Homoclinic solutions of non-autonomous difference equations arising in hydrodynamics

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Abstract. The paper deals with the second-order non-autonomous difference equation

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \Big(x(n) - x(n-1) + h^2 f(x(n))\Big), \quad n \in \mathbb{N},$$

where $h > 0$ is a parameter and f is Lipschitz continuous and has three real zeros $L_0 < 0 < L$.

We provide conditions for f under which for each sufficiently small $h > 0$ there exists a homoclinic solution of the above equation. The homoclinic solution is a sequence $\{x(n)\}_{n=0}^{\infty}$ satisfying the equation and such that $\{x(n)\}_{n=1}^{\infty}$ is increasing, $x(0) = x(1) \in (L_0, 0)$ and $\lim_{n\to\infty} x(n) = L$. The problem is motivated by some models arising in hydrodynamics.

Keywords. Non-autonomous second-order difference equation, homoclinic solutions, strictly increasing solutions.

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1 Introduction

In hydrodynamics or in the nonlinear field theory we can find differential models which can be reduced, after some substitution, to the form

$$(t^2 u')' = 4\lambda^2 t^2 (u+1) u (u - \xi), \tag{1.1}$$

$$u'(0) = 0, \quad u(\infty) = \xi, \tag{1.2}$$

where $\lambda \in (0, \infty)$ and $\xi \in (0, 1)$ are parameters. See e.g. [5], [6], [8], [10], [11].

Consider the following generalization of equation (1.1)

$$(t^2 u')' = t^2 f(u) \tag{1.3}$$

and construct a discretization of problem (1.3), (1.2). Choose $h > 0$ and a sequence $\{t_n\}_{n=0}^{\infty} \subset [0, \infty)$ such that

$$t_0 = 0, \quad t_{n+1} - t_n = h, \ n \in \mathbb{N}, \quad \lim_{n \to \infty} t_n = \infty. \tag{1.4}$$

Denote $x(0) = u(0)$ and $x(n) = u(t_n)$ for $n \in \mathbb{N}$. Then the discrete analogy of problem (1.3), (1.2) has the form of the following difference problem

$$\frac{1}{h^2} \Delta(t_n^2 \Delta x(n-1)) = t_n^2 f(x(n)), \ n \in \mathbb{N}, \tag{1.5}$$

$$\Delta x(0) = 0, \quad \lim_{n \to \infty} x(n) = \xi. \tag{1.6}$$

Here $\Delta x(n-1) = x(n) - x(n-1)$ is the forward difference operator and $t_n = hn$, $n \in \mathbb{N}$.

2 Formulation of problem

Equation (1.5) has an equivalent form

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \Big(x(n) - x(n-1) + h^2 f(x(n))\Big), \quad n \in \mathbb{N}. \tag{2.1}$$

We will investigate equation (2.1) under the assumption that f fulfils

$$L_0 < 0 < L, \quad f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad f(L_0) = f(0) = f(L) = 0, \tag{2.2}$$

$$x f(x) < 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \tag{2.3}$$

$$\exists \bar{B} \in (L_0, 0) \text{ such that } \int_{\bar{B}}^{L} f(z) \, \mathrm{d}z = 0. \tag{2.4}$$

Let us note that $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ means that for each $[A_0, A] \subset \mathbb{R}$ there exists $K > 0$ such that $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in [A_0, A]$. We see that the function $f(x) = 4\lambda^2(x+1)x(x-\xi)$ of equation (1.1) with $\lambda \in (0, \infty)$ and $\xi \in (0, 1)$ satisfies conditions (2.2)–(2.4) for $L_0 = -1$ and $L = \xi$.

A sequence $\{x(n)\}_{n=0}^{\infty}$ which satisfies (2.1) is called a solution of equation (2.1). For each values $B, B_1 \in [L_0, \infty)$ there exists a unique solution $\{x(n)\}_{n=0}^{\infty}$ of equation (2.1) satisfying the initial conditions

$$x(0) = B, \quad x(1) = B_1. \tag{2.5}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called a solution of problem (2.1), (2.5).

Strictly increasing solutions with just one zero play a fundamental role in the differential models (1.1), (1.2). According to this we search for solutions $\{x(n)\}_{n=0}^{\infty}$ of equation (2.1) satisfying

$$x(0) = x(1), \quad \lim_{n \to \infty} x(n) = L, \quad \{x(n)\}_{n=1}^{\infty} \text{ is increasing}. \tag{2.6}$$

To this aim (see Lemma 3.1) we will study solutions of problem (2.1), (2.7), where

$$x(0) = B, \quad x(1) = B, \quad B \in (L_0, 0). \tag{2.7}$$

Using our results of [17] and [18], we will prove that for each sufficiently small $h > 0$ there exists at least one $B \in (L_0, 0)$ such that the corresponding solution of problem (2.1), (2.7) fulfils (2.6). Note that an autonomous case of (2.1) was studied in [16]. We mention also some recent papers investigating the solvability of other second-order discrete boundary value problems, for example [1], [2], [9], [13]–[15], [20].

3 Four types of solutions

Lemma 3.1 shows that it suffices to consider $B \in (L_0, 0)$ in order to find a solution fulfilling (2.6).

Lemma 3.1 *Let $B \in [L_0, L]$ and $\{x(n)\}_{n=0}^{\infty}$ be the corresponding solution of equation* (2.1) *satisfying $x(0) = x(1) = B$. If $B \notin (L_0, 0)$, then $\{x(n)\}_{n=1}^{n_0}$ is not increasing for any $n_0 \in \mathbb{N}$, $n_0 > 1$.*

Proof. Due to (2.1), $\{x(n)\}_{n=0}^{\infty}$ fulfils

$$\Delta x(n) = \left(\frac{n}{n+1}\right)^2 \Big(\Delta x(n-1) + h^2 f(x(n))\Big), \quad n \in \mathbb{N}. \tag{3.1}$$

(i) Let $B \in (0, L)$. By (2.3) and (2.7) we have $f(x(1)) = f(B) < 0$, and (3.1) yields $\Delta x(1) < 0$. Hence $x(1) > x(2)$ and $\{x(n)\}_{n=1}^{n_0}$ is not increasing for any $n_0 > 1$.

(ii) Let $B \in \{L_0, 0, L\}$. Then (2.1) and (2.2) imply that $\{x(n)\}_{n=0}^{\infty}$ is the constant sequence with $x(n) = B$, $n \in \mathbb{N}$. Hence $\{x(n)\}_{n=1}^{n_0}$ is not increasing for any $n_0 > 1$. □

Definition 3.2 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (2.1), (2.7) such that

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n\to\infty} x(n) = 0. \tag{3.2}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a damped solution.*

Remark 3.3 The differential equation (1.3) for $t \in (0, \infty)$ corresponds to the difference equation (2.1). If we consider equation (1.3) for $t \in (-\infty, 0)$, then its discrete analogy can have the form (compare with (1.5))

$$\frac{1}{h^2}\Delta(t_{-n-1}^2 \Delta x(-n-1)) = t_{-n}^2 f(x(-n)), \ n \in \mathbb{N}, \tag{3.3}$$

where $\Delta x(-n-1) = x(-n-1) - x(-n)$, $t_{-n} = -hn$, $n \in \mathbb{N}$. Then (3.3) has an equivalent form

$$x(-n-1) = x(-n) + \left(\frac{n}{n+1}\right)^2 \Big(x(-n) - x(-n+1) \\ +h^2 f(x(-n))\Big), \quad n \in \mathbb{N}. \tag{3.4}$$

Assume that $B^* \in (L_0, 0)$ is such that the solution $\{x^*(n)\}_{n=0}^{\infty}$ of problem (2.1), (2.7) with $B = B^*$ satisfies $\lim_{n\to\infty} x^*(n) = L$. Now, consider the sequence $\{x^*(-n)\}_{n=0}^{\infty}$ which fulfils (3.4) and $x^*(-1) = x^*(0) = B^*$. Comparing (2.1) and (3.4) we see that $x^*(n) = x^*(-n)$ for $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} x^*(-n) = \lim_{n\to\infty} x^*(n) = L. \tag{3.5}$$

Motivated by (3.5) we will use the following definition.

Definition 3.4 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (2.1), (2.7) which fulfils

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n\to\infty} x(n) = L. \tag{3.6}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a homoclinic solution.*

Lemma 3.7 needs next two definitions.

Definition 3.5 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (2.1), (2.7). Assume that there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and

$$x(b) \le L < x(b+1). \tag{3.7}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *an escape solution.*

Definition 3.6 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (2.1), (2.7). Assume that there exists $b \in \mathbb{N}$, $b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and

$$0 < x(b) < L, \quad x(b+1) \le x(b). \tag{3.8}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a non-monotonous solution.*

We present some results of [17] and [18] which we use in next sections.

Lemma 3.7 [17] (On four types of solutions)
Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (2.1), (2.7). Then $\{x(n)\}_{n=0}^{\infty}$ is just one of the following four types:

(I) $\{x(n)\}_{n=0}^{\infty}$ *is an escape solution;*

(II) $\{x(n)\}_{n=0}^{\infty}$ *is a homoclinic solution;*

(III) $\{x(n)\}_{n=0}^{\infty}$ *is a damped solution;*

(IV) $\{x(n)\}_{n=0}^{\infty}$ *is a non-monotonous solution.*

Lemma 3.8 [17] (On the existence of non-monotonous or damped solutions)
Let $B \in (\bar{B}, 0)$, where $\bar{B}$ is defined by (2.4). *There exists $h_B > 0$ such that if $h \in (0, h_B]$, then the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem* (2.1), (2.7) *is non-monotonous or damped.*

Remark 3.9 Our main task is to prove the existence of $B \in (L_0, 0)$ such that $\{x(n)\}_{n=0}^{\infty}$ a homoclinic solution of problem (2.1), (2.7) with this B. Such solution fulfils $L_0 < B \le x(n) < L$ for $n \in \mathbb{N} \cup \{0\}$. Therefore we may assume without loss of generality that

$$f(x) = 0 \quad \text{for } x \in (-\infty, L_0) \cup (L, \infty). \tag{3.9}$$

By Remark 3.9, we assume that, in addition to (2.2)–(2.4), f fulfils moreover (3.9) in Lemma 3.10.

Lemma 3.10 [18] (On the existence of escape solutions)
There exists $h^* > 0$ such that for any $h \in (0, h^*]$ there exists an escape solution $\{x_\ell(n)\}_{n=0}^{\infty}$ of problem (2.1), (2.7) *for some $B = B_\ell \in (L_0, \bar{B})$.*

4 Estimates of solutions

In this section, f is supposed to fulfil (2.2)–(2.4) and (3.9).

Lemma 4.1 *Let $\{x(n)\}_{n=0}^{\infty}$ be an escape solution of problem* (2.1), (2.7). *Then $\{x(n)\}_{n=1}^{\infty}$ is increasing and*

$$\lim_{n\to\infty} x(n) \in (L, \infty). \tag{4.1}$$

Proof. According to Definition 3.5 there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and (3.7) holds. By (3.9) we get $f(x(b+1)) = 0$. Consequently, by (3.1) and (3.7), $\Delta x(b+1) = \left(\frac{b+1}{b+2}\right)^2 \Delta x(b) > 0$ and $f(x(b+2)) = 0$. Similarly $\Delta x(b+j) = \left(\frac{b+j}{b+1+j}\right)^2 \Delta x(b+j-1)$ and

$$\Delta x(b+j) = \left(\frac{b+1}{b+1+j}\right)^2 \Delta x(b), \quad j \in \mathbb{N}. \tag{4.2}$$

This yields that $\{x(n)\}_{n=1}^{\infty}$ is increasing.

Summing (4.2) for $j = 1, \dots, k$, we obtain

$$x(b+k+1) = x(b+1) + (b+1)^2 \Delta x(b) \sum_{j=1}^{k} \frac{1}{(b+1+j)^2}, \quad k \in \mathbb{N}.$$

Consequently

$$\lim_{n\to\infty} x(n) = x(b+1) + (b+1)^2 \Delta x(b) \sum_{j=1}^{\infty} \frac{1}{(b+1+j)^2}.$$

We have $\sum_{n=1}^{\infty} \frac{1}{(b+1+n)^2} < \infty$ and (4.1) follows. □

Lemma 4.2 [18] *Let* $\{x(n)\}_{n=0}^{\infty}$ *be a solution of problem* (2.1), (2.7). *Then there exists a maximal* $b \in \mathbb{N} \cup \{\infty\}$ *satisfying*

$$x(n) \in [B, L) \quad \text{for } n = 1, \dots, b, \tag{4.3}$$

and, if moreover $b > 1$, *then*

$$\{x(n)\}_{n=1}^{b} \quad \text{is increasing.} \tag{4.4}$$

In addition

$$\Delta x(n) < h\sqrt{(L-2L_0)M_0} + h^2 M_0, \quad n = 1, \dots, b-1, \tag{4.5}$$

where

$$M_0 = \max\{|f(x)|\colon x \in [L_0, L]\}. \tag{4.6}$$

Corollary 4.3 *Let* $h \in (0,1)$. *If* $\{x(n)\}_{n=0}^{\infty}$ *is a damped solution of problem* (2.1), (2.7), *then*

$$\frac{\Delta x(n)}{h} < \sqrt{2|L_0|M_0}, \quad n \in \mathbb{N}. \tag{4.7}$$

If $\{x(n)\}_{n=0}^{\infty}$ *is an escape solution of problem* (2.1), (2.7), *then*

$$\frac{\Delta x(n)}{h} < \sqrt{(L-2L_0)M_0} + 2M_0, \quad n \in \mathbb{N}. \tag{4.8}$$

Proof. Equation (2.1) has an equivalent form

$$\Delta x(n) - \Delta x(n-1) + \frac{2n+1}{n^2}\Delta x(n) = h^2 f(x(n)), \quad n \in \mathbb{N}. \tag{4.9}$$

Multiplying (4.9) by $\Delta x(n) + \Delta x(n-1)$, we obtain

$$\begin{aligned} (\Delta x(n))^2 - (\Delta x(n-1))^2 + \frac{2n+1}{n^2}\Delta x(n)(\Delta x(n) + \Delta x(n-1)) \\ = h^2 f(x(n))(x(n+1) - x(n-1)), \quad n \in \mathbb{N}. \end{aligned} \tag{4.10}$$

Summing (4.10) from 1 to $n \in \mathbb{N}$, we have

$$\begin{aligned} (\Delta x(n))^2 + \sum_{j=1}^{n} \frac{2j+1}{j^2}\Delta x(j)(\Delta x(j) + \Delta x(j-1)) \\ = h^2 \sum_{j=1}^{n} f(x(j))(x(j+1) - x(j-1)), \quad n \in \mathbb{N}. \end{aligned} \tag{4.11}$$

If $\{x(n)\}_{n=0}^{\infty}$ is a damped solution of problem (2.1), (2.7), then by (3.2) and (4.6) we get

$$\Delta x(n) < h\sqrt{2|B|M_0} < h\sqrt{2|L_0|M_0}, \quad n \in \mathbb{N}. \tag{4.12}$$

Let $\{x(n)\}_{n=0}^{\infty}$ be an escape solution. By Definition 3.5, $\{x(n)\}_{n=1}^{\infty}$ is increasing and there exists $b \in \mathbb{N}$ such that $x(b) \le L < x(b+1)$. By (4.5) we have

$$\Delta x(b-1) < h\sqrt{(L-2L_0)M_0} + h^2 M_0, \tag{4.13}$$

and, by (3.1) and (4.6),

$$\Delta x(b) = \left(\frac{b}{b+1}\right)^2 \left(\Delta x(b-1) + h^2 f(x(b))\right) < \Delta x(b-1) + h^2 M_0. \tag{4.14}$$

Further, $x(n) > L$ for $n \ge b+1$ and hence, due to (3.9), $f(x(n)) = 0$. Therefore

$$\Delta x(n) = \left(\frac{n-1}{n}\right)^2 \Delta x(n-1) < \Delta x(n-1), \quad n \ge b+1. \tag{4.15}$$

Consequently (4.13)–(4.15) give (4.8). □

Lemma 4.4 [18] *Choose an arbitrary $\varrho > 0$. Let $B_1, B_2 \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ and $\{y(n)\}_{n=0}^{\infty}$ be solutions of problem (2.1), (2.7) with $B = B_1$ and $B = B_2$, respectively. Let K by the Lipschitz constant for f on $[L_0, L]$. Then*

$$|x(n) - y(n)| \le |B_1 - B_2| e^{\varrho^2 K}, \tag{4.16}$$

$$\left|\frac{\Delta x(n) - \Delta y(n)}{h}\right| \le |B_1 - B_2| \varrho K\, e^{\varrho^2 K}, \tag{4.17}$$

where $n \in \mathbb{N}$, $n \le \frac{\varrho}{h}$.

Corollary 4.5 *Let the assumptions of Lemma 4.4 be fulfilled and let $b_0 \in \mathbb{N}$, $b_0 > 1$, $h \in (0,1)$. Then for $n \in \mathbb{N}$, $n \le b_0$, the following inequalities hold:*

$$|x(n) - y(n)| \le |B_1 - B_2| e^{b_0^2 K}, \tag{4.18}$$

$$\left|\frac{\Delta x(n) - \Delta y(n)}{h}\right| \le |B_1 - B_2| b_0 K\, e^{b_0^2 K}, \tag{4.19}$$

$$\begin{gathered}\left|\frac{\Delta x(n)}{h} \cdot \frac{\Delta x(n) + \Delta x(n-1)}{2h} - \frac{\Delta y(n)}{h} \cdot \frac{\Delta y(n) + \Delta y(n-1)}{2h}\right| \\ \le |B_1 - B_2| \Lambda,\end{gathered} \tag{4.20}$$

where

$$\Lambda = 2\left(\sqrt{(L-2L_0)M_0} + M_0\right) b_0 K\, e^{b_0^2 K}. \tag{4.21}$$

Proof. Inequalities (4.18) and (4.19) follow directly from (4.16) and (4.17). Inequality (4.20) is based on (4.7), (4.8), (4.19) and on the inequality

$$\left|\frac{\Delta x(n)}{h}\cdot\frac{\Delta x(n)+\Delta x(n-1)}{2h}-\frac{\Delta y(n)}{h}\cdot\frac{\Delta y(n)+\Delta y(n-1)}{2h}\right|$$

$$\leq\left|\frac{\Delta x(n)-\Delta y(n)}{h}\right|\cdot\left|\frac{\Delta y(n)+\Delta y(n-1)}{2h}\right|$$

$$+\left|\frac{\Delta x(n)}{h}\right|\cdot\left|\frac{\Delta x(n)-\Delta y(n)}{2h}\right|+\left|\frac{\Delta x(n)}{h}\right|\cdot\left|\frac{\Delta x(n-1)-\Delta y(n-1)}{2h}\right|.$$

□

5 Further properties of solutions

In order to prove the existence of a homoclinic solution we will need the following lemmas. Here f fulfils (2.2)–(2.4) and (3.9).

Lemma 5.1 *Let $\{x_\sharp(n)\}_{n=0}^\infty$ be a non-monotonous (an escape) solution of problem* (2.1), (2.7) *with $B=B_\sharp\in(L_0,0)$. Then there exists $\varepsilon>0$ such that for each $B\in(B_\sharp-\varepsilon,B_\sharp+\varepsilon)$ the corresponding solution $\{x(n)\}_{n=0}^\infty$ of problem* (2.1), (2.7) *is also a non-monotonous (an escape) solution.*

Proof. Let K be the Lipschitz constant for f on $[L_0,L]$ and let $\{x(n)\}_{n=0}^\infty$ be a solution of problem (2.1), (2.7) with $B\neq B_\sharp$. For $b\in\mathbb{N}$ put $\varrho=h(b+2)$. According to Lemma 4.4,

$$|x_\sharp(n)-x(n)|\leq|B_\sharp-B|e^{\varrho^2K},\quad n\leq b+2. \tag{5.1}$$

(i) Assume that $\{x_\sharp(n)\}_{n=0}^\infty$ is a non-monotonous solution. By Definition 3.6 there exists $b\in\mathbb{N}$, $b>1$, such that $\{x_\sharp(n)\}_{n=1}^b$ is increasing and

$$0<x_\sharp(b)<L,\quad x_\sharp(b+1)\leq x_\sharp(b).$$

We can find $\delta_1,\delta_2>0$ such that

$$0<x_\sharp(b)-\delta_1,\quad x_\sharp(b)+\delta_1<L, \tag{5.2}$$

and for $n\leq b-1$

$$\delta_2<\frac{1}{2}(x_\sharp(n+1)-x_\sharp(n)). \tag{5.3}$$

Let $x_\sharp(b+1)=x_\sharp(b)$. Then $x_\sharp(b+2)<x_\sharp(b+1)$ because, by (3.1),

$$\Delta x_\sharp(b+1)=\left(\frac{b+1}{b+2}\right)^2\Big(\Delta x_\sharp(b)+h^2f(x_\sharp(b+1))\Big)<0.$$

We choose $\delta_3 > 0$ such that

$$\delta_3 < \frac{1}{2}(x_\sharp(b+1) - x_\sharp(b+2)). \tag{5.4}$$

Let $x_\sharp(b+1) < x_\sharp(b)$. Then we choose $\delta_3 > 0$ such that

$$\delta_3 < \frac{1}{2}(x_\sharp(b) - x_\sharp(b+1)). \tag{5.5}$$

Now, for $x_\sharp(b+1) \le x_\sharp(b)$, put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, $\varepsilon = \mathrm{e}^{-\varrho^2 K}\delta$ and assume that $|B_\sharp - B| < \varepsilon$. Then, by (5.1), we get

$$|x_\sharp(n) - x(n)| \le \delta, \quad n \le b+2. \tag{5.6}$$

Therefore, by (5.2), $0 < x_\sharp(b) - \delta \le x(b)$ and $x(b) \le x_\sharp(b) + \delta < L$. So $0 < x(b) < L$. Further, by (5.3) and (5.6), for $n \le b-1$,

$$x(n) \le x_\sharp(n) + \delta < x_\sharp(n+1) - \delta \le x(n+1).$$

Therefore $\{x(n)\}_{n=1}^{b}$ is increasing.

Let $x_\sharp(b+1) = x_\sharp(b)$. If $x(b+1) \le x(b)$, we see that $\{x(n)\}_{n=0}^{\infty}$ is non-monotonous. So assume that $x(b+1) > x(b)$. Then $\{x(n)\}_{n=1}^{b+1}$ is increasing. Further, by (5.4) and (5.6),

$$x(b+2) \le x_\sharp(b+2) + \delta < x_\sharp(b+1) - \delta \le x(b+1).$$

Hence $x(b+2) < x(b+1)$ which yields that $\{x(n)\}_{n=0}^{\infty}$ is non-monotonous in this case, as well.

If $x_\sharp(b+1) < x_\sharp(b)$, we deduce by (5.5) and (5.6) that $x(b+1) < x(b)$ and get that $\{x(n)\}_{n=0}^{\infty}$ is non-monotonous.

(ii) Assume that $\{x_\sharp(n)\}_{n=0}^{\infty}$ is an escape solution. By Definition 3.5 there exists $b \in \mathbb{N}$ such that $\{x_\sharp(n)\}_{n=1}^{b+1}$ is increasing and $L < x_\sharp(b+1)$. Then we can find $\delta_1, \delta_2 > 0$ such that

$$L < x_\sharp(b+1) - \delta_1, \tag{5.7}$$

and inequality (5.3) holds for $n \le b$. Put $\delta = \min\{\delta_1, \delta_2\}$, $\varepsilon = \mathrm{e}^{-\varrho^2 K}\delta$ and assume that $|B_\sharp - B| < \varepsilon$. Then, (5.6) holds and using (5.7) and (5.3) we deduce as in part (i) that $\{x(n)\}_{n=1}^{b+1}$ is increasing and $L < x(b+1)$. Consequently, $\{x(n)\}_{n=0}^{\infty}$ is an escape solution. □

Lemma 5.2 *There exists $h^* > 0$ such that if $h \in (0, h^*]$, $B_0 \in (L_0, 0)$ and $\{x_0(n)\}_{n=0}^{\infty}$ is a damped solution of problem* (2.1), (2.7) *with $B = B_0$, then there exists $\delta_{B_0} > 0$ such that for each $B \ne B_0$, $B \in (B_0 - \delta_{B_0}, B_0 + \delta_{B_0}) \cap (L_0, 0)$, the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem* (2.1), (2.7) *cannot be an escape solution.*

Proof. By (2.2), f is integrable on $[L_0, L]$ and we can choose c_0, ε and η^* such that

$$0 < c_0 < \frac{1}{3}\left|\int_0^L f(z)\,\mathrm{d}z\right|, \quad 0 < \varepsilon < \frac{c_0}{3}, \tag{5.8}$$

$$|B - B_0| < 2\eta^* \Longrightarrow \left|\int_B^{B_0} f(z)\,\mathrm{d}z\right| < \varepsilon, \quad B, B_0 \in [L_0, 0]. \tag{5.9}$$

Step 1. By (2.2) and (3.9), for each $B \in [L_0, 0]$ there exists $\delta_B > 0$ such that each increasing sequence $\{x(j)\}_{j=1}^{n+1}$, $n \in \mathbb{N}$, fulfils the following implication: If

$$\begin{gathered} x(1) \in (B - \delta_B, B + \delta_B), \quad x(0) = x(1), \quad -\delta_B < x(n+1) < 0, \\ \tfrac{x(j+1) - x(j-1)}{2} < \delta_B, \quad j = 1, \ldots, n, \end{gathered} \tag{5.10}$$

then

$$\left|\sum_{j=1}^n f(x(j))\frac{x(j+1) - x(j-1)}{2} - \int_{x(1)}^0 f(z)\,\mathrm{d}z\right| < \varepsilon. \tag{5.11}$$

Let $\mathcal{M} = \bigcup_{B\in[L_0,0]}(B-\delta_B, B+\delta_B)$. Then $[L_0, 0] \subset \mathcal{M}$ and since $[L_0, 0]$ is compact, we can choose a finite number ν of intervals $(B_k - \delta_{B_k}, B_k + \delta_{B_k})$ such that

$$[L_0, 0] \subset \bigcup_{k=1}^{\nu} (B_k - \delta_{B_k}, B_k + \delta_{B_k}). \tag{5.12}$$

Consider M_0 of (4.6) and choose $h_k > 0$ such that

$$h_k\sqrt{2|L_0|M_0} < \delta_{B_k}, \quad k = 1, \ldots, \nu. \tag{5.13}$$

Step 2. Consider η^* of (5.9). By (2.2) and (3.9), for each $B \in [L_0, 0]$ there exists $\eta_B \in (0, \eta^*)$ such that each increasing sequence $\{x(j)\}_{j=1}^{n+1}$, $n \in \mathbb{N}$, fulfils the following implication: If

$$\begin{gathered} x(1) \in (B - \eta_B, B + \eta_B), \quad x(0) = x(1), \quad L < x(n+1), \\ \tfrac{x(j+1) - x(j-1)}{2} < \eta_B, \quad j = 1, \ldots, n, \end{gathered} \tag{5.14}$$

then

$$\left|\sum_{j=1}^n f(x(j))\frac{x(j+1) - x(j-1)}{2} - \int_{x(1)}^L f(z)\,\mathrm{d}z\right| < \varepsilon. \tag{5.15}$$

As in Step 1 we deduce that there is a finite number μ of intervals $(B_\ell - \eta_{B_\ell}, B_\ell + \eta_{B_\ell})$ such that

$$[L_0, 0] \subset \bigcup_{\ell=1}^{\mu} (B_\ell - \eta_{B_\ell}, B_\ell + \eta_{B_\ell}), \tag{5.16}$$

and we choose $\tilde{h}_\ell > 0$ such that

$$\tilde{h}_\ell \left(\sqrt{(L - 2L_0)M_0} + 2M_0 \right) < \eta_{B_\ell}, \quad \ell = 1, \dots, \mu. \tag{5.17}$$

In what follows we assume that

$$h \in (0, h^*], \quad h^* = \min\{1, h_1, \dots, h_\nu, \tilde{h}_1, \dots, \tilde{h}_\mu\}. \tag{5.18}$$

Step 3. Let $B_0 \in (L_0, 0)$ be such that $\{x_0(n)\}_{n=0}^\infty$ is a damped solution of problem (2.1), (2.7) with $B = B_0$. By (5.12), $B_0 \in (B_k - \delta_{B_k}, B_k + \delta_{B_k})$ for some $k \in \{1, \dots, \nu\}$. Therefore, by (4.7), (5.13) and (5.18), $\{x_0(j)\}_{j=1}^{n+1}$, $n \in \mathbb{N}$, satisfies (5.10) for B_k in place of B, and consequently

$$\left| \sum_{j=1}^n f(x_0(j)) \frac{x_0(j+1) - x_0(j-1)}{2} - \int_{B_0}^0 f(z)\,\mathrm{d}z \right| < \varepsilon.$$

Letting $n \to \infty$ we get

$$\left| \sum_{j=1}^\infty f(x_0(j)) \frac{x_0(j+1) - x_0(j-1)}{2} - \int_{B_0}^0 f(z)\,\mathrm{d}z \right| \le \varepsilon. \tag{5.19}$$

Further, $\{x_0(n)\}_{n=0}^\infty$ satisfies (4.11) and hence

$$\begin{aligned} \frac{1}{2} \left(\frac{\Delta x_0(n)}{h} \right)^2 &+ \sum_{j=1}^n \frac{2j+1}{j^2} \cdot \frac{\Delta x_0(j)}{h} \cdot \frac{\Delta x_0(j) + \Delta x_0(j-1)}{2h} \\ &= \sum_{j=1}^n f(x_0(j)) \frac{x_0(j+1) - x_0(j-1)}{2}, \quad n \in \mathbb{N}. \end{aligned}$$

Letting $n \to \infty$ and having in mind that $\lim_{n\to\infty} \Delta x_0(n) = 0$, we get

$$\begin{aligned} \sum_{j=1}^\infty \frac{2j+1}{j^2} &\cdot \frac{\Delta x_0(j)}{h} \cdot \frac{\Delta x_0(j) + \Delta x_0(j-1)}{2h} \\ &= \sum_{j=1}^\infty f(x_0(j)) \frac{x_0(j+1) - x_0(j-1)}{2}. \end{aligned}$$

This together with (5.19) give

$$\left| \sum_{j=1}^\infty \frac{2j+1}{j^2} \cdot \frac{\Delta x_0(j)}{h} \cdot \frac{\Delta x_0(j) + \Delta x_0(j-1)}{2h} - \int_{B_0}^0 f(z)\,\mathrm{d}z \right| \le \varepsilon. \tag{5.20}$$

Consequently, there exists $b_0 \in \mathbb{N}$ such that

$$\sum_{j=b_0+1}^\infty \frac{2j+1}{j^2} \cdot \frac{\Delta x_0(j)}{h} \cdot \frac{\Delta x_0(j) + \Delta x_0(j-1)}{2h} < c_0. \tag{5.21}$$

Define Λ by (4.21). By virtue of (5.16), we have $B_0 \in (B_\ell - \eta_{B_\ell}, B_\ell + \eta_{B_\ell})$ for some $\ell \in \{1, \dots, \mu\}$. Therefore there exists $\delta_{B_0} \in (0, \eta_{B_\ell})$ such that $(B_0 - \delta_{B_0}, B_0 + \delta_{B_0}) \subset (B_\ell - \eta_{B_\ell}, B_\ell + \eta_{B_\ell})$ and

$$\delta_{B_0}\Lambda < c_0. \tag{5.22}$$

Step 4. Assume on the contrary that for some $B \in (B_0 - \delta_{B_0}, B_0 + \delta_{B_0}) \cap (L_0, 0)$, $B \neq B_0$, a sequence $\{x(n)\}_{n=0}^\infty$ is an escape solution of problem (2.1), (2.7). Then $\{x(n)\}_{n=1}^\infty$ is increasing and there exists $b \in \mathbb{N}$ such that $x(b) \le L < x(b+1)$. By (4.8), (5.17) and (5.18), we get that $\{x(j)\}_{j=1}^{n+1}$, $n \ge b$, satisfies (5.14) for B_ℓ in place of B, and consequently, inequality (5.15) holds for $n \in \mathbb{N}$, $n \ge b$.

Let $n \ge \max\{b_0, b\}$. Using successively (5.15), (4.11), (4.20), (5.21), (5.22), (5.20) and (5.9), we get

$$\varepsilon + \int_B^L f(z)\,\mathrm{d}z > \sum_{j=1}^n f(x(j))\frac{x(j+1) - x(j-1)}{2} =$$

$$\frac{1}{2}\left(\frac{\Delta x(n)}{h}\right)^2 + \sum_{j=1}^n \frac{2j+1}{j^2} \cdot \frac{\Delta x(j)}{h} \cdot \frac{\Delta x(j) + \Delta x(j-1)}{2h} >$$

$$\sum_{j=1}^{b_0} \frac{2j+1}{j^2} \cdot \frac{\Delta x(j)}{h} \cdot \frac{\Delta x(j) + \Delta x(j-1)}{2h} \ge$$

$$\sum_{j=1}^{b_0} \frac{2j+1}{j^2} \cdot \frac{\Delta x_0(j)}{h} \cdot \frac{\Delta x_0(j) + \Delta x_0(j-1)}{2h} - |B - B_0|\Lambda =$$

$$\sum_{j=1}^{\infty} \frac{2j+1}{j^2} \cdot \frac{\Delta x_0(j)}{h} \cdot \frac{\Delta x_0(j) + \Delta x_0(j-1)}{2h} -$$

$$\sum_{j=b_0+1}^{\infty} \frac{2j+1}{j^2} \cdot \frac{\Delta x_0(j)}{h} \cdot \frac{\Delta x_0(j) + \Delta x_0(j-1)}{2h} - |B - B_0|\Lambda \ge$$

$$\sum_{j=1}^{\infty} \frac{2j+1}{j^2} \cdot \frac{\Delta x_0(j)}{h} \cdot \frac{\Delta x_0(j) + \Delta x_0(j-1)}{2h} - 2c_0 \ge$$

$$\int_{B_0}^0 f(z)\,\mathrm{d}z - \varepsilon - 2c_0 > \int_B^0 f(z)\,\mathrm{d}z - 2\varepsilon - 2c_0.$$

Hence,

$$\int_B^L f(z)\,\mathrm{d}z > \int_B^0 f(z)\,\mathrm{d}z - 3\varepsilon - 2c_0,$$

and using (2.3) and (5.8) we get

$$3c_0 > -\int_0^L f(z)\,\mathrm{d}z = \left|\int_0^L f(z)\,\mathrm{d}z\right| > 3c_0,$$

a contradiction. □

6 Existence of homoclinic solutions

Now, we are ready to state and prove the main result provided f fulfils only our basic assumptions (2.2)–(2.4).

Theorem 6.1 (On the existence of homoclinic solutions)
There exists $h^* > 0$ such that for any $h \in (0, h^*]$ there exists a homoclinic solution $\{x^*(n)\}_{n=0}^{\infty}$ of problem (2.1), (2.7)*, that is $\{x^*(n)\}_{n=1}^{\infty}$ is increasing and $\lim_{n\to\infty} x^*(n) = L$.*

Proof. First, consider an equation

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \Big(x(n) - x(n-1) + h^2 f^*(x(n))\Big), \quad n \in \mathbb{N}, \qquad (6.1)$$

where

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in [L_0, L] \\ 0 & \text{if } x \notin [L_0, L] \end{cases}.$$

Hence f^* fulfils (2.2)–(2.4) and (3.9). Let us choose $h_1^* > 0$ such that the assertion of Lemma 5.2 is valid for problem (6.1), (2.7). By Lemma 3.8 and Lemma 3.10, we can find $h^* \in (0, h_1^*]$ such that if $h \in (0, h^*]$, than for some $B_{\text{es}} \in (L_0, \bar{B})$, the solution of (6.1), (2.7) with $B = B_{\text{es}}$ is an escape solution, and for some $B_{\text{nd}} \in (\bar{B}, 0)$, the solution of (6.1), (2.7) with $B = B_{\text{nd}}$ is non-monotonous or damped.

By Lemma 5.1, there exists $\varepsilon > 0$ such that for each $B \in (B_{\text{es}}, B_{\text{es}} + \varepsilon)$, the corresponding solution of (6.1), (2.7) is an escape solution. Let ε^* be the supremum of such epsilons and put $B^* := B_{\text{es}} + \varepsilon^*$. Then $L_0 < B^* \le B_{\text{nd}} < 0$. Denote $\{x^*(n)\}_{n=0}^{\infty}$ the solution of (6.1), (2.7) with $B = B*$.

(i) Let $\{x^*(n)\}_{n=0}^{\infty}$ be non-monotonous. Then, by Lemma 5.1, there is $\tilde{\varepsilon}_1 > 0$ such that for each $B \in (B^* - \tilde{\varepsilon}_1, B^*)$, the corresponding solution is also non-monotonous. This contradicts the definition of ε^*.

(ii) Let $\{x^*(n)\}_{n=0}^{\infty}$ be an escape solution. Then, by Lemma 5.1, there is $\tilde{\varepsilon}_2 > 0$ such that for each $B \in (B^*, B^* + \tilde{\varepsilon}_2)$, the corresponding solution is also escape. This contradicts the maximality of ε^*.

(iii) Let $\{x^*(n)\}_{n=0}^{\infty}$ be a damped solution. Then, by Lemma 5.2, there is $\tilde{\varepsilon}_3 > 0$ such that for each $B \in (B^* - \tilde{\varepsilon}_3, B^*)$, the corresponding solution cannot be an escape solution. This contradicts the definition of ε^*.

By Lemma 3.7, $\{x^*(n)\}_{n=0}^{\infty}$ must be a homoclinic solution. Since $L_0 < B^* \le x^*(n) < L$ for $n \in \mathbb{N}$, the homoclinic solution $\{x^*(n)\}_{n=0}^{\infty}$ of problem (6.1), (2.7) is also a solution of problem (2.1), (2.7). □

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