{L}(J)$ for each compact set $K \subset \mathbb{R}^2$. For a Banach space X and a set $M \subset$ X, $cl(M)$ stands for the closure of M and ∂M denotes the boundary of M. If Ω is an open bounded subset in $\mathrm{C}^1(J)$ and the operator $T : cl(\Omega) \to \mathrm{C}^1(J)$ is compact, then $deg(I - T, \Omega)$ denotes the Leray-Schauder topological degree of $I - T$ with respect to Ω, where I stands for the identity operator on $\mathrm{C}^1(J)$. For a definiton and properties of the degree see e.g. [1]-[4].

By *a solution* of problem (1.1), (1.2) we understand a function $u \in \mathrm{AC}^1(J)$ satisfying (1.1) for a.e. $t \in J$ and fulfilling conditions (1.2).

A function $\sigma_1 \in \mathrm{AC}^1(J)$ is said to be *a lower function* of (1.1), (1.2), if

$$\sigma_1'' + a(t)\sigma_1' + b(t)\sigma_1 \geq f(t, \sigma_1, \sigma_1') \quad \text{a.e. on } J,$$

$$\sigma_1(0) = \sigma_1(2\pi), \quad \sigma_1'(0) \geq \sigma_1'(2\pi).$$

A function $\sigma_2 \in \mathrm{AC}^1(J)$ is called *an upper function* of (1.1), (1.2), if

$$\sigma_2'' + a(t)\sigma_2' + b(t)\sigma_2 \leq f(t, \sigma_2, \sigma_2') \quad \text{a.e. on } J,$$

$$\sigma_2(0) = \sigma_2(2\pi), \quad \sigma_2'(0) \leq \sigma_2'(2\pi).$$

A lower function σ_1 of (1.1), (1.2) is called *strict*, if σ_1 does not satisfy (1.1) a.e. on J and if there exists $\varepsilon \in (0, \infty)$ such that

$$\sigma_1'' + a(t)y + b(t)x \geq f(t, x, y)$$

holds a.e. on J and for all $(x, y) \in [\sigma_1(t), \sigma_1(t) + \varepsilon] \times [\sigma_1'(t) - \varepsilon, \sigma_1'(t) + \varepsilon]$.

An upper function σ_2 of (1.1), (1.2) is called *strict*, if σ_2 does not satisfy (1.1) a.e. on J and if there exists $\varepsilon \in (0, \infty)$ such that

$$\sigma_2'' + a(t)y + b(t)x \le f(t, x, y) \tag{1.3}$$

holds a.e. on J and for all $(x, y) \in [\sigma_2(t) - \varepsilon, \sigma_2(t)] \times [\sigma_2'(t) - \varepsilon, \sigma_2'(t) + \varepsilon]$.

Now, let us define operators which will make possible to write problem (1.1), (1.2) in an operator form. Denote

$$domL = \{x \in \mathrm{AC}^1(J) : x \text{ satisfies (1.2) }\}. \tag{1.4}$$

We can see that

$$L : domL \to \mathrm{L}(J),\ x \mapsto x'' + a(\cdot)x' + b(\cdot)x \tag{1.5}$$

is a linear bounded operator and

$$F : \mathrm{C}^1(J) \to \mathrm{L}(J),\ x \mapsto f(\cdot, x(\cdot), x'(\cdot)) \tag{1.6}$$

is a continuous (nonlinear in general) operator, and problem (1.1), (1.2) is equivalent to the operator equation

$$Lx = Fx. \tag{1.7}$$

To determine an operator the degree of which will be studied we need to distinguish two cases: $KerL = \{0\}$ and $KerL \ne \{0\}$.

We will say that problem (1.7) is *resonance* if $KerL \ne \{0\}$. If $KerL = \{0\}$ the problem is called *nonresonance*.

Both the cases are investigated in Section 2.

2 Nonresonance and resonance problems

I. First, let us consider the nonresonance case $KerL = \{0\}$. It means that the homogeneous linear boundary value problem corresponding to (1.1), (1.2)

$$x'' + a(t)x' + b(t)x = 0,\ x(0) = x(2\pi),\ x'(0) = x'(2\pi) \tag{2.1}$$

has the trivial solution, only. One class of nonresonance problems (1.1), (1.2) is characterized in the next lemma.

Lemma 2.1 *Let us suppose that $a, b \in \mathrm{L}(J)$ and that b satisfies*

$$b(t) \le 0 \ a.e. \ on \ J \tag{2.2}$$

and

$$\int_0^{2\pi} b(t)dt \ne 0. \tag{2.3}$$

Then problem (2.1) has only the trivial solution, i.e. $KerL = \{0\}$.

Proof. Suppose on the contrary that $KerL \neq \{0\}$. Then there exists a nontrivial solution u of (2.1) and, having in mind condition (1.2) and the fact that $-u \in KerL$, we can assume without loss of generality that

$$\max_{t\in J} u(t) = u(t_0) > 0, \; u'(t_0) = 0, \; t_0 \in [0, 2\pi). \tag{2.4}$$

Further, if we extend the functions a, b and u to 2π-periodic on $\mathbb{R}$ functions, we get for all $t \in \mathbb{R}$

$$u'(t) = -e^{-A(t)} \int_{t_0}^{t} b(s)u(s)e^{A(s)}ds, \tag{2.5}$$

where $A(t) = \int_{t_0}^{t} a(s)ds$. Conditions (2.4) and (2.5) yield

$$u(t) > 0 \text{ and } u'(t) \geq 0 \text{ for all } t \in [t_0, \infty). \tag{2.6}$$

On the other hand, in view of conditon (2.3) we see that u cannot be a constant function. This together with the periodicity of u imply that u' has to change its sign on each interval of the length 2π, which contradicts (2.6). Thus problem (2.1) has only the trivial solution. □

Remark 2.2 Condition (2.3) in Lemma 2.1 cannot be omitted because problem (2.1) with $b(t) = 0$ a.e. on J has constant nontrivial solutions.

If $KerL = \{0\}$, then the Green function G of (2.1) exists and we can find the inverse (to L) operator

$$L^{-1} : \mathrm{L}(J) \to domL, \; y \mapsto \int_0^{2\pi} G(t,s)y(s)ds. \tag{2.7}$$

If we denote

$$L^{+} = iL^{-1} : \mathrm{L}(J) \to \mathrm{C}^1(J), \tag{2.8}$$

where $i : \mathrm{AC}^1(J) \to \mathrm{C}^1(J)$ is the embedding operator, then the operator L^+F is absolutely continuous and problem (1.1), (1.2) is equivalent to the operator equation $(I - L^+F)x = 0, \; x \in domL$. The degree theory implies that provided for some open bounded set $\Omega \subset \mathrm{C}^1(J)$ the relation

$$deg(I - L^+F, \Omega) \neq 0 \tag{2.9}$$

is true, then the operator L^+F has a fixed point in Ω. This means, in view of (2.7), (2.8), that this fixed point belongs to $domL$ and so problem (1.1), (1.2) has a solution in Ω. We will see in Section 4 that such a set Ω can be found by means of strict lower and upper functions of problem (1.1), (1.2).

II. Now, we will consider resonance problems having $KerL \neq \{0\}$. Using Lemma 2.1 we can transform such problems on nonresonance ones by means of auxiliary operators L_μ and H_μ.

So, let $KerL \neq \{0\}$ and let $domL$ be given by (1.4). Then for a $\mu \in (-\infty, 0)$ we define a linear operator

$$L_\mu : domL \to \mathrm{L}(J),\ x \mapsto x'' + a(\cdot)x' + \mu x \tag{2.10}$$

and an operator

$$H_\mu : \mathrm{C}^1(J) \to \mathrm{L}(J), x \mapsto h_\mu(\cdot, x(\cdot), x'(\cdot)), \tag{2.11}$$

where

$$h_\mu(t, x, y) = f(t, x, y) + (\mu - b(t))x.$$

We see that L_μ and H_μ are continuous and problem $(1.1), (1.2)$ is equivalent to the operator equation

$$L_\mu x = H_\mu x. \tag{2.12}$$

According to Lemma 2.1 problem (2.12) is nonresonance, i.e. $KerL_\mu = \{0\}$. Therefore we can argue as in Part I and get the inverse (to L_μ) operator

$$L_\mu^{-1} : \mathrm{L}(J) \to domL,\ y \mapsto \int_0^{2\pi} G_\mu(\cdot, s)y(s)ds,$$

where G_μ is the Green function of

$$x'' + a(t)x' + \mu x = 0,\ x(0) = x(2\pi), x'(0) = x'(2\pi). \tag{2.13}$$

As before, denoting

$$L_\mu^+ = iL_\mu^{-1} : \mathrm{L}(J) \to \mathrm{C}^1(J), \tag{2.14}$$

we arrive to the operator equation

$$(I - L_\mu^+ H_\mu)x = 0,\ x \in domL, \tag{2.15}$$

which is equivalent to $(1.1), (1.2)$. Since $L_\mu^+ H_\mu$ is absolutely continuous, we can use the degree theory again and deduce that if

$$deg(I - L_\mu^+ H_\mu, \Omega) \neq 0 \tag{2.16}$$

for some open bounded set $\Omega \subset \mathrm{C}^1(J)$, then equation (2.15) has a solution in $\Omega \cap domL$ which implies that problem $(1.1), (1.2)$ has a solution in Ω.

To summarize, for the existence of a solution to $(1.1), (1.2)$ in Ω we need to prove:
(I) $deg(I - L^+ F, \Omega) \neq 0$ if $KerL = \{0\}$.
(II) $deg(I - L_\mu^+ H_\mu, \Omega) \neq 0$ for some negative μ if $KerL \neq \{0\}$.

3 Values of the Leray-Schauder degree

In this section we prove several theorems with statements of the type (2.9) or (2.16). For definitions of operators see (1.5), (1.6), (2.8), (2.10), (2.11) and (2.14).

Proposition 3.1 *Let Ker $L = \{0\}$. Further suppose that there exist numbers $c, r_1 \in (0, \infty)$ such that for any $\lambda \in [0,1]$ each solution u of the equation*

$$(I - \lambda L^+ F)x = 0, \; x \in domL \tag{3.1}$$

satisfies

$$|u(t_u)| < c \text{ for some } t_u \in J, \; ||u'||_{\mathrm{C}} < r_1. \tag{3.2}$$

Denote $r_0 = c + 2\pi r_1$ and

$$\Omega = \{x \in \mathrm{C}^1(J) : ||x||_{\mathrm{C}} < r_0, \; ||x'||_{\mathrm{C}} < r_1\}. \tag{3.3}$$

Then

$$deg(I - L^+F, \Omega) = 1.$$

Proof. Let us choose $\lambda \in [0,1]$ and let u be a corresponding solution of (3.1) with this λ. Then u fulfils (3.2) and so $|u(t)| \le |u(t_u)| + |\int_{t_u}^t u'(s)ds| < c + \int_0^{2\pi} |u'(s)|ds < r_0$ for each $t \in J$. Therefore $u \notin \partial\Omega$ and so the operator $I - \lambda L^+ F$ is the homotopy on $cl(\Omega) \times [0,1]$ which implies that $deg(I - L^+F, \Omega)$=$deg(I, \Omega) = 1$. □

Proposition 3.2 *Let Ker $L \neq \{0\}$ and let $\mu \in (-\infty, 0)$. Moreover, let us suppose that there are positive numbers c, r_1 such that for any $\lambda \in [0,1]$ each solution u of the equation*

$$(I - \lambda L_\mu^+ H_\mu)x = 0, \; x \in domL$$

satisfies (3.2). *Then*

$$deg(I - L_\mu^+ H_\mu, \Omega) = 1,$$

where Ω is given by (3.3) *and $r_0 = c + 2\pi r_1$.*

Proof. We can argue as in the proof of Proposition 3.1. □

Using the homotopy argument as before we get the following modification of Proposition 3.1.

Proposition 3.3 *Let Ker $L = \{0\}$ and let there exist $\rho^* \in (0, \infty)$ such that for any $\lambda \in [0,1]$ each solution u of* (3.1) *satisfies $||u||_{\mathrm{C}^1} \le \rho^*$. Then for each $\rho > \rho^*$*

$$deg(I - L^+F, K(\rho)) = 1, \tag{3.4}$$

where

$$K(\rho) = \{x \in \mathrm{C}^1(J) : ||x||_{\mathrm{C}^1} < \rho\}. \tag{3.5}$$

We see that a priori estimates of solutions of problems under consideration are essential for the determination of Ω and for the degree computation. In contrast to Propositions 3.1–3.3, where we assumed such estimates directly, now, we will show conditions which can be imposed on f to ensure the needed estimates.

Theorem 3.4 *Let Ker L= $\{0\}$ and let there exist $e \in \mathrm{L}(J)$ such that*

$$|f(t,x,y)| \le e(t) \ for \ a.e. \ t \in J \ and \ each \ x, y \in \mathbb{R}. \tag{3.6}$$

Then there exists $\rho^* \in (0,\infty)$ such that $(3.4), (3.5)$ *are true for each $\rho > \rho^*$.*

Proof. Let u be a solution of (3.1) for some $\lambda \in [0,1]$. Then

$$u(t) = \lambda \int_0^{2\pi} G(t,s) f(s, u(s), u'(s)) ds,$$

where G is the Green function of (2.1). Denote

$$\gamma = \max\{|G(t,s)| : t, s \in J\}, \ \delta = \max\{\left|\frac{\partial G(t,s)}{\partial t}\right| : t, s \in J\}.$$

Then $||u||_{\mathrm{C}^1} \le (\gamma + \delta)||e||_1 = \rho^*$ and we can use Proposition 3.3. □

Remark 3.5 In the case $KerL \neq \{0\}$, condition (3.6) need not be sufficient for the existence of solutions of (1.1), (1.2), which is obvious if we choose (1.1) in the form $x'' = 1$. (Clearly problem $x'' = 0, \ x(0) = x(2\pi), x'(0) = x'(2\pi)$ has nontrivial solutions and problem $x'' = 1, \ x(0) = x(2\pi), x'(0) = x'(2\pi)$ is not solvable.) Moreover, having $KerL \neq \{0\}$, the Green function G of (2.1) does not exist and we cannot argue as in the proof of Theorem 3.4 and hence, without additional assumptions, we are not able to get an assertion about the degree as before. In this case it may be the method of lower and upper functions, which is used in Section 4, an profitable instrument.

4 The Leray-Schauder degree and lower and upper functions

Let us consider problem (1.1), (1.2) and functions $\sigma_1, \sigma_2 \in \mathrm{AC}^1(J)$. Further, for any $\mu \in (-\infty, 0)$ let G_μ be the Green function of (2.13) and let the operators L_μ, L_μ^+, H_μ be given by (2.10), (2.14) and (2.11). We denote

$$r_i = \max\{||\sigma_1^{(i)}||_{\mathrm{C}}, ||\sigma_2^{(i)}||_{\mathrm{C}}\}, \ i = 0, 1, \ \gamma_\mu = \max_{J \times J} |\frac{\partial G_\mu(t,s)}{\partial t}|. \tag{4.1}$$

Proposition 4.1 *Let σ_1, σ_2 be strict lower and upper functions of* (1.1), (1.2) *such that*

$$\sigma_1 < \sigma_2 \text{ on } J, \tag{4.2}$$

and let there exist $e \in \mathrm{L}(J)$ satisfying

$$|f(t,x,y)| < e(t) \text{ for a.e. } t \in J \text{ and each } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}. \tag{4.3}$$

Then for any $\mu \in (-\infty, 0)$

$$deg(I - L_\mu^+ H_\mu, \Omega_\mu) = 1, \tag{4.4}$$

where

$$\Omega_\mu = \{x \in \mathrm{C}^1(J) : \sigma_1 < x < \sigma_2 \text{ on } J, ||x'||_{\mathrm{C}} < M_\mu\}, \tag{4.5}$$

and $M_\mu \geq \gamma_\mu(3||e||_1 + (||b||_1 - 2\pi\mu)r_0 + ||a||_1)$.

Proof. Let us choose $\mu \in (-\infty, 0)$ and put for a.e. $t \in J$ and for each $(x,y) \in \mathbb{R}^2$

$$q_\mu(t,x,y) = f(t, \sigma(x), y) + (\mu - b(t))\sigma(x),$$

where

$$\sigma(x) = \begin{cases} \sigma_2(t) & if \;\; \sigma_2(t) < x \\ x & if \;\; \sigma_1(t) \leq x \leq \sigma_2(t) \\ \sigma_1(t) & if \;\; x < \sigma_1(t). \end{cases}$$

Further, define

$$p_\mu(t,x,y) = \begin{cases} q_\mu(t,x,y) + \omega\left(t, \frac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}\right) & if \;\; \sigma_2(t) < x \\ q_\mu(t,x,y) & if \;\; \sigma_1(t) \leq x \leq \sigma_2(t) \\ q_\mu(t,x,y) - \omega\left(t, \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}\right) & if \;\; x < \sigma_1(t), \end{cases} \tag{4.6}$$

and for $\varepsilon \in [0,1]$

$$\omega(t, \varepsilon) = \sup_{(x,y,z) \in D_{t,\varepsilon}} \{|f(t,x,y) - f(t,x,z)| + |a(t)(y-z)|\},$$

where $D_{t,\varepsilon} = \{(x,y,z) \in \mathbb{R}^3 : \sigma_1(t) \leq x \leq \sigma_2(t), |y| \leq 1 + |\sigma_1'(t)| + |\sigma_2'(t)|, |y - z| \leq \varepsilon\}$. We can see that $\omega \in \mathrm{Car}(J \times [0,1])$ is non-negative and non-decreasing in the second variable, $\omega(t,0) = 0$ a.e. on J. Moreover, for a.e. $t \in J$ and any $y \in \mathbb{R}$ satisfying $|y - \sigma_i'(t)| \leq 1$ the inequality

$$|f(t, \sigma_i, \sigma_i') - f(t, \sigma_i, y)| + |a(t)(y - \sigma_i')| \leq \omega(t, |y - \sigma_i'|), \;\; i = 1, 2, \tag{4.7}$$

is true. In view of (4.6), for a.e. $t \in J$ and for all $(x,y) \in \mathbb{R}^2$, we have

$$|p_\mu(t,x,y)| < 3e(t) + (|b(t)| - \mu)r_0 + |a(t)|. \tag{4.8}$$

Recall that L_μ is defined by (2.10) and define an operator

$$P_\mu : \mathrm{C}^1(J) \to \mathrm{L}(J),\ x \mapsto p_\mu(\cdot, x(\cdot), x'(\cdot)).$$

With respect to Lemma 2.1 we have $Ker L_\mu = \{0\}$. Therefore, according to (4.8), Theorem 3.4 ensures the existence of $\rho^* \in (r_0 + M_\mu, \infty)$ such that for each $\rho > \rho^*$

$$deg(I - L_\mu^+ P_\mu, K(\rho)) = 1, \tag{4.9}$$

where $K(\rho) = \{x \in \mathrm{C}^1(J) : ||x||_{\mathrm{C}^1} < \rho\}$. Let us consider an arbitrary solution $u \in domL$ of the equation $(I - L_\mu^+ P_\mu)x = 0$ and let us prove that $u \in \Omega_\mu$. Since $u(t) = \int_0^{2\pi} G_\mu(t,s) p_\mu(s, u(s), u'(s))ds$ for all $t \in J$, we have that

$$u'' + a(t)u' + \mu u = p_\mu(t, u, u')$$

for a.e. $t \in J$. By (4.1) and (4.8), we get

$$||u'||_{\mathrm{C}} \le \max_{t \in J} \int_0^{2\pi} \left| \frac{\partial G_\mu(t,s)}{\partial t} \right| |p_\mu(s, u(s), u'(s))| ds < M_\mu. \tag{4.10}$$

Let us show that

$$\sigma_1 < u < \sigma_2 \text{ on } J. \tag{4.11}$$

Put $v = u - \sigma_2$ on J and assume on the contrary that

$$\max_{t \in J}\{v(t)\} = v(t_0) \ge 0.$$

Then, having in mind conditions (1.2), we can assume without loss of generality that $v'(t_0) = 0$ and $t_0 \in [0, 2\pi)$.

First, let $v(t_0) > 0$. Then there is $\delta > 0$ such that for a.e. $t \in (t_0, t_0 + \delta)$

$$v(t) > 0,\ |v'(t)| < \frac{v(t)}{v(t)+1} < 1. \tag{4.12}$$

Therefore we have for a.e. $t \in (t_0, t_0 + \delta)$

$$v''(t) = u''(t) - \sigma_2''(t) \ge f(t, \sigma_2, u') + (\mu - b(t))\sigma_2 + \omega\left(t, \frac{x - \sigma_2}{x - \sigma_2 + 1}\right)$$

$$-a(t)u' - \mu u - f(t, \sigma_2, \sigma_2') + a(t)\sigma_2' + b(t)\sigma_2,$$

and using (4.7), (4.12), we get $v''(t) > 0$ for a.e. $t \in (t_0, t_0 + \delta)$. Hence,

$$0 < \int_{t_0}^{t} v''(s)ds \le v'(t) \text{ for all } t \in (t_0, t_0 + \delta),$$

which contradicts the fact that $v(t_0)$ is the maximal value of v on J. Thus, $u \le \sigma_2$ on J. The inequality $\sigma_1 \le u$ on J can be proved analogously putting $v = \sigma_1 - u$ on J. So, we have

$$\sigma_1 \le u \le \sigma_2 \text{ on } J. \tag{4.13}$$

It remains to prove that the inequalities in (4.13) must be strict. Suppose that $v(t_0) = 0$. Since σ_2 is a strict upper function of (1.1), (1.2), there is $\varepsilon > 0$ such that (1.3) is valid a.e. on J and for all $x \in [\sigma_2(t)-\varepsilon, \sigma_2(t)]$, $y \in [\sigma_2'(t)-\varepsilon, \sigma_2'(t)+\varepsilon]$. Moreover, since σ_2 is not a solution of (1.1), there is $\delta > 0$ such that for each $t \in [t_0, t_0 + \delta)$ the inequalities $-\varepsilon \le v(t) \le 0$, $|v'(t)| \le \varepsilon$ are satisfied and we can assume without loss of generality that there exists $\xi \in (t_0, t_0 + \delta)$ such that $v'(\xi) < 0$. On the other hand, according to (1.3), we have

$$v''(t) = u''(t) - \sigma_2''(t) = f(t, u, u') - a(t)u'(t) - b(t)u(t) - \sigma_2''(t) \ge 0$$

for a.e. $t \in (t_0, t_0 + \delta)$, thus

$$0 \le \int_{t_0}^{\xi} v''(s)ds = v'(\xi) < 0,$$

a contradiction. Therefore $u < \sigma_2$ on J. The inequality $\sigma_1 < u$ on J can be proved similarly for $v = \sigma_1 - u$ on J. Thus, we have proved (4.10) and (4.11), which means that u belongs to Ω_μ. But then, by (4.9) and the excission property of the degree, we get

$$deg(I - L_\mu^+ P_\mu, \Omega_\mu) = 1,$$

and since $P_\mu = H_\mu$ on $cl(\Omega_\mu)$, assertion (4.4) is valid. □

Corollary 4.2 *Let the assumptions of Proposition 4.1 be fulfilled and moreover let Ker L=* $\{0\}$. *Further, suppose that G is the Green function of* (2.1) *and the operators* L^+, F *are given by by* (1.5), (1.6). *Then*

$$deg(I - L^+F, \Omega) = 1, \tag{4.14}$$

where

$$\Omega = \{x \in \mathrm{C}^1(J) : \sigma_1 < x < \sigma_2 \text{ on } J,\ ||x'||_{\mathrm{C}} < M\}$$

and $M = \max_{J\times J}\{|\frac{\partial G(t,s)}{\partial t}|\}||e||_1$.

Proof. We can argue similarly as in the proof of Proposition 4.1, working with G, L, F and $q(t, x, y) = f(t, \sigma(x), y)$ instead of G_μ, L_μ, H_μ and q_μ. □

Remark 4.3 Comparing Theorem 3.4 and Corollary 4.2 we see that we have used different sets in their assertions (3.5) and (4.14) about degree values. In (3.5) we work with a ball $K(\rho)$ the radius of which is not specified, it is sufficiently large, only, while the set Ω in (4.14) is described by means of lower and upper functions σ_1 and σ_2. Such specification of the set Ω will be useful for the multiplicity result in Section 5.

Using a proper lemma on a priori estimates, we can weaken condition (4.3) in Proposition 4.1. Let us show one of such lemmas.

Lemma 4.4 *Suppose that* $r \in (0, \infty)$, $q \in \mathrm{L}_\infty(J)$, $a, b, p \in \mathrm{L}(J)$, q, p *positive a.e. on* J. *Further, let a constant* r^* *satisfy* $r^* \geq (e^M - A)A$, *where* $A = \exp(||a||_1)$ *and* $M = r(2||q||_\infty + ||b||_1) + ||a||_1 + ||p||_1$. *Then for each* $x \in \mathrm{AC}^1(J)$ *fulfilling conditions* (1.2),

$$||x||_{\mathrm{C}} < r \tag{4.15}$$

and

$$x'' + a(t)x' + b(t)x \leq (1 + |x'|)(q(t)|x'| + p(t)) \text{ for a.e. } t \in J, \tag{4.16}$$

the estimate

$$||x'||_{\mathrm{C}} < r^* \tag{4.17}$$

is valid.

Proof. Suppose that $x \in \mathrm{AC}^1(J)$ satisfies conditions (1.2), (4.15) and (4.16) and extend x, q, a, b, p on $\mathbb{R}$ as 2π-periodic functions. Let us assume that $\max\{x'(t) : t \in J\} = x'(t_0) > 0$. Then we can find $\tau_0 < t_0$ such that $t_0 - \tau_0 < 2\pi$, $x'(\tau_0) = 0$ and $x'(t) > 0$ on $(\tau_0, t_0]$. With respect to (4.16) we have for a.e. $t \in [\tau_0, t_0]$

$$x'' + a(t)x' \leq (1 + x')(q(t)x' + p(t) + |b(t)|r).$$

Multiply this inequality by $\exp(\int_{\tau_0}^t a(s)ds)$ and put $z(t) = x'(t)\exp(\int_{\tau_0}^t a(s)ds)$. Then, integrating from τ_0 to t_0, we get

$$\int_{\tau_0}^{t_0} \frac{z'(t)dt}{A + z(t)} < 2r||q||_\infty + ||p||_1 + ||b||_1 r.$$

Therefore $z(t_0) < e^M - A$ and so $x'(t_0) < r^*$.

Similarly, if we assume that $\min\{x'(t) : t \in J\} = x'(t_1) < 0$, we can find $\tau_1 > t_1$ with $\tau_1 - t_1 < 2\pi$, $x'(\tau_1) = 0$, $x'(t) < 0$ on $[t_1, \tau_1)$. Then (4.16) yields a.e. on $[t_1, \tau_1]$

$$x'' + a(t)x' \leq (1 - x')(-q(t)x' + p(t) + |b(t)|r).$$

Multiply this inequality by $\exp(\int_{\tau_1}^t a(s)ds)$ and put $z(t) = -x'(t)\exp(\int_{\tau_1}^t a(s)ds)$. Then, integrating from t_1 to τ_1, we get

$$-\int_{t_1}^{\tau_1} \frac{z'(t)dt}{A + z(t)} < 2r||q||_\infty + ||p||_1 + ||b||_1 r.$$

Therefore $z(t_1) < e^M - A$, and so $x'(t_1) > -r^*$. □

Consider the constant r^* from Lemma 4.4 and put

$$e^*(t) = \sup\{|f(t, x, y)| : x \in [\sigma_1(t), \sigma_2(t)], y \in [-2r^*, 2r^*]\}. \tag{4.18}$$

Clearly $e^* \in \mathrm{L}(J)$ and using Proposition 4.1 and Lemma 4.4 we can prove the following theorem.

Theorem 4.5 *Let σ_1 and σ_2 be strict lower and upper functions of* (1.1), (1.2) *satisfying* (4.2). *Further, suppose that there exist functions $q \in \mathrm{L}_\infty(J)$, $d \in \mathrm{L}(J)$ which are positive a.e. on J and such that for a.e. $t \in J$ and for all $x \in [\sigma_1(t), \sigma_2(t)], y \in \mathbb{R}$*

$$f(t,x,y) \leq (1+|y|)(q(t)|y| + d(t)). \tag{4.19}$$

Then for any $\mu \in (-\infty, 0)$

$$deg(I - L_\mu^+ H_\mu, \Omega^*) = 1, \tag{4.20}$$

where

$$\Omega^* = \{x \in \mathrm{C}^1(J) : \sigma_1 < x < \sigma_2 \ on\ J, ||x'||_{\mathrm{C}} < r^*\}, \tag{4.21}$$

with r^* from Lemma 4.4. (For L_μ^+ and H_μ see (2.14) *and* (2.11)*.)*

Proof. Let us take r_0 and r_1 according to (4.1), put

$$r = r_0, \ p = d \text{ a.e. on } J \tag{4.22}$$

and assume that r^* from Lemma 4.4 satisfies $r^* > r_1$. For $y \in \mathbb{R}$ define

$$\chi(y, r^*) = \begin{cases} 1 & if \ \ |y| \leq r^* \\ 2 - |y|/r^* & if \ \ r^* < |y| < 2r^* \\ 0 & if \ \ |y| \geq 2r^*, \end{cases}$$

and consider the equation

$$x'' + a(t)x' + b(t)x = f^*(t, x, x'), \tag{4.23}$$

where $f^*(t,x,y) = \chi(y, r^*) f(t,x,y)$ for a.e. $t \in J$ and all $x, y \in \mathbb{R}$. We can see that σ_1 and σ_2 are strict lower and upper functions for (4.23), (1.2), and that

$$|f^*(t,x,y)| < e^*(t) \text{ for a.e. } t \in J \text{ and for all } x \in [\sigma_1(t), \sigma_2(t)], y \in \mathbb{R},$$

where e^* is given by (4.18). So, for any $\mu \in (-\infty, 0)$, we can define an operator

$$H_\mu^* : \mathrm{C}^1(J) \to \mathrm{L}(J), \ x \mapsto f^*(\cdot, x(\cdot), x'(\cdot)) + (\mu - b(\cdot))x$$

and a set Ω_μ by (4.5) with $M_\mu = r^* + \gamma_\mu(3||e^*||_1 + (||b||_1 - 2\pi\mu)r_0 + ||a||_1)$. Then, applying Proposition 4.1 on problem (4.23), (1.2), we get

$$deg(I - L_\mu^+ H_\mu^*, \Omega_\mu) = 1. \tag{4.24}$$

Let $u \in \Omega_\mu$ be a solution of (4.23), (1.2). Then, by (4.22), (4.19), we have $||u||_{\mathrm{C}} < r$ and

$$u'' + a(t)u' + b(t)u = \chi(u', r^*) f(t,u,u') \leq (1 + |u'|)(q(t)|u'| + p(t)) \text{ a.e. on } J.$$

Therefore, by Lemma 4.4, $||u'||_{\mathrm{C}} < r^*$ and so, in view of (4.21), $u \in \Omega^*$. Using (4.24) and the excission property of the degree we get $deg(I - L_\mu^+ H_\mu^*, \Omega^*) = 1$, which together with the fact that $H_\mu = H_\mu^*$ on $cl(\Omega^*)$ imply (4.20). $\square$

Corollary 4.6 *Let the assertions of Theorem 4.5 be fulfilled and moreover let Ker L=* $\{0\}$*. Further, suppose that the operators* L^+*,* F *are given by* (1.5), (1.6)*. Then*

$$deg(I - L^+F, \Omega^*) = 1,$$

with Ω^* *by Theorem 4.5.*

Proof. We can argue similarly as in the proof of Theorem 4.5, working with L, F, $F^* : \mathrm{C}^1(J) \to \mathrm{L}(J),\ x \mapsto f^*(\cdot, x(\cdot), x'(\cdot))$ and Corollary 4.2 instead of L_μ, H_μ, H^*_μ and Proposition 4.1, respectively. □

5 Main results

Using properties of the Leray-Schauder degree we get the following existence result as the direct consequence of Theorem 4.5 or Corollary 4.6.

Theorem 5.1 *Let* σ_1 *and* σ_2 *be strict lower and upper functions of* (1.1), (1.2) *satisfying* (4.2)*. Further, suppose that there exist functions* $q \in \mathrm{L}_\infty(J)$*,* $d \in \mathrm{L}(J)$ *which are positive a.e. on* J *and such that for a.e.* $t \in J$ *and for all* $x \in [\sigma_1(t), \sigma_2(t)], y \in \mathbb{R}$ *condition* (4.19) *is satisfied. Then problem* (1.1), (1.2) *has at least one solution* x *such that* $\sigma_1 < x < \sigma_2$ *on* J*.*

Remark 5.2 The existence of a solution to $(1.1), (1.2)$ can be proved under weaker assumptions than those in Theorem 5.1. Particularly, σ_1 and σ_2 need not be strict and we can assume that $\sigma_1 \leq \sigma_2$ on J. Then $(1.1), (1.2)$ has a solution x satisfying $\sigma_1 \leq x \leq \sigma_2$ on J. For the proof of this generalization we can modify corresponding proofs in [6].

Now, we will prove our main result about the existence of three solutions of problem $(1.1), (1.2)$. To this aim we will consider reverse ordered lower and upper functions σ_1 and σ_2 of this problem , i.e. we will suppose

$$\sigma_2 < \sigma_1 \text{ on } J. \tag{5.1}$$

Theorem 5.3 *Let* σ_1 *and* σ_2 *be strict lower and upper functions of* (1.1), (1.2) *satisfying* (5.1)*. Let the inequalities*

$$\liminf_{x\to\infty}(f(t,x,0) - b(t)x) > 0, \limsup_{x\to-\infty}(f(t,x,0) - b(t)x) < 0 \tag{5.2}$$

be fulfilled uniformly for a.e. $t \in J$*. Finally, suppose that there exist functions* $q \in \mathrm{L}_\infty(J)$*,* $d \in \mathrm{L}(J)$ *which are positive a.e. on* J *and such that condition* (4.19) *holds for a.e.* $t \in J$ *and for all* $x, y \in \mathbb{R}$*. Then problem* (1.1), (1.2) *has at least three different solutions.*

Proof. According to inequalities (5.2) we can find a number $\rho > \max\{||\sigma_1||_{\mathrm{C}}, ||\sigma_2||_{\mathrm{C}}\}$ such that

$$f(t,\rho,0) - b(t)\rho > 0 \; f(t,-\rho,0) + b(t)\rho < 0, \text{ a.e. on } J. \tag{5.3}$$

For a.e. $t \in J$ and for all $x, y \in \mathbb{R}$ define functions

$$g(t,x,y) = f(t,x,y) - a(t)y - b(t)x,$$

$$h(t,x,y) = \begin{cases} g(t,-\rho,y) - \omega_1(t, \frac{-\rho-x}{-\rho-x+1}) & if \;\; x < -\rho \\ g(t,x,y) & if \;\; |x| \le \rho \\ g(t,\rho,y) + \omega_2(t, \frac{x-\rho}{x-\rho+1}) & if \;\; x > \rho, \end{cases}$$

and for $\varepsilon > 0$ put

$$\omega_i(t,\varepsilon) = \sup_{z\in[-\varepsilon,\varepsilon]} \{|g(t,(-1)^i\rho,0) - g(t,(-1)^i\rho,z)|\}, \; i = 1,2.$$

We will study the auxiliary equation

$$x'' = h(t,x,x'). \tag{5.4}$$

Choose an arbitrary number $\eta > 0$ and put $\tilde{\sigma}_2(t) = \rho + \eta$, $\tilde{\sigma}_1(t) = \rho - \eta$ for all $t \in J$. Then, in view of (5.3),

$$h(t,\rho+\eta,0) = g(t,\rho,0) + \omega_2(t, \frac{\eta}{\eta+1}) > 0$$

is valid for a.e. $t \in J$. This means that $\tilde{\sigma}_2$ is an upper function of (5.4), (1.2) and that it is not a solution of (5.4). Further, put $\varepsilon = (\eta/2)(\eta/2+1)^{-1}$ and choose arbitrary $x \in [\tilde{\sigma}_2 - \varepsilon, \tilde{\sigma}_2]$, $y \in [\tilde{\sigma}_2' - \varepsilon, \tilde{\sigma}_2' + \varepsilon]$. Then

$$x \in (\rho + \frac{\eta}{2}, \rho + \eta], \; y \in [-\varepsilon, \varepsilon], \; |y| < \frac{x-\rho}{x-\rho+1}, \tag{5.5}$$

wherefrom

$$\omega_2(|y|) \le \omega_2(t, \frac{x-\rho}{x-\rho+1}).$$

Thus, according to (5.5), we have

$$h(t,x,y) = g(t,\rho,y)$$

$$+\omega_2(t, \frac{x-\rho}{x-\rho+1}) \ge g(t,\rho,0) - |g(t,\rho,y) - g(t,\rho,0)| + \omega_2(t,|y|) > 0,$$

and we proved that $\tilde{\sigma}_2$ is a strict upper function of (5.4), (1.2). Similarly we can get that $\tilde{\sigma}_1$ is a strict lower function of (5.4), (1.2).

Equation (5.4) can be written in the form

$$x'' + a(t)x' + b(t)x = \tilde{f}(t,x,x'),$$

where $\tilde{f}(t,x,y) = h(t,x,y) + a(t)y + b(t)x$. Put $p(t) = d(t) + |b(t)|\eta + \omega_2(\frac{\eta}{\eta+1})$ a.e. on J. Then, by (4.19), for a.e. $t \in J$ and for all $(x,y) \in [\tilde{\sigma}_1, \tilde{\sigma}_2] \times \mathbb{R}$ the inequality $\tilde{f}(t,x,y) \le (1+|y|)(q(t)|y| + p(t))$ is satisfied. Therefore any solution x of problem (5.4), (1.2) which fulfils $||x||_{\mathrm{C}} \le \rho + \eta$, satisfies condition (4.16). So, if we put $r = \rho + \eta$, we can use Lemma 4.4 and get r^* such that estimate (4.17) is valid. According to this r^* we define sets

$$D = \{x \in \mathrm{C}^1(J) : ||x||_{\mathrm{C}} < \rho + \eta,\ ||x'||_{\mathrm{C}} < r^*\},$$

$$D_1 = \{x \in D : \sigma_1 < x \text{ on } J\},\ D_2 = \{x \in D : x < \sigma_2 \text{ on } J\},$$

and

$$D_3 = \{x \in D : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x) \text{ for a } t_x \in J\}.$$

Choose $\mu \in (-\infty, 0)$ and define an operator

$$\tilde{H}_\mu : \mathrm{C}^1(J) \to \mathrm{L}(J),\ x \mapsto \tilde{f}(\cdot, x(\cdot), x'(\cdot)) + (\mu - b(\cdot))x.$$

Then Theorem 4.5 guarantees that

$$deg(I - L_\mu^+ \tilde{H}_\mu, D_1) = 1,\ deg(I - L_\mu^+ \tilde{H}_\mu, D_2) = 1, \tag{5.6}$$

and

$$deg(I - L_\mu^+ \tilde{H}_\mu, D) = 1.$$

(For L_μ^+ see (2.14).) Now, we use the aditivity of the degree. Since $D_3 = D - cl(D_1 \cup D_2)$, where $D_1, D_2 \subset D$ are disjoint sets, we have

$$deg(I - L_\mu^+ \tilde{H}_\mu, D) = deg(I - L_\mu^+ \tilde{H}_\mu, D_1) + deg(I - L_\mu^+ \tilde{H}_\mu, D_2) + deg(I - L_\mu^+ \tilde{H}_\mu, D_3).$$

Therefore

$$deg(I - L_\mu^+ \tilde{H}_\mu, D_3) = -1. \tag{5.7}$$

Conditions (5.6) and (5.7) imply that problem (5.4), (1.2) has solutions $x_i \in D_i$, $i = 1,2,3$. Since D_1, D_2 and D_3 are disjoint, solutions x_1, x_2 and x_3 are different.

It remains to prove that any solution x of (5.4), (1.2) satisfies

$$||x||_{\mathrm{C}} \le \rho. \tag{5.8}$$

Suppose that x is an arbitrary solution of (5.4), (1.2) and that $\max_{t \in J} x(t) = x(t_0) > \rho$. Without loss of generality we can suppose that there is an interval $[t_0, \tau] \subset [0, 2\pi)$ such that

$$x'(t_0) = 0,\ x(t) > \rho \text{ and } |x'(t)| < \frac{x(t) - \rho}{x(t) - \rho + 1} \text{ for all } t \in [t_0, \tau].$$

Then for a.e. $t \in [t_0, \tau]$

$$x'' = h(t, x, x') = g(t, \rho, x')$$

$$+\omega_2(t, \frac{x(t) - \rho}{x(t) - \rho + 1}) > g(t, \rho, 0) - |g(t, \rho, x') - g(t, \rho, 0)| + \omega_2(t, |x'|) > 0,$$

which implies that $x'(t) > 0$ for all $t \in (t_0, \tau]$. But this contradicts the fact that $x(t_0)$ is the maximum value on J. The estimate $x \geq -\rho$ on J can be proved analogously. Thus the solutions x_1, x_2 and x_3 satisfy estimate (5.8) and so they are solutions of problem (1.1), (1.2), as well. This completes the proof. □

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