On a homoclinic point of an autonomous second-order difference equation

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Abstract. The paper deals with the second-order difference equation.

$$x(n+1) = 2x(n) - x(n-1) + h^2 f(x(n)), \quad n \in \mathbb{N},$$

where $h > 0$ is a parameter and f has continuous first derivative and three zeros on the real line. The main result is that for each sufficiently small h the above equation has a homoclinic point.

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1 Introduction

We consider the autonomous second-order difference equation

$$x(n+1) = 2x(n) - x(n-1) + h^2 f(x(n)), \quad n \in \mathbb{N}, \tag{1.1}$$

where $h \in (0, \infty)$ is a parameter. A sequence $\{x(n)\}_{n=0}^{\infty}$ which satisfies (1.1) is called *a solution* of equation (1.1). We assume that

$$L_0 < 0 < L, \quad f \in C^1[L_0, L], \quad f(L_0) = f(0) = f(L) = 0, \tag{1.2}$$

$$x f(x) < 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \quad f'(L_0) > 0, \; f'(0) < 0, \; f'(L) > 0, \tag{1.3}$$

$$\exists \bar{B} \in (L_0, 0) \text{ such that } \int_{\bar{B}}^{L} f(z)\,\mathrm{d}z = 0. \tag{1.4}$$

Equation (1.1) represents an autonomous discrete case of some models arising in hydrodynamics. See [7], [9], [13], [16]. For monographs dealing with difference equations we refer to [1], [2], [3], [8], [12], [14]. We mention also some recent papers investigating the solvability of second-order discrete boundary value problems, for example [4]–[6], [10], [11], [15], [17]–[24].

The main result of our paper is the existence of a homoclinic point of equation (1.1). The results presented here can be also useful when analysing the discretization of corresponding boundary value problems for ordinary differential equations, in particular, by finite-difference methods. To elucidate the geometry of the dynamics of (1.1) it is convenient to convert it to an equivalent planar map. To this end we let $x_1^n = x(n-1)$, $x_2^n = x(n)$ and we obtain the equivalent first-order system of difference equations

$$\begin{aligned} x_1^{n+1} &= x_2^n \\ x_2^{n+1} &= 2x_2^n - x_1^n + h^2 f(x_2^n), \end{aligned}$$

which can be written as the iteration of the map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ 2x_2 - x_1 + h^2 f(x_2) \end{pmatrix}. \tag{1.5}$$

Let us choose $B \in (L_0, 0)$ and denote

$$\mathbf{x^0} = \begin{pmatrix} B \\ B \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{F}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 2x_2 - x_1 + h^2 f(x_2) \end{pmatrix}. \tag{1.6}$$

Then (1.5) has the form $\mathbf{x} \mapsto \mathbf{F}(\mathbf{x})$, and the positive orbit $\gamma^+(\mathbf{x^0})$ is the sequence

$$\gamma^+(\mathbf{x^0}) = \{\mathbf{x^0}, \mathbf{F}(\mathbf{x^0}), \ldots, \mathbf{F}^n(\mathbf{x^0}), \ldots\}.$$

The map $\mathbf{F}$ is invertible and

$$\mathbf{F}^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 + h^2 f(x_1) \\ x_1 \end{pmatrix}.$$

Hence the negative orbit $\gamma^-(\mathbf{x^0})$ is the sequence

$$\gamma^-(\mathbf{x^0}) = \{\mathbf{x^0}, \mathbf{F}^{-1}(\mathbf{x^0}), \ldots, \mathbf{F}^{-n}(\mathbf{x^0}), \ldots\},$$

and the orbit $\gamma(\mathbf{x^0}) = \gamma^+(\mathbf{x^0}) \cup \gamma^-(\mathbf{x^0})$ is uniquely determined for each $B \in (L_0, 0)$. Under the assumption that $h > 0$ is sufficiently small we prove that $(L, L)^T$ is a saddle point of $\mathbf{F}$ and that there exists $B^* \in (L_0, L)$ such that $(B^*, B^*)^T$ is a homoclinic point for $\mathbf{F}$, that is the orbit $\gamma(\mathbf{x}^*)$, when $\mathbf{x}^* = (B^*, B^*)^T$, satisfies

$$\lim_{n\to\infty} \mathbf{F}^n(\mathbf{x}^*) = \lim_{n\to\infty} \mathbf{F}^{-n}(\mathbf{x}^*) = \begin{pmatrix} L \\ L \end{pmatrix}. \tag{1.7}$$

2 Fixed points

Due to (1.2) the map $\mathbf{F}$ given by (1.6) has three fixed points $(L_0, L_0)^T$, $(0,0)^T$ and $(L, L)^T$ in the set $[L_0, L] \times [L_0, L]$. The Jacobian matrix of $\mathbf{F}$ has the form

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ -1 & 2 + h^2 f'(x_2) \end{pmatrix}.$$

The assumption (1.3) gives $\frac{1}{2}h^2 f'(L) =: \varepsilon > 0$, and hence

$$D\mathbf{F}\begin{pmatrix} L \\ L \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 + 2\varepsilon \end{pmatrix}$$

has the eigenvalues $\lambda_{1,2} = 1 + \varepsilon \pm \sqrt{\varepsilon^2 + 2\varepsilon}$. So, for a sufficiently small $h > 0$, one eigenvalue has modulus greater than 1 and the other less than 1. Therefore $(L, L)^T$ is an unstable hyperbolic fixed point—a saddle point. The same is true for $(L_0, L_0)^T$. On the other hand, (1.3) yields $\frac{1}{2}h^2 f'(0) =: -\delta < 0$, and hence

$$D\mathbf{F}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 - 2\delta \end{pmatrix}$$

has the eigenvalues $\lambda_{1,2} = 1 - \delta \pm i\sqrt{1 - (1-\delta)^2}$ with moduli equal to 1. Therefore $(0,0)^T$ is an elliptic fixed point which is a centre in the phase portrait of the approximate linear map

$$\mathbf{x} \mapsto D\mathbf{F}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\mathbf{x}.$$

The stability and type of the fixed point $(0,0)^T$ of the nonlinear map $\mathbf{x} \to \mathbf{F}(\mathbf{x})$ cannot be determined solely from linearization and the effects of the nonlinear terms in local dynamics must be accounted for.

3 Increasing solutions

For each values $A_0, A_1 \in [L_0, L]$ there exists a unique solution $\{x(n)\}_{n=0}^{\infty}$ of equation (1.1) satisfying the initial conditions

$$x(0) = A_0, \quad x(1) = A_1. \tag{3.1}$$

Such sequence $\{x(n)\}_{n=0}^{\infty}$ is called *a solution of problem* (1.1), (3.1). In order to find a point $\mathbf{x}^* = (B^*, B^*)^T$ satisfying (1.7) we choose $B \in (L_0, 0)$ and study solutions of problem (1.1), (3.2), where

$$x(0) = B, \quad x(1) = B. \tag{3.2}$$

Lemma 3.1 *Let* $B \in (L_0, 0)$ *and let* $\{x(n)\}_{n=0}^{\infty}$ *be a solution of problem* (1.1), (3.2). *Then there exists* $r \in \mathbb{N}$, $r > 1$, *such that*

$$x(1) < x(2) < \cdots < x(r-1) < 0 \le x(r) \quad \text{if } r > 2, \tag{3.3}$$

$$x(1) < 0 \le x(2) \quad \text{if } r = 2. \tag{3.4}$$

Proof. Choose $B \in (L_0, 0)$ and consider the solution $\{x(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2). Then $\{x(n)\}_{n=0}^{\infty}$ fulfils

$$\Delta x(n) = \Delta x(n-1) + h^2 f(x(n)), \quad n \in \mathbb{N}, \tag{3.5}$$

$$x(0) = B, \quad \Delta x(0) = 0, \tag{3.6}$$

where $\Delta x(n-1) = x(n) - x(n-1)$ is the forward difference operator. By (1.3) and (3.2), we have $f(x(0)) = f(x(1)) = f(B) > 0$, and (3.5) yields $\Delta x(1) > 0$. Hence $x(1) < x(2)$. If $x(2) \ge 0$ we get (3.4). Otherwise $x(1) < x(2) < 0$ and we repeat the above arguments to get $\Delta x(3) > \Delta x(2)$ and $x(2) < x(3)$. If $x(3) \ge 0$, we put $r = 3$ and get (3.3). Otherwise we continue as before and prove that after a finite number r of steps we get (3.3) and

$$\Delta x(1) < \Delta x(2) < \cdots < \Delta x(r-1). \tag{3.7}$$

Assume on the contrary that r is not finite, that is $x(n) < 0$ for each $n \in \mathbb{N}$. By (1.3), the inequality $f(x(n)) > 0$ holds for each $n \in \mathbb{N}$, and the sequence $\{\Delta x(n)\}_{n=1}^{\infty}$ is positive and increasing. Therefore

$$\lim_{n\to\infty} \Delta x(n) > 0. \tag{3.8}$$

The positivity of $\{\Delta x(n)\}_{n=1}^{\infty}$ implies that $\{x(n)\}_{n=1}^{\infty}$ is increasing. Since $\{x(n)\}_{n=1}^{\infty}$ is bounded above by 0, there exists a finite $\lim_{n\to\infty} x(n)$, contrary to (3.8). So, we have proved that (3.3) holds for some $r \in \mathbb{N}$. $\square$

Lemma 3.2 *Let* $B \in (L_0, 0)$ *and let* $\{x(n)\}_{n=0}^{\infty}$ *be a solution of problem* (1.1), (3.2). *If* $\{x(n)\}_{n=1}^{\infty}$ *is increasing and* $x(n) < L$ *for* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} x(n) = L, \quad \lim_{n\to\infty} \Delta x(n) = 0. \tag{3.9}$$

Proof. Since $\{x(n)\}_{n=1}^{\infty}$ is increasing and bounded above by L, there exists $\lim_{n\to\infty} x(n) = L_1 \le L$. Consequently $\lim_{n\to\infty} \Delta x(n) = 0$. By Lemma 3.1, we have $0 < x(r) < x(r+1)$ and $L_1 > 0$. If $L_1 < L$, then by virtue of (3.5), $\lim_{n\to\infty} \Delta x(n) = \lim_{n\to\infty} \Delta x(n-1) + h^2 \lim_{n\to\infty} f(x(n))$, and hence $0 = 0 + h^2 f(L_1) < 0$, a contradiction. Therefore $L_1 = L$ and (3.9) is proved. $\square$

Definition 3.3 A solution satisfying the conditions of Lemma 3.2 is called *a homoclinic solution.*

Remark 3.4 Our main task is to prove the existence of a homoclinic solution $\{x^*(n)\}_{n=0}^{\infty}$ of (1.1), (3.2) for some $B = B^* \in (L_0, 0)$. Since $L_0 < B \le x^*(n) < L$ for $n \in \mathbb{N} \cup \{0\}$, we may assume without loss of generality that

$$f(x) = 0 \quad \text{for } x \in (-\infty, L_0) \cup (L, \infty). \tag{3.10}$$

Note that if we have the above homoclinic solution and put $\mathbf{x}^* = (B^*, B^*)^T$, then the map $\mathbf{F}$ given by (1.6) satisfies (1.7), and hence the point $(B^*, B^*)^T$ is a homoclinic point for $\mathbf{F}$.

In what follows (Sec. 3–6) we assume that, in addition to (1.2)–(1.4), f fulfils moreover (3.10).

Lemma 3.5 *Let $B \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem* (1.1), (3.2). *Assume that there exists $b \in \mathbb{N}$, $b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and*

$$x(b) < L < x(b+1) \quad \textit{or} \quad x(b) = L. \tag{3.11}$$

Then $\{x(n)\}_{n=1}^{\infty}$ is increasing and

$$\lim_{n\to\infty} x(n) = \infty, \quad \lim_{n\to\infty} \Delta x(n) = \Delta x(b) > 0. \tag{3.12}$$

Proof. Choose $B \in (L_0, 0)$ and consider the solution $\{x(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2) which is increasing for $1 \le n \le b$. If the first condition in (3.11) holds, then $\Delta x(b) > 0$. Let $x(b) = L$. Then (3.5) yields $\Delta x(b) = \Delta x(b-1) + h^2 f(L) = \Delta x(b-1) > 0$. Therefore (3.11) gives $\Delta x(b) > 0$ in both cases. By (3.10) and (3.11), $f(x(b+1)) = 0$. Consequently, by (3.5), $\Delta x(b+1) = \Delta x(b) + h^2 f(x(b+1)) = \Delta x(b)$, and similarly $\Delta x(n) = \Delta x(b)$ for $n > b+1$. This gives $\lim_{n\to\infty} \Delta x(n) = \Delta x(b) > 0$. Therefore $\{x(n)\}_{n=1}^{\infty}$ is increasing and $\lim_{n\to\infty} x(n) = \infty$. □

Definition 3.6 A solution satisfying the conditions of Lemma 3.5 is called *an escape solution.*

Theorem 3.7 (On three types of solutions)
Let $B \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (3.2). *Then $\{x(n)\}_{n=0}^{\infty}$ is just one of the following three types:*

(I) *$\{x(n)\}_{n=0}^{\infty}$ is a homoclinic solution;*

(II) *$\{x(n)\}_{n=0}^{\infty}$ is an escape solution;*

(III) *there exists $b \in \mathbb{N}$, $b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and*

$$0 < x(b) < L, \quad x(b+1) \le x(b). \tag{3.13}$$

Proof. Choose $B \in (L_0, 0)$ and consider the solution $\{x(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2). By Lemma 3.1, there exists $r \in \mathbb{N}$, $r > 1$, such that $\{x(n)\}_{n=1}^{r}$ is increasing and $x(r) \geq 0$. Let $x(r) \geq L$. Then, due to (3.3), (3.4) and Lemma 3.5, $\{x(n)\}_{n=0}^{\infty}$ is an escape solution. Now, assume that $x(r) < L$ and that $\{x(n)\}_{n=0}^{\infty}$ is neither a homoclinic solution nor an escape solution. Then, by Lemma 3.2 and Lemma 3.5, the sequence $\{x(n)\}_{n=1}^{\infty}$ cannot be increasing and cannot fulfil (3.9) or (3.11). Therefore there exists $b \geq r$ such that $\{x(n)\}_{n=1}^{b}$ is increasing and $x(b+1) \leq x(b)$. Clearly $x(b) < L$. Otherwise (3.5) gets $x(b+1) > x(b)$, a contradiction. We have proved that $\{x(n)\}_{n=0}^{\infty}$ is a solution of the type (III). □

4 Estimates of solutions

Lemma 4.1 *Let $B \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem* (1.1), (3.2). *If $h > 0$ is sufficiently small, then there exist constants $r > 2$, $m \geq r$ and $L_1 \in (0, L)$ such that*

$$\begin{aligned} x(1) < x(2) < \cdots < x(r-1) < 0 \leq x(r) < \cdots < x(m) = L_1 \quad \text{if } m > r, \\ x(1) < x(2) < \cdots < x(r-1) < 0 < x(r) = L_1 \quad \text{if } m = r. \end{aligned} \tag{4.1}$$

Moreover

$$\Delta x(j) < h\sqrt{2|B|M_0} + h^2 M_0, \quad j = 1, \ldots, m-1, \tag{4.2}$$

where $M_0 = \max\{|f(x)|\colon x \in [L_0, L]\}$.

Proof. By Lemma 3.1 there exists $r \in \mathbb{N}$, $r > 1$ such that either (3.3) or (3.4) holds. In particular, we have $x(1) < x(2)$. By (3.2) and (3.5), $x(2) = B + h^2 f(B) \leq B + h^2 M_0$. So, if we choose h such small that $h^2 M_0 < |B|$, we have $x(1) < x(2) < 0$. Consequently $r > 2$ holds, and inequalities in (3.3) are fulfilled. Multiplying (3.5) by $\Delta x(n) + \Delta x(n-1)$, we obtain

$$(\Delta x(n))^2 - (\Delta x(n-1))^2 = h^2 f(x(n))(x(n+1) - x(n-1)), \quad n \in \mathbb{N}. \tag{4.3}$$

Summing (4.3) from 1 to $r-2$, we have

$$(\Delta x(r-2))^2 = h^2 \sum_{j=1}^{r-2} f(x(j))(x(j+1) - x(j-1)) < 2h^2 |B| M_0,$$

and

$$\Delta x(r-2) < h\sqrt{2|B|M_0}. \tag{4.4}$$

(i) Let $x(r) = 0$. By (3.5) we get $\Delta x(r) = \Delta x(r-1) \leq \Delta x(r-2) + h^2 M_0$. Hence, (4.4) implies

$$\Delta x(r-1) < h\sqrt{2|B|M_0} + h^2 M_0, \quad x(r+1) < h\sqrt{2|B|M_0} + h^2 M_0. \tag{4.5}$$

(ii) Let $x(r) > 0$. By (3.5) we get $\Delta x(r-1) \leq \Delta x(r-2) + h^2 M_0$. Using (3.3) we get $x(r) \leq x(r-1) + \Delta x(r-2) + h^2 M_0 < \Delta x(r-2) + h^2 M_0$. So, by (4.4), we can choose $h > 0$ such small that $x(r) < L$ and $\Delta x(r) = \Delta x(r-1) + h^2 f(x(r)) < \Delta x(r-1)$. Further, using (4.4), we obtain

$$\Delta x(r-1) < h\sqrt{2|B|M_0} + h^2 M_0, \quad x(r+1) < 2h\sqrt{2|B|M_0} + 2h^2 M_0. \tag{4.6}$$

Estimates (4.5) and (4.6) imply that we can find $h > 0$ such small that $x(r+1) < L$, as well. If $x(r) \geq x(r+1)$, we put $m = r$.

Let $x(r) < x(r+1)$. If $x(r+1) \geq x(r+2)$ or $x(r+2) \geq L$, we put $m = r+1$.

Let $x(r) < x(r+1) < x(r+2) < L$. If $x(r+2) \geq x(r+3)$ or $x(r+3) \geq L$, we put $m = r+2$. Otherwise we continue as before. Due to Theorem 3.7, after a finite number of steps, we get $m > r+2$ fulfilling (4.1).

According to (3.7), the finite sequence $\{\Delta x(j)\}_{j=1}^{r-1}$ is increasing. Similarly, by (1.3), $f(x(r)) \leq 0$ and $f(x(j)) < 0$ for $j = r+1, \ldots, m$, provided $m \geq r+1$. Therefore, by (3.5), $\Delta x(r-1) \geq \Delta x(r)$. If $m > r+1$, the finite sequence $\{\Delta x(j)\}_{j=r}^{m-1}$ is decreasing. Consequently (4.5) and (4.6) give (4.2). □

Lemma 4.2 *Choose an arbitrary $c > 0$. Let $B_1, B_2 \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ and $\{y(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (3.2) with $B = B_1$ and $B = B_2$, respectively. Then*

$$|x(n) - y(n)| \leq |B_1 - B_2| e^{c^2 K_0} \quad \text{for } n \in \mathbb{N},\ n \leq \frac{c}{h} + 1, \tag{4.7}$$

where K_0 is the Lipschitz constant for f on $[L_0, L]$.

Proof. By (3.5) we have $\Delta x(k) = \Delta x(k-1) + h^2 f(x(k))$, $k \in \mathbb{N}$. Summing it from 1 to k, we get by (3.2), $\Delta x(k) = h^2 \sum_{j=1}^{k} f(x(j))$, $k \in \mathbb{N}$. Summing it again from 1 to $n-1$, we get

$$x(n) = B_1 + h^2 \sum_{k=1}^{n-1} \sum_{j=1}^{k} f(x(j)), \quad n \in \mathbb{N},$$

and similarly

$$y(n) = B_2 + h^2 \sum_{k=1}^{n-1} \sum_{j=1}^{k} f(y(j)), \quad n \in \mathbb{N}.$$

Therefore

$$|x(n) - y(n)| \leq |B_1 - B_2| + h^2 \sum_{k=1}^{n-1} \sum_{j=1}^{k} |f(x(j)) - f(y(j))|$$

$$\leq |B_1 - B_2| + (n-1) h^2 K_0 \sum_{j=1}^{n-1} |x(j) - y(j)|, \quad n \in \mathbb{N}.$$

By the discrete analogue of the Gronwall-Bellman inequality (see e.g. [8], Lemma 4.34), we get

$$|x(n) - y(n)| \leq |B_1 - B_2| e^{(n-1)^2 h^2 K_0} \quad \text{for } n \in \mathbb{N}.$$

So, (4.7) is proved. □

5 Existence of non-monotonous solutions

Definition 5.1 A solution of problem (1.1), (3.2) satisfying conditions (III) of Theorem 3.7 is called *a non-monotonous solution*.

Lemma 5.2 *Let* $\{x(n)\}_{n=0}^{\infty}$ *be a non-monotonous solution. Then there exists* $c \in \mathbb{N}$, $c \geq b$, *such that* $\{x(n)\}_{n=b}^{c+1}$ *is decreasing and*

$$x(c) > 0 > x(c+1) \quad \text{or} \quad x(c) = 0. \tag{5.1}$$

Proof. Consider b of Theorem 3.7 (III). If $x(b+1) < 0$, we put $b = c$ and (3.13) yields $x(c) > 0 > x(c+1)$. Clearly $\{x(n)\}_{n=b}^{c+1}$ is decreasing. If $x(b+1) = 0$, then for $b+1 = c$ we have $x(b) > x(c) = 0$. Further, by (3.5) and (3.13), $\Delta x(c) = \Delta x(c-1) + h^2 f(x(c)) = \Delta x(c-1) < 0$. So, $x(c+1) < 0$ and $\{x(n)\}_{n=b}^{c+1}$ is decreasing. Let $x(b+1) > 0$. Then (3.5) and (3.13) yield $\Delta x(b+1) = \Delta x(b) + h^2 f(x(b+1)) < \Delta x(b) \leq 0$, and hence $x(b+2) < x(b+1)$. We see that $\{x(n)\}_{n=b}^{c+1}$ and $\{\Delta x(n)\}_{n=b}^{c}$ are decreasing as long as $x(c) \geq 0$. If $x(n) > 0$ for all $n > b$, then $\lim_{n\to\infty} \Delta x(n) < 0$ which gives $\lim_{n\to\infty} x(n) = -\infty$, a contradiction. Therefore a finite c fulfilling (5.1) must exist. □

Theorem 5.3 *Let* $B \in (\bar{B}, 0)$. *There exists* $h_B > 0$ *such that if* $h \in (0, h_B]$, *then the corresponding solution* $\{x(n)\}_{n=0}^{\infty}$ *of problem* (1.1), (3.2) *is non-monotonous.*

Proof. Choose $B \in (\bar{B}, 0)$. Then, by (1.3) and (1.4), we can find $\varepsilon > 0$ such that

$$\int_B^L f(z)\,\mathrm{d}z + \varepsilon < 0. \tag{5.2}$$

Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (3.2) with this B.

(i) Assume that $\{x(n)\}_{n=0}^{\infty}$ is an escape solution. Then there exists $b \in \mathbb{N}$, $b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and (3.11) holds. Therefore, summing (4.3) from 1 to $b-1$ and multiplying by $\frac{1}{2}$, we get

$$0 < \frac{1}{2}\left(\frac{\Delta x(b-1)}{h}\right)^2 = \sum_{j=1}^{b-1} f(x(j)) \frac{x(j+1) - x(j-1)}{2}. \tag{5.3}$$

By (1.2), f integrable on $[L_0, L]$ and hence there exists $\delta > 0$ such that if $(x(j+1) - x(j-1))/2 < \delta$, then

$$\left| \sum_{j=1}^{b-1} f(x(j)) \frac{x(j+1) - x(j-1)}{2} - \int_B^L f(z)\,\mathrm{d}z \right| < \varepsilon. \tag{5.4}$$

Let $h_B \in (0, 1)$ be such that

$$h_B < \frac{\delta}{\sqrt{2|B|M_0} + M_0}, \tag{5.5}$$

where $M_0 = \max\{|f(x)|\colon x \in [L_0, L]\}$. Now, choose $h \in (0, h_B]$. Then (4.2) implies $(x(j+1) - x(j-1))/2 < \delta$ and we obtain (5.4). Consequently, by (5.2), (5.3) and (5.4), we get

$$0 < \int_B^L f(z)\,\mathrm{d}z + \varepsilon < 0,$$

a contradiction. So, $\{x(n)\}_{n=0}^{\infty}$ is not an escape solution provided $B \in (\bar{B}, 0)$ and $h \in (0, h_B]$.

(ii) Assume that $\{x(n)\}_{n=0}^{\infty}$ is a homoclinic solution, that is $\{x(n)\}_{n=1}^{\infty}$ is increasing, $x(n) < L$ for $n \in \mathbb{N}$, and (3.9) holds. Summing (4.3) from 1 to n and multiplying by $\frac{1}{2}$, we get

$$0 < \frac{1}{2} \left(\frac{\Delta x(n)}{h} \right)^2 = \sum_{j=1}^{n} f(x(j)) \frac{x(j+1) - x(j-1)}{2}, \quad n \in \mathbb{N}. \tag{5.6}$$

Let δ, h_B and h be as in part (i). By (3.9), we can choose $n_0 \in \mathbb{N}$ such that $|x(n+1) - L| < \delta$ for $n \geq n_0$. Then, as in part (i), we conclude that

$$\left| \sum_{j=1}^{n} f(x(j)) \frac{x(j+1) - x(j-1)}{2} - \int_B^L f(z)\,\mathrm{d}z \right| < \varepsilon, \quad n \geq n_0. \tag{5.7}$$

By (5.2), (5.6) and (5.7) we get a contradiction as in (i). We have proved that $\{x(n)\}_{n=0}^{\infty}$ is not a homoclinic solution provided $B \in (\bar{B}, 0)$ and $h \in (0, h_B]$.

Therefore, by virtue of Theorem 3.7, $\{x(n)\}_{n=0}^{\infty}$ has to be a non-monotonous solution provided $B \in (\bar{B}, 0)$ and $h \in (0, h_B]$. □

6 Existence of escape solutions

Theorem 6.1 *Let $B_{es} \in (L_0, \bar{B})$. There exists $h_{B_{es}} > 0$ such that if $h \in (0, h_{B_{es}}]$, then the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem* (1.1), (3.2) *with $B = B_{es}$ is an escape solution.*

Proof. Choose $B_{es} \in (L_0, \bar{B})$. Then, by (1.3) and (1.4), we can find $\varepsilon > 0$ and $c_0 > 0$ such that

$$\int_{B_{es}}^{L} f(z)\,\mathrm{d}z - \varepsilon = c_0^2. \tag{6.1}$$

Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (3.2) with $B = B_{es}$.

(i) Assume that $\{x(n)\}_{n=0}^{\infty}$ is a non-monotonous solution. Then there exists $b \in \mathbb{N}$ such that $\{x(n)\}_{n=1}^{b}$ is increasing and (3.13) holds. As in the proof of Theorem 5.3 we can find $\delta > 0$ such that (5.4) holds with $B = B_{es}$. Choose $h_B \in (0, 1)$ such that (5.5) is valid. Assume that $h \in (0, h_B]$. We derive (5.3) as in the proof of Theorem 5.3. Consequently we get

$$\begin{aligned} \frac{\Delta x(b-1)}{h\sqrt{2}} &= \sqrt{\sum_{j=1}^{b-1} f(x(j)) \frac{x(j+1) - x(j-1)}{2}} \\ &> \sqrt{\int_{B_{es}}^{L} f(z)\,\mathrm{d}z - \varepsilon} = c_0 > 0. \end{aligned} \tag{6.2}$$

Further, by (1.3) and (3.13) it holds $f(x(b)) < 0$, $\Delta x(b-1) > 0$ and $\Delta x(b) \le 0$. Therefore (3.5) leads to $|\Delta x(b)| + \Delta x(b-1) = h^2 |f(x(b))|$ and

$$\frac{\Delta x(b-1)}{h} \le h M_0, \quad M_0 = \max\{|f(x)| \colon x \in [L_0, L]\}.$$

Choose $h_{B_{es}} \in (0, h_B]$ such that $h_{B_{es}} M_0 < c_0$. Then for each $h \in (0, h_{B_{es}}]$ we get $\Delta x(b-1)/h < c_0$, contrary to (6.2). So, $\{x(n)\}_{n=0}^{\infty}$ is not a non-monotonous solution provided $B_{es} \in (L_0, \bar{B})$ and $h \in (0, h_{B_{es}}]$.

(ii) Assume that $\{x(n)\}_{n=0}^{\infty}$ is a homoclinic solution. We choose $h_B \in (0, 1)$ and $n_0 \in \mathbb{N}$ as in the proof of Theorem 5.3 part (ii) and arguing similarly we get (5.6) and (5.7) with $B = B_{es}$. Using (6.1) we obtain

$$\begin{aligned} \frac{\Delta x(n)}{h\sqrt{2}} &= \sqrt{\sum_{j=1}^{n} f(x(j)) \frac{x(j+1) - x(j-1)}{2}} \\ &> \sqrt{\int_{B_{es}}^{L} f(z)\,\mathrm{d}z - \varepsilon} = c_0 > 0 \quad \text{for } n \ge n_0. \end{aligned} \tag{6.3}$$

By (3.9), for any fixed $h \in (0, h_B]$, we have

$$\lim_{n \to \infty} \frac{\Delta x(n)}{h} = 0,$$

contrary to (6.3). Put $h_{B_{es}} = h_B$. Then $\{x(n)\}_{n=0}^{\infty}$ cannot be a homoclinic solution provided $B_{es} \in (L_0, \bar{B})$ and $h \in (0, h_{B_{es}}]$.

Therefore, by virtue of Theorem 3.7, $\{x(n)\}_{n=0}^{\infty}$ has to be an escape solution provided $B_{es} \in (L_0, \bar{B})$ and $h \in (0, h_{B_{es}}]$. □

7 Existence of homoclinic solutions

Theorem 7.1 *Let f fulfil* (1.2)–(1.4). *There exists $h_0 > 0$ such that for each $h^* \in (0, h_0]$ there exists $B^* \in (L_0, 0)$ such that the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem* (1.1), (3.2) *with $B = B^*$ is a homoclinic solution.*

Proof. First, assume that, in addition, f fulfils (3.10). By Theorems 5.3 and 6.1 there exists $h_0 \in (0, 1)$ such that if we choose an arbitrary $h \in (0, h_0]$, then it holds:

(a) Let $B_{non} \in (\bar{B}, 0)$. Then the corresponding solution $\{x_{non}(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2) with $B = B_{non}$ is a non-monotonous solution.

(b) Let $B_{es} \in (L_0, \bar{B})$. Then the corresponding solution $\{x_{es}(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2) with $B = B_{es}$ is an escape solution.

Choose $h \in (0, h_0]$, $B_{non} \in (\bar{B}, 0)$ and the solution $\{x_{non}(n)\}_{n=0}^{\infty}$. By Lemma 5.2 there exists $c \in \mathbb{N}$ satisfying (5.1) for $x = x_{non}$. That is $x_{non}(c + 1) < 0$. Let $B \in (L_0, B_{non})$ and $\{x(n)\}_{n=0}^{\infty}$ be the corresponding solution of problem (1.1), (3.2). By Lemma 4.2,

$$|x_{non}(n) - x(n)| \le |B_{non} - B| e^{c^2 K_0}, \quad n \in \mathbb{N},\ n \le \frac{c}{h} + 1. \tag{7.1}$$

Since $h < 1$, (7.1) yields for $n = c + 1$

$$|x_{non}(c + 1) - x(c + 1)| \le |B_{non} - B| e^{c^2 K_0}.$$

Therefore we can find $\delta > 0$ such small that if $B \in (B_{non} - \delta, B_{non}]$, then $x(c+1) < 0$. Consequently $\{x(n)\}_{n=0}^{\infty}$ is a non-monotonous solution. According to (b) there exists the minimal number $B^* \in (L_0, B_{non})$ such that for $B \in (B^*, B_{non}]$ the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ is a non-monotonous.

Let $\{x^*(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (3.2) with $B = B^*$. Assume that $\{x^*(n)\}_{n=0}^{\infty}$ is non-monotonous. Then using the same arguments as above we can find $\delta > 0$ such small that for $B \in (B^* - \delta, B^*]$ the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ is also non-monotonous. This contradicts the minimality of B^*.

Assume that $\{x^*(n)\}_{n=0}^{\infty}$ is an escape solution. By Lemma 3.5 there exists $b \in \mathbb{N}$ satisfying (3.11) for $x = x^*$. That is $x^*(b + 1) > L$. Consider a solution $\{x(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2) for $B \in (B^*, 0)$. We can use Lemma 4.2 again and get

$$|x^*(n) - x(n)| \le |B^* - B| e^{b^2 K_0}, \quad n \in \mathbb{N},\ n \le \frac{b}{h} + 1. \tag{7.2}$$

Since $h < 1$, (7.2) yields for $n = b + 1$

$$|x^*(b + 1) - x(b + 1)| \le |B^* - B| e^{b^2 K_0}.$$

Therefore we can find $\delta > 0$ such small that if $B \in [B^*, B^* + \delta)$ then $x(b+1) > L$. This yields that $\{x(n)\}_{n=0}^{\infty}$ is an escape solution, contrary to the definition of B^*.

We have proved that $\{x^*(n)\}_{n=0}^\infty$ is a homoclinic solution of equation (1.1). Since $L_0 < x^*(n) < L$ for $n \in \mathbb{N}$, we can omit the assumption (3.10). $\square$

Remark 7.2 The proof of Theorem 7.1 implies that the homoclinic point $\mathbf{x}^* = (B^*, B^*)^T$ of equation (1.1) can be found in the following way. Choose $h \in (0, h_0]$. We have two subsets $\mathcal{M}_{non}$ and $\mathcal{M}_{es}$ of the interval $(L_0, 0)$. $\mathcal{M}_{non}$ consists of all B such that the corresponding solutions $\{x(n)\}_{n=0}^\infty$ of problem (1.1), (3.2) are non-monotonous. $\mathcal{M}_{non}$ is non-empty by Theorem 5.3 and open by Lemma 4.2. $\mathcal{M}_{es}$ consists of all B such that the corresponding solutions $\{x(n)\}_{n=0}^\infty$ of problem (1.1), (3.2) are escape solutions. $\mathcal{M}_{es}$ is non-empty by Theorem 6.1 and open by Lemma 4.2. Each B^* lying on the common boundary of $\mathcal{M}_{non}$ and $\mathcal{M}_{es}$ forms the homoclinic point $\mathbf{x}^* = (B^*, B^*)^T$ satisfying (1.7).

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