Singular Dirichlet second order BVPs with impulses *

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Abstract: The paper deals with the existence of solutions to singular second order differential equations with impulse effects and with the Dirichlet boundary conditions. The right hand side of the differential equation can be singular in its phase variable.
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1 Introduction

For $[a, b] \subset \mathbb{R}$ we denote by $C^1[a, b]$ the set of functions having continuous first derivatives on $[a, b]$ and by $L_1[a, b]$ the set of functions Lebesgue integrable on $[a, b]$. Choose $p \in \mathbb{N}$ and consider the division $\mathcal{D} = \{t_i\}_{i=1}^p$ of the interval $[0, T] \subset \mathbb{R}$, where $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$. Further denote by $C_{\mathcal{D}}^1$ the set of functions $x : [0, T] \to \mathbb{R}$

$$x(t) = \begin{cases} x_{(0)}(t) & \text{for} & t \in [0, t_1] \\ x_{(1)}(t) & \text{for} & t \in (t_1, t_2] \\ \ldots & \ldots & \ldots \\ x_{(p)}(t) & \text{for} & t \in (t_p, T], \end{cases} \tag{1.1}$$

where $x_{(i)} \in C^1[t_i, t_{i+1}]$, $0 \leq i \leq p$, and denote by $AC_{\mathcal{D}}^1$ the set of functions $x \in C_{\mathcal{D}}^1$ having first derivatives absolutely continuous on (t_i, t_{i+1}), $0 \leq i \leq p$. For $x \in C_{\mathcal{D}}^1$ we will use the notation

$$x'(t_i) = \lim_{t \to t_i-} x'(t), \ 1 \leq i \leq p+1, \quad x'(0) = \lim_{t \to 0+} x'(t). \tag{1.2}$$

In this paper we study the existence of positive solutions of the following *singular Dirichlet boundary value problem* (BVP) with *impulses*

$$-u''(t) = f(t, u(t), u'(t)), \tag{1.3}$$

$$u(t_i+) = J_i(u(t_i)), \quad u'(t_i+) = M_i(u'(t_i)), \ 1 \le i \le p, \tag{1.4}$$

$$u(0) = u(T) = 0, \tag{1.5}$$

where $u'(t_i), 1 \le i \le p$, are given by (1.2). Here we suppose that $J_i : \mathbb{R} \to \mathbb{R}$, $M_i : \mathbb{R} \to \mathbb{R}$, $1 \le i \le p$, are continuous and increasing functions and that f fulfils the *Carathéodory conditions* on the set $[0, T] \times ((0, \infty) \times \mathbb{R})$, which means that

(i) for each $(x, y) \in (0, \infty) \times \mathbb{R}$ the function $f(\cdot, x, y)$ is measurable on $[0, T]$;

(ii) for a.e. $t \in [0, T]$ the function $f(t, \cdot, \cdot)$ is continuous on $(0, \infty) \times \mathbb{R}$;

(iii) for each compact set $K \subset (0, \infty) \times \mathbb{R}$ the function $m_K(t) = \sup\{|f(t, x, y)| : (x, y) \in K\}$ is Lebesgue integrable on $[0, T]$.

Definition 1.1. By *a solution of BVP* (1.3)-(1.5) we understand a function $x \in AC_{\mathcal{D}}^1$ which satisfies the impulsive conditions (1.4) and the Dirichlet boundary conditions (1.5) and for a.e. $t \in [0, T]$ fulfils the equation (1.3). If x is positive on $(0, T)$ it is called *a positive* solution of BVP (1.3)-(1.5).

The fact that BVP (1.3)-(1.5) is singular means that the right hand side f of the differential equation does not fulfil the Carathéodory conditions on the whole region where we seek for solutions, i.e. on $[0, T] \times [0, \infty) \times \mathbb{R}$. Here, the Carathéodory conditions can be broken in the phase variable x. Particularly, for a.e. $t \in [0, T]$ and all $y \in \mathbb{R}$ the function $f(t, x, y)$ can be unbounded for $x \to 0+$. Such singular problems without impulse effects have been solved before for example in [1]-[4], [7], [9]-[12], [14]-[17], [19]-[25]. But as far as we know the solvability of singular problems with impulses has not been studied yet. In this paper we want to fill in this gap and extend the existence results on the case of singular problems with impulses. Our main goal is to find conditions for $f, J_i, M_i, 1 \le i \le p$, which guarantee the existence of at least one solution of problem (1.3)-(1.5). The proofs are based on the method of *a priori estimates*, on the *regularization* technique, on the *topological degree* arguments and on the *Vitali* convergence theorem.

In what follows we assume that $C^1[a, b]$ and $L_1[a, b]$ is respectively equipped with the norm

$$\|x\|_{C^1} = \max\{|x(t)| + |x'(t)| : t \in [a, b]\}, \quad \text{and} \quad \|y\|_{L_1} = \int_a^b |y(t)|dt.$$

Then $C^1[a, b]$ and $L_1[a, b]$ become Banach spaces. For any measurable set $\mathcal{M} \subset \mathbb{R}$, $\mu(\mathcal{M})$ denotes the Lebesgue measure of $\mathcal{M}$.

Definition 1.2. A collection $\mathcal{A} \subset L_1[a, b]$ is called *uniformly absolutely continuous* on $[a, b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi \in \mathcal{A}$ and $\mathcal{M} \subset [a, b]$ with $\mu(\mathcal{M}) < \delta$, then

$$\int_{\mathcal{M}} |\varphi(t)|dt < \varepsilon.$$

Theorem 1.3. (Vitali convergence theorem, [5], p. 178-180.) *Let $[a,b] \subset \mathbb{R}$ and let $\{f_n\}$ be a sequence in $L_1[a,b]$ which is convergent to f for a.e. $t \in [a,b]$. Then the following statements are equivalent:*
(a) $f \in L_1[a,b]$ and $\lim_{n \to \infty} \|f_n - f\|_{L_1} = 0$.
(b) The set $\{f_n : n \in \mathbb{N}\}$ is uniformly absolutely continuous on $[a,b]$.

2 Main assumptions

In the study of problem (1.3)-(1.5) we will work with assumptions:

(H_1) The impulse functions $J_i : \mathbb{R} \to \mathbb{R}$, $M_i : \mathbb{R} \to \mathbb{R}$, $1 \le i \le p$, are continuous and increasing and

$$J_i(0) = 0, \quad M_i(0) = 0, \quad 1 \le i \le p. \tag{2.1}$$

(H_2)

$$\lim_{x \to \infty} \frac{J_i(x)}{x} < \infty, \quad 1 \le i \le p, \tag{2.2}$$

$$\lim_{x \to \infty} \frac{M_1(x)}{x} > 0, \quad \lim_{x \to \infty} \frac{M_i(x)}{x} > 1, \quad 2 \le i \le p, \tag{2.3}$$

$$\lim_{x \to -\infty} \frac{x}{M_p(x)} > 0, \quad \lim_{x \to -\infty} \frac{x}{M_i(x)} > 1, \quad 1 \le i \le p-1. \tag{2.4}$$

(H_3) The function f satisfies the Carathéodory conditions on the set $[0,T] \times ((0,\infty) \times \mathbb{R})$ and there exists a function ψ Lebesgue integrable on $[0,T]$ such that

$$0 < \psi(t) \le f(t,x,y) \tag{2.5}$$

for a.e. $t \in [0,T]$ and each $x \in (0,\infty), y \in \mathbb{R}$.

(H_4) A function h satisfies the Carathéodory conditions on the set $[0,T] \times [0,\infty)$, h is nonnegative and nondecreasing in its second argument and

$$\lim_{z \to \infty} \frac{1}{z} \int_0^T h(t,z)dt = 0, \tag{2.6}$$

a function q is nonnegative and essentially bounded on $[0,T]$ and a function ω is positive, nonincreasing on $(0,\infty)$ and

$$\int_0^T \omega(s)ds < \infty. \tag{2.7}$$

(H_5) For a.e. $t \in [0,T]$ and for each $x \in (0,\infty), y \in \mathbb{R}$

$$f(t,x,y) \le h(t,x+|y|) + q(t)\omega(x), \tag{2.8}$$

where functions h, q, ω satisfy (H_4).

Remark 2.1. Since ω in (H_4) is positive and nonincreasing, the assumption (2.7) implies that $\int_0^V \omega(s)ds < \infty$ for each $V \in \mathbb{R}_+$.

3   A priori estimates

In order to construct auxiliary regular problems and to use convergence theorems we need a priori estimates of solutions both below and above. Such types of estimates are proved in lemmas of this section.

Lemma 3.1. *Suppose that* (H_1) *holds and that* $u \in AC_D^1$ *fulfils* (1.4), (1.5) *and*

$$0 < -u''(t) \quad \text{for a.e. } t \in [0, T]. \tag{3.1}$$

Then u *is positive on* $(0, T)$ *and there exists* $\xi \in (0, T)$ *such that*

$$u'(\xi) = 0, \quad u' > 0 \text{ on } [0, \xi), \quad u' < 0 \text{ on } (\xi, T]. \tag{3.2}$$

Proof. Let $u \in AC_D^1$ fulfil (1.4), (1.5) and (3.1). Then

$$u' \text{ is decreasing on } (t_i, t_{i+1}], \quad 0 \le i \le p. \tag{3.3}$$

(i) Suppose $u'(0) \le 0$. Then, by (3.3), $u'(t) < 0$ for $t \in (0, t_1]$ and $u'(t_1+) = M_1(u'(t_1)) < M_1(0) = 0$. Therefore $u'(t_1+) < 0$ and, by (3.3), $u'(t) < 0$ for $t \in (t_1, t_2]$. Repeating these arguments we get

$$u'(t) < 0 \quad \text{for} \quad t \in (0, T]. \tag{3.4}$$

Then, according to (1.5), $u(t) < 0$ for $t \in (0, t_1]$ and $u(t_1+) = J_1(u(t_1)) < J_1(0) = 0$. Therefore $u(t_1+) < 0$ and, by (3.4), $u(t) < 0$ for $t \in (t_1, t_2]$. Repeating it we get $u(t) < 0$ for $t \in (0, T]$, contrary to (1.5). Therefore we have proved

$$u'(0) > 0. \tag{3.5}$$

(ii) Suppose $u'(T) \ge 0$. Then, by (3.3), $u'(t) > 0$ for $t \in (t_p, T)$ and $u'(t_p+) > 0$, which gives $M_p(u'(t_p)) > 0 = M_p(0)$. Since M_p is increasing, we get $u'(t_p) > 0$. By (3.3) we conclude that $u'(t) > 0$ for $t \in (t_{p-1}, t_p]$. Repeating these arguments we get

$$u'(t) > 0 \quad \text{for } t \in [0, T). \tag{3.6}$$

Then, according to (1.5), $u(t) < 0$ for $t \in (t_p, T)$ and $u(t_p+) < 0$. Therefore $J_p(0) = 0 > u(t_p+) = J_p(u(t_p))$, and having in mind that J_p is increasing, we get $u(t_p) < 0$, which together with (3.6) gives $u(t) < 0$ on $(t_{p-1}, t_p]$. Repeating it we deduce that $u(t) < 0$ for $t \in [0, T)$, contrary to (1.5). Therefore we have proved

$$u'(T) < 0. \tag{3.7}$$

(iii) (H_1) implies that

$$\text{if } x \in \mathbb{R} \setminus \{0\}, \text{ then } \operatorname{sgn}(M_i(x)) = \operatorname{sgn}x, \ 1 \le i \le p. \tag{3.8}$$

(a) Suppose that there is $i_0 \in \{1, \ldots, p\}$ such that $u'(t_{i_0}) = 0$. Then, by (H_1), $u'(t_{i_0}+) = M_{i_0}(0) = 0$. By virtue of (3.3) and (3.8), we have $u'(t) > 0$ for $t \in [0, t_{i_0})$, $u'(t) < 0$ for $t \in (t_{i_0}, T]$. Therefore (3.2) is satisfied with $\xi = t_{i_0}$.
(b) Let $u'(t_i) \neq 0, 1 \le i \le p$. Then, by (3.8), $\operatorname{sgn}u'(t_i+) = \operatorname{sgn}u'(t_i), 1 \le i \le p$. Thus, by (3.3), (3.5) and (3.7), there exists $\xi \in \cup_{i=0}^{p}(t_i, t_{i+1})$ satisfying (3.2). $\qquad\square$

Definition 3.2. Suppose that $u \in AC_{\mathcal{D}}^1$ and $\xi \in (0, T)$ are from Lemma 3.1. Then ξ is called *a critical point of u*.

Lemma 3.3. *Suppose that (H_1) holds and that $\psi \in L_1[0, T]$. Then there exists a constant $\gamma > 0$ such that for any function $u \in AC_{\mathcal{D}}^1$ having a critical point $\xi = \xi(u)$ and fulfilling* $(1.4), (1.5)$ *and*

$$0 < \psi(t) \le -u''(t) \quad \text{for a.e.} \quad t \in [0, T], \tag{3.9}$$

the following estimates hold:
(I) if $\xi \in \mathcal{D}$, i.e. $\xi = t_{j+1}$ for some $j = j(u) \in \{0, \ldots, p-1\}$, then

$$u(t) \ge \begin{cases} \dfrac{t - t_i}{t_{i+1} - t_i}\gamma & \text{for } t \in (t_i, t_{i+1}], \quad 0 \le i \le j \\[2mm] \dfrac{t_{i+1} - t}{t_{i+1} - t_i}\gamma & \text{for } t \in (t_i, t_{i+1}], \quad j+1 \le i \le p; \end{cases} \tag{3.10}$$

(II) if $\xi \in (0, T) \setminus \mathcal{D}$, i.e. $\xi \in (t_j, t_{j+1})$ for some $j = j(u) \in \{0, \ldots, p\}$, then

$$u(t) \ge \begin{cases} \dfrac{t - t_i}{t_{i+1} - t_i}\gamma & \text{for } t \in (t_i, t_{i+1}], \quad 0 \le i \le j-1 \\[2mm] \dfrac{t - t_j}{\xi - t_j}\gamma & \text{for } t \in (t_j, \xi], \\[2mm] \dfrac{t_{j+1} - t}{t_{j+1} - \xi}\gamma & \text{for } t \in (\xi, t_{j+1}], \\[2mm] \dfrac{t_{i+1} - t}{t_{i+1} - t_i}\gamma & \text{for } t \in (t_i, t_{i+1}], \quad j+1 \le i \le p. \end{cases} \tag{3.11}$$

Proof. Let $u \in AC_{\mathcal{D}}^1$ satisfy (1.4), (1.5) and (3.9). Denote $u(t_i) = c_i, 0 \le i \le p+1$. Then, by Lemma 3.1,

$$c_0 = c_{p+1} = 0, \ c_i > 0, \ J_i(c_i) > 0, \ 1 \le i \le p. \tag{3.12}$$

Let $G_i(t, s)$ be the Green's function of the problem

$$-v''(t) = 0, \ v(t_i) = v(t_{i+1}) = 0, \ 0 \le i \le p,$$

i.e.

$$G_i(t, s) = \begin{cases} \dfrac{(s - t_i)(t_{i+1} - t)}{t_{i+1} - t_i}, & t_i \le s \le t \le t_{i+1}, \\[3mm] \dfrac{(t - t_i)(t_{i+1} - s)}{t_{i+1} - t_i}, & t_i \le t < s \le t_{i+1}. \end{cases}$$

Denote $J_0(c_0) = c_0$ and let for $0 \le i \le p$, $u_{(i)} \in C^1[t_i, t_{i+1}]$ correspond to u in the sense of (1.1), i.e.

$$u_{(0)}(0) = 0 = u(0), \ u_{(i)}(t) = u(t) \text{ on } (t_i, t_{i+1}], \ u_{(i)}(t_i) = u(t_i+). \tag{3.13}$$

We have

$$-u''_{(i)}(t) = -u''(t) \text{ a.e. on } (t_i, t_{i+1}), \ 0 \le i \le p,$$

$$u_{(i)}(t_i) = J_i(c_i), \quad u_{(i)}(t_{i+1}) = c_{i+1}, \ 0 \le i \le p.$$

Now, let us choose an arbitrary $i \in \{0, \dots, p\}$. Hence, for $t \in [t_i, t_{i+1}]$, we get

$$u_{(i)}(t) = \frac{t - t_i}{t_{i+1} - t_i} c_{i+1} + \frac{t_{i+1} - t}{t_{i+1} - t_i} J_i(c_i) - \int_{t_i}^{t_{i+1}} G_i(t, s) u''(s) ds,$$

which, by virtue of (3.12) and (3.9), yields

$$u_{(i)}(t) \ge \int_{t_i}^{t_{i+1}} G_i(t, s) \psi(s) ds \text{ for } t \in [t_i, t_{i+1}]. \tag{3.14}$$

Put

$$\Phi_i(t, s) = \frac{G_i(t, s)}{(t - t_i)(t_{i+1} - t)} \text{ on } [t_i, t_{i+1}] \times [t_i, t_{i+1}]. \tag{3.15}$$

Then,

$$\Phi_i > 0 \quad \text{on } (t_i, t_{i+1}) \times (t_i, t_{i+1}) \tag{3.16}$$

and for $s \in (t_i, t_{i+1})$

$$\lim_{t \to t_i+} \Phi_i(t, s) = \frac{1}{t_{i+1} - t_i} \frac{\partial G_i(t, s)}{\partial t} \Big|_{(t,s)=(t_i,s)} = \frac{t_{i+1} - s}{(t_{i+1} - t_i)^2} > 0,$$

$$\lim_{t \to t_{i+1}-} \Phi_i(t, s) = \frac{-1}{t_{i+1} - t_i} \frac{\partial G_i(t, s)}{\partial t} \Big|_{(t,s)=(t_{i+1},s)} = \frac{s - t_i}{(t_{i+1} - t_i)^2} > 0.$$

Therefore, by (3.16), we can extend the function Φ_i at $t = t_i, t = t_{i+1}$ such that for each $s \in (t_i, t_{i+1})$ the function $\Phi_i(\cdot, s)$ is continuous and positive on $[t_i, t_{i+1}]$. Put

$$F_i(t) = \int_{t_i}^{t_{i+1}} \Phi_i(t, s) \psi(s) ds \quad \text{for } t \in [t_i, t_{i+1}]. \tag{3.17}$$

Then, by virtue of (3.9), the function F_i is continuous and positive on $[t_i, t_{i+1}]$ and so we can find $\delta_i > 0$ such that $F_i(t) \ge \delta_i$ for $t \in [t_i, t_{i+1}]$. Therefore, by (3.14), (3.15) and (3.17)

$$u_{(i)}(t) \ge (t - t_i)(t_{i+1} - t) \int_{t_i}^{t_{i+1}} \Phi_i(t, s) \psi(s) ds =$$

$$= (t - t_i)(t_{i+1} - t) F_i(t) \ge (t - t_i)(t_{i+1} - t) \delta_i \text{ for } t \in [t_i, t_{i+1}].$$

So,

$$\max\{u_{(i)}(t) : t \in [t_i, t_{i+1}]\} \ge \frac{1}{4}(t_{i+1} - t_i)^2 \delta_i = \gamma_i > 0.$$

If we put $\gamma = \min\{\gamma_i, 0 \le i \le p\}$, we have

$$\max\{u_{(i)}(t) : t \in [t_i, t_{i+1}]\} \ge \gamma, \ 0 \le i \le p. \tag{3.18}$$

Now, we are going to prove the assertion *(I)*.
Suppose that u has a critical point ξ and $\xi = t_{j+1}$ for some $j \in \{0, \ldots, p-1\}$. Choose an arbitrary $i \in \{0, \ldots, j\}$. Then, by (3.9), (3.2) and (3.13), $u_{(i)}$ is concave and increasing on $[t_i, t_{i+1}]$. Moreover $u_{(i)}(t_i) \ge 0$ and $u_{(i)}(t_{i+1}) \ge \gamma$, by (3.18). Hence we conclude that

$$u(t) = u_{(i)}(t) \ge \frac{t - t_i}{t_{i+1} - t_i}\gamma \quad \text{for } t \in (t_i, t_{i+1}], \ 0 \le i \le j.$$

Now consider the case $j+1 \le i \le p$. For such i the function $u_{(i)}$ is concave and decreasing on $[t_i, t_{i+1}]$, $u_{(i)}(t_i) \ge \gamma$, $u_{(i)}(t_{i+1}) \ge 0$. Therefore the estimate

$$u(t) = u_{(i)}(t) \ge \frac{t_{i+1} - t}{t_{i+1} - t_i}\gamma \quad \text{for } t \in (t_i, t_{i+1}], \ j+1 \le i \le p,$$

is true.

To prove the assertion *(II)* we assume that $\xi \in (t_j, t_{j+1})$ for some $j \in \{0, \ldots, p\}$ and get (3.11) by means of the concavity and monotonicity arguments as in the proof of (3.10). $\square$

Lemma 3.4. *Let $\psi \in L_1[0,T]$. Suppose that (H_1), (H_2) and (H_4) with $\tilde{h}$ instead of h hold. Then there exists constants $A^*, B^* > 0$ such that for any function $u \in AC_{\mathcal{D}}^1$ satisfying (1.4), (1.5), (3.9) and*

$$-u''(t) \le \tilde{h}(t, u(t) + |u'(t)|) + q(t)\omega(u(t)) \ \text{for a.e. } t \in [0,T], \tag{3.19}$$

the estimates

$$\sup\{u(t) : t \in [0,T]\} < A^*, \quad \sup\{|u'(t)| : t \in [0,T]\} < B^* \tag{3.20}$$

are valid.
Proof. Let $u \in AC_{\mathcal{D}}^1$ satisfy (1.4), (1.5), (3.9), and (3.19). According to Lemma 3.1 and Definition 3.2 u has a critical point $\xi \in (0,T)$ satisfying (3.2). We distinguish two cases: *(i)* ξ belongs to the division $\mathcal{D}$ and *(ii)* ξ does not belong to $\mathcal{D}$.

Case (i). Suppose that there is $j \in \{0, \ldots, p-1\}$ such that $\xi = t_{j+1}$. Denote

$$u'(t_i) = \rho_i, \quad 0 \le i \le p+1. \tag{3.21}$$

Then

$$u'(t_i+) = M_i(\rho_i), \quad 1 \le i \le p, \tag{3.22}$$

and by (3.2) and (H_1)

$$\begin{cases} \rho_i > 0, \ M_i(\rho_i) > 0 \text{ for } 1 \le i \le j, \ \rho_0 > 0, \\ \rho_{j+1} = 0, \ M_{j+1}(\rho_{j+1}) = 0, \\ \rho_i < 0, \ M_i(\rho_i) < 0 \text{ for } j+2 \le i \le p, \ \rho_{p+1} < 0. \end{cases} \tag{3.23}$$

Further, due to (3.3),
$$\rho_0 > \rho_1, \quad M_i(\rho_i) > \rho_{i+1}, \ 1 \le i \le p \tag{3.24}$$

and
$$\begin{cases} \sup\{|u'(t)| : t \in (0, t_1)\} = u'(0) = \rho_0, \\ \sup\{|u'(t)| : t \in (t_i, t_{i+1})\} = u'(t_i+) = M_i(\rho_i), \ 1 \le i \le j, \\ \sup\{|u'(t)| : t \in (t_i, t_{i+1})\} = |u'((t_{i+1})| = |\rho_{i+1}|, \ j+1 \le i \le p. \end{cases} \tag{3.25}$$

Part 1. First, we are going to find bounds for u and $|u'|$ on $[0, t_{j+1}]$. Denote

$$\alpha = \max\{t_{i+1} - t_i, 0 \le i \le j\}, \quad M_0(\rho_0) = \rho_0,$$
$$Q = \text{supess}\{q(t) : t \in [0, T]\}, \quad \bar{\omega} = \tfrac{1}{\gamma} \int_0^\gamma \omega(t)dt. \tag{3.26}$$

Here $\gamma > 0$ is from Lemma 3.3.

According to Lemma 3.1, u is increasing on $(t_i, t_{i+1}], 0 \le i \le j$, and $u(t_0) = u(0) = 0$, which together with (3.25) yields

$$\begin{cases} \sup\{u(t) : t \in (0, t_1)\} = u(t_1) \le \alpha\rho_0 = r_1 > 0, \\ \\ \sup\{u(t) : t \in (t_i, t_{i+1})\} = u(t_{i+1}) \le J_i(r_i) + \alpha M_i(\rho_i) = r_{i+1} > 0, \\ \text{for} \quad 1 \le i \le j. \end{cases} \tag{3.27}$$

Integrating (3.19) on $[0, t_{j+1}]$ we obtain, by virtue of (3.21) and (3.22)

$$\rho_0 + \sum_{i=1}^j M_i(\rho_i) - \sum_{i=1}^j \rho_i \le \int_0^{t_{j+1}} \left(\tilde{h}(t, u(t) + |u'(t)|) + q(t)\omega(u(t)) \right) dt,$$

wherefrom, due to (3.25)-(3.27),

$$\sum_{i=0}^j M_i(\rho_i) \le \sum_{i=1}^j \rho_i +$$
$$+ \sum_{i=0}^j \int_{t_i}^{t_{i+1}} \tilde{h}(t, r_{i+1} + M_i(\rho_i))dt + Q \int_0^{t_{j+1}} \omega(u(t))dt. \tag{3.28}$$

Applying statement *(I)* of Lemma 3.3 we get

$$\int_0^{t_{j+1}} \omega(u(t))dt \le \sum_{i=0}^j \int_{t_i}^{t_{i+1}} \omega\left(\frac{t - t_i}{t_{i+1} - t_i}\gamma\right)dt = \frac{1}{\gamma} \sum_{i=0}^j (t_{i+1} - t_i) \int_0^\gamma \omega(t)dt,$$

hence, by (3.26),

$$\int_0^{t_{j+1}} \omega(u(t))dt \le T\bar{\omega}. \tag{3.29}$$

Emploing (3.28) and (3.29), it follows that

$$1 \le \frac{1}{\sum_{i=0}^j M_i(\rho_i)} \left(\sum_{i=1}^j \rho_i + \sum_{i=0}^j \int_{t_i}^{t_{i+1}} \tilde{h}(t, r_{i+1} + M_i(\rho_i))dt + QT\bar{\omega} \right). \tag{3.30}$$

Assume for the sake of contradiction that:

(a₁) there is a sequence of functions $\{u_m\}$, satisfying (1.4), (1.5), (3.9), (3.19) such that each u_m, $m \in \mathbb{N}$, has its critical point equal to t_{j+1};

(a₂) if we put (according to (3.21))

$$u'_m(t_i) = \rho_{i,m}, \quad 0 \le i \le j+1, \ m \in \mathbb{N}, \tag{3.31}$$

then there is a $k \in \{0, \ldots, j\}$ such that

$$\lim_{m \to \infty} \rho_{k,m} = \infty. \tag{3.32}$$

Assuming *(a₁)* and *(a₂)* we derive a contradiciton in the following way. Let k be the largest number satisfying (3.32), i.e. if $k < j$, then $\{\rho_{i,m}\}, k+1 \le i \le j$, are bounded. First, let us show that (3.32) implies

$$\lim_{m \to \infty} \rho_{i,m} = \infty, \ 0 \le i \le k. \tag{3.33}$$

Really, if $k = 1$ then (3.33) follows from the first inequality in (3.24). If $k \ge 2$, then, by the second inequality in (3.24), $\lim_{m \to \infty} M_{k-1}(\rho_{k-1,m}) = \infty$. By virtue of (H_1), it follows that $\lim_{m \to \infty} \rho_{k-1,m} = \infty$. Continue inductively we get (3.33). Now, according to (3.27), denote

$$\begin{cases} r_{1,m} = \alpha \rho_{0,m}, \\ r_{i+1,m} = J_i(r_{i,m}) + \alpha M_i(\rho_{i,m}), \quad 1 \le i \le j, \ m \in \mathbb{N}. \end{cases} \tag{3.34}$$

Then, by virtue of (3.26), (3.30) and the assumption *(a₁)*, we put $M_0(\rho_{0,m}) = \rho_{0,m}$ and have

$$1 \le \frac{\sum_{i=1}^{j} \rho_{i,m}}{\sum_{i=0}^{j} M_i(\rho_{i,m})} +$$

$$+ \frac{\sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \tilde{h}(t, r_{i+1,m} + M_i(\rho_{i,m})) dt + QT\bar{\omega}}{\sum_{i=0}^{j} M_i(\rho_{i,m})}, \quad m \in \mathbb{N}. \tag{3.35}$$

Consider the first member on the right-hand side of (3.35). Note that due to (2.3) and (3.33) we can find $\delta > 0$ and $m_0 \in \mathbb{N}$ such that for $m \ge m_0$

$$\begin{cases} M_1(\rho_{1,m}) > \rho_{1,m}\delta, \\ M_i(\rho_{i,m}) > (1+\delta)\rho_{i,m}, \quad \text{if } k \ge 2, \quad 2 \le i \le k. \end{cases} \tag{3.36}$$

Therefore, using also the first inequality in (3.24), we get

$$S_m = \frac{\sum_{i=1}^{j} \rho_{i,m}}{\sum_{i=0}^{j} M_i(\rho_{i,m})} < \frac{\sum_{i=1}^{k} \rho_{i,m}}{(1+\delta) \sum_{i=1}^{k} \rho_{i,m}} + \frac{\sum_{i=k+1}^{j} \rho_{i,m}}{\rho_{0,m}}$$

(if $k = j$, the last member is zero), which yields

$$\lim_{m \to \infty} S_m \le \frac{1}{1+\delta}. \tag{3.37}$$

Now, consider the second member in (3.35) and put

$$z_{i,m} = r_{i+1,m} + M_i(\rho_{i,m}), \ 0 \le i \le j, \ m \in \mathbb{N}. \tag{3.38}$$

Then conditions (3.33), (3.36) and (3.34) imply that

$$\begin{cases} \lim_{m\to\infty} r_{i,m} = \infty, & 1 \le i \le k+1, \\ \lim_{m\to\infty} z_{i,m} = \infty, & 0 \le i \le k. \end{cases} \tag{3.39}$$

It is immediate from (2.6), that for $0 \le i \le k$

$$\lim_{m\to\infty} \frac{1}{z_{i,m}} \int_{t_i}^{t_{i+1}} \tilde{h}(t, z_{i,m}) dt = 0. \tag{3.40}$$

Further, for $1 \le i \le j$, by virtue of (3.34) and (3.38), it follows

$$\frac{z_{i,m}}{\sum_{l=0}^{j} M_l(\rho_{l,m})} < \frac{J_i(r_{i,m})}{\sum_{l=0}^{j} M_l(\rho_{l,m})} + 1 + \alpha =$$

$$= \frac{J_i(r_{i,m})}{r_{i,m}} \cdot \frac{r_{i,m}}{\sum_{l=0}^{j} M_l(\rho_{l,m})} + 1 + \alpha <$$

$$< \frac{J_i(r_{i,m})}{r_{i,m}} \left(\frac{J_{i-1}(r_{i-1,m})}{r_{i-1,m}} \cdot \frac{r_{i-1,m}}{\sum_{l=0}^{j} M_l(\rho_{l,m})} + \alpha \right) + 1 + \alpha.$$

Continue inductively we get for $1 \le i \le j$

$$\frac{z_{i,m}}{\sum_{l=0}^{j} M_l(\rho_{l,m})} <$$

$$\frac{J_i(r_{i,m})}{r_{i,m}} \left(\frac{J_{i-1}(r_{i-1,m})}{r_{i-1,m}} \left(\frac{J_{i-2}(r_{i-2,m})}{r_{i-2,m}} \left(\dots \frac{J_1(\alpha\rho_{0,m})}{\alpha\rho_{0,m}} \alpha + \alpha \dots \right) + \alpha \right) + \alpha \right) + 1 + \alpha.$$

Finally,

$$\frac{z_{0,m}}{\sum_{l=0}^{j} M_l(\rho_{l,m})} = \frac{(1+\alpha)\rho_{0,m}}{\rho_{0,m} + \sum_{l=1}^{j} M_l(\rho_{l,m})} < 1 + \alpha.$$

Therefore, by virtue of (2.2), for $0 \le i \le k$,

$$\lim_{m\to\infty} \frac{z_{i,m}}{\sum_{l=0}^{j} M_l(\rho_{l,m})} \le$$

$$\le A_i(A_{i-1}(A_{i-2}(\dots A_1\alpha + \alpha \dots) + \alpha) + \alpha + 1 + \alpha, \tag{3.41}$$

where

$$A_i = \lim_{z\to\infty} \frac{J_i(z)}{z} < \infty, \quad 1 \le i \le k.$$

Hence, using (3.38), (3.40) and (3.41), we get that for $0 \le i \le k$

$$\lim_{m\to\infty} \frac{\int_{t_i}^{t_{i+1}} \tilde{h}(t, r_{i+1,m} + M_i(\rho_{i,m})) dt}{\sum_{l=0}^{j} M_l(\rho_{l,m})} = 0. \tag{3.42}$$

Now, we are going to show that if $k < j$, then (3.42) is valid for $k + 1 \le i \le j$, as well. There are two cases to consider. These are $\{r_{k+2,m}\}$ bounded and $\{r_{k+2,m}\}$ unbounded.

In the first case we see that $\{r_{i,m}\}$ is bounded for $k + 2 \leq i \leq j + 1$ by (3.34), and $\{z_{i,m}\}$ is bounded for $k + 1 \leq i \leq j$, by (3.38). Therefore $\{\int_{t_i}^{t_{i+1}} h(t, z_{i,m}) dt\}$ is bounded which together with (3.33) yields (3.42) for $k + 1 \leq i \leq j$. In the second case we can suppose that $\lim_{m \to \infty} r_{k+2,m} = \infty$. Thus, due to (3.38), $\lim_{m \to \infty} z_{k+1,m} = \infty$ and so, (3.40), (3.41) and consequently (3.42) are valid for $i = k + 1$. Continue inductively we conclude that (3.42) is true for $k + 1 \leq i \leq j$ in the second case, as well. To summarize, we have proved that (3.42) is valid for $0 \leq i \leq j$.

Let us go back to (3.35). The condition (3.37) and (3.42) imply that

$$1 \leq \lim_{m \to \infty} \frac{\sum_{i=1}^j \rho_{i,m}}{\sum_{i=0}^j M_i(\rho_{i,m})} +$$

$$+ \lim_{m \to \infty} \frac{\sum_{i=0}^j \int_{t_i}^{t_{i+1}} \tilde{h}(t, r_{i+1,m} + M_i(\rho_{i,m})) dt + QT\bar{\omega}}{\sum_{i=0}^j M_i(\rho_{i,m})} \leq \frac{1}{1 + \delta},$$

a contradiction. It means that there exists $B_j > 0$ such that

$$\sup\{|u'(t)| : t \in [0, t_{j+1}]\} < B_j, \tag{3.43}$$

for each function $u \in AC_{\mathcal{D}}^1$ satisfying (1.4), (1.5), (3.9), (3.19) and having its critical point equal to t_{j+1}. This implies that there is $A_j > 0$ such that

$$\sup\{u(t) : t \in [0, t_{j+1}]\} < A_j. \tag{3.44}$$

Part 2. It remains to estimate u and $|u'|$ on $(t_{j+1}, T]$. According to Lemma 3.1, u is decreasing on (t_i, t_{i+1}), $j + 1 \leq i \leq p$, and $u(t_{p+1}) = u(T) = 0$, which together with (3.44) yields

$$\sup\{|(u(t)| : t \in (t_{j+1}, t_{j+2})\} = u(t_{j+1}+) \leq J_{j+1}(A_j) = c_{j+2} > 0,$$

$$\sup\{|u(t)| : t \in (t_i, t_{i+1})\} = u(t_i+) \leq J_i(c_i) = c_{i+1} > 0, \; j + 2 \leq i \leq p,$$

and we can find a constant $\tilde{A}_j > 0$ (independent on u) satisfying

$$\sup\{u(t) : t \in [t_{j+1}, T]\} < \tilde{A}_j, \quad j + 2 \leq i \leq p + 1. \tag{3.45}$$

Integrating (3.19) on $[t_{j+1}, T]$ we obtain by virtue of (3.21), (3.22)

$$\sum_{i=j+2}^p M_i(\rho_i) - \sum_{i=j+2}^{p+1} \rho_i \leq \int_{t_{j+1}}^T \left(\tilde{h}(t, u(t) + |u'(t)|) + q(t)\omega(u(t)) \right) dt,$$

wherefrom, due to (3.25), (3.45) and (3.26)

$$\sum_{i=j+2}^{p+1} |\rho_i| \leq \sum_{i=j+2}^p |M_i(\rho_i)| +$$

$$+ \sum_{i=j+1}^p \int_{t_i}^{t_{i+1}} \tilde{h}(t, \tilde{A}_j + |\rho_{i+1}|) dt + Q \int_{t_{j+1}}^T \omega(u(t)) dt. \tag{3.46}$$

Applying statement *(I)* of Lemma 3.3, we get

$$\int_{t_{j+1}}^T \omega(u(t)) dt \leq \sum_{i=j+1}^p \int_{t_i}^{t_{i+1}} \omega\left(\frac{t_{i+1} - t}{t_{i+1} - t_i} \gamma \right) dt = \frac{1}{\gamma} \sum_{i=j+1}^p (t_{i+1} - t_i) \int_0^\gamma \omega(t) dt,$$

hence

$$\int_{t_{j+1}}^{T} \omega(u(t))dt \le T\bar{\omega}. \tag{3.47}$$

Emploing (3.46) and (3.47), it follows that

$$1 \le \frac{1}{\sum_{i=j+2}^{p+1} |\rho_i|} \left(\sum_{i=j+2}^{p} |M_i(\rho_i)| + \sum_{i=j+1}^{p} \int_{t_i}^{t_{i+1}} \tilde{h}(t, \tilde{A}_j + |\rho_{i+1}|)dt + QT\bar{\omega} \right). \tag{3.48}$$

Assume for the sake of contradiction (a_1) as in *Part 1* and instead of (a_2) we suppose:

(a_3) if we put (according to (3.21))

$$u_m'(t_i) = \rho_{i,m}, \ j+1 \le i \le p+1, \ m \in \mathbb{N}, \tag{3.49}$$

then there is $k \in \{j+2, \ldots, p+1\}$ such that

$$\lim_{m \to \infty} \rho_{k,m} = -\infty. \tag{3.50}$$

Let $\{j+2 \ldots, p+1\} = \mathbb{J} \cup \mathbb{K}$, where

$$\lim_{m \to \infty} \rho_{i,m} = -\infty \quad \text{for } i \in \mathbb{K} \tag{3.51}$$

and

$$\{\rho_{i,m}\} \quad \text{is bounded for } i \in \mathbb{J}.$$

Further, let $\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2$, where

$$\lim_{m \to \infty} M_i(\rho_{i,m}) = -\infty \quad \text{for } i \in \mathbb{K}_1$$

and

$$\{M_i(\rho_{i,m})\} \quad \text{is bounded for } i \in \mathbb{K}_2.$$

By virtue of (3.45), (3.48) and the assumption (a_1), we have

$$1 \le \frac{\sum_{i=j+2}^{p} |M_i(\rho_{i,m})|}{\sum_{i=j+2}^{p+1} |\rho_{i,m}|} +$$

$$+ \frac{\sum_{i=j+1}^{p} \int_{t_i}^{t_{i+1}} \tilde{h}(t, \tilde{A}_j + |\rho_{i+1,m})|)dt + QT\bar{\omega}}{\sum_{i=j+2}^{p+1} |\rho_{i,m}|}, \ m \in \mathbb{N}. \tag{3.52}$$

Consider the first member on the right-hand side of (3.52). Note, that due to (2.4) and (3.51) we can find $\delta > 0$ and $m_0 \in \mathbb{N}$ such that for $m \ge m_0$

$$\begin{cases} \ |\rho_{p,m}| > \delta |M_p(\rho_{p,m})| & \text{if } p \in \mathbb{K}, \\ |\rho_{i,m}| > (1+\delta)|M_i(\rho_{i,m})| & \text{if } i < p, \ i \in \mathbb{K}. \end{cases} \tag{3.53}$$

Therefore, using also the inequality in (3.24) for $i = p$, we get

$$R_m = \frac{\sum_{i=j+2}^{p} |M_i(\rho_{i,m})|}{\sum_{i=j+2}^{p+1} |\rho_{i,m}|} < \frac{\sum_{i \in \mathbb{J} \cup \mathbb{K}_2} |M_i(\rho_{i,m})|}{\sum_{i \in \mathbb{K}} |\rho_{i,m}|} + \frac{\sum_{i \in \mathbb{K}_1} |M_i(\rho_{i,m})|}{(1+\delta) \sum_{i \in \mathbb{K}_1} |M_i(\rho_{i,m})|}$$

which yields

$$\lim_{m \to \infty} R_m \le \frac{1}{1+\delta}. \tag{3.54}$$

Now, consider the second member in (3.52) and put

$$d_{i+1,m} = \tilde{A}_j + |\rho_{i+1,m}|, \ \ j+1 \le i \le p, \ m \in \mathbb{N}. \tag{3.55}$$

Then, for $k \in \mathbb{K}$

$$\lim_{m \to \infty} d_{k,m} = \infty, \quad \lim_{m \to \infty} \frac{d_{k,m}}{\sum_{i=j+2}^{p+1} |\rho_{i,m}|} \le 1. \tag{3.56}$$

Further, it is immediate from (2.6) that for $k \in \mathbb{K}$

$$\lim_{m \to \infty} \frac{1}{d_{k,m}} \int_{t_{k-1}}^{t_k} \tilde{h}(t, d_{k,m}) dt = 0. \tag{3.57}$$

Therefore, according to (3.56) and (3.57),

$$\lim_{m \to \infty} \frac{\sum_{k \in \mathbb{K}} \int_{t_{k-1}}^{t_k} \tilde{h}(t, d_{k,m}) dt}{\sum_{i=j+2}^{p+1} |\rho_{i,m}|} = 0. \tag{3.58}$$

Finally, for $k \in \mathbb{J}$ the sequences $\{d_{k,m}\}$ and $\{\int_{t_{k-1}}^{t_k} \tilde{h}(t, d_{k,m}) dt\}$ are bounded which yields

$$\lim_{m \to \infty} \frac{\sum_{k \in \mathbb{J}} \int_{t_{k-1}}^{t_k} \tilde{h}(t, d_{k,m}) dt}{\sum_{i=j+2}^{p+1} |\rho_{i,m}|} = 0. \tag{3.59}$$

Let us go back to (3.52). The conditions (3.58), (3.59) and (3.54) imply that

$$1 \le \lim_{m \to \infty} \frac{\sum_{i=j+2}^{p} |M_i(\rho_{i,m})|}{\sum_{i=j+2}^{p+1} |\rho_{i,m}|} +$$

$$+ \lim_{m \to \infty} \frac{\sum_{i=j+1}^{p} \int_{t_i}^{t_{i+1}} \tilde{h}(t, \tilde{A}_j + |\rho_{i+1,m}|) dt + QT\bar{\omega}}{\sum_{i=j+2}^{p+1} |\rho_{i,m}|} \le \frac{1}{1+\delta},$$

a contradiction. It means that there exists $\tilde{B}_j > 0$ such that

$$\sup\{|u'(t)| : t \in [t_{j+1}, T]\} < \tilde{B}_j, \tag{3.60}$$

for each function $u \in AC_{\mathcal{D}}^1$ satisfying (1.4), (1.5), (3.9), (3.19) and having its critical point equal to t_{j+1}.

  If we find $A_j, \tilde{A}_j$ satisfying (3.44), (3.45) and $B_j, \tilde{B}_j$ satisfying (3.43), (3.60) for each $j \in \{0, \dots, p-1\}$ and put

$$A^* = \max\{A_j, \tilde{A}_j, 0 \le j \le p-1\}, \quad B^* = \max\{B_j, \tilde{B}_j, \ 0 \le j \le p-1\},$$

we get (3.20) for each function $u \in AC_{\mathcal{D}}^1$ satisfying (1.4), (1.5), (3.9), (3.19) and having its critical point in $\mathcal{D}$.

  Case (ii). Suppose that $u \in AC_{\mathcal{D}}^1$ fulfils (1.4), (1.5), (3.9), (3.19) and has its critical point $\xi \in (0, T) \setminus \mathcal{D}$. It means that there is $j \in \{0, \dots, p\}$ such that $\xi \in (t_j, t_{j+1})$. Then we argue similarly as in *Case (i)*. Particularly, in *Part 1* we take the interval $(t_j, \xi]$ instead of $(t_j, t_{j+1}]$ and use assertion *(II)* of Lemma 3.3 instead of assertion *(I)*. In *Part 2* we have in addition the interval $(\xi, t_{j+1}]$, where u has the same properties as on $(t_{j+1}, t_{j+2}]$ in *Case (i)*. We also use assertion *(II)* of Lemma 3.3 and then argue as in *Case (i)*.   □

4   Uniform absolute continuity

Let us denote by $\mathcal{B}$ the set of all functions $x \in AC_{\mathcal{D}}^1$ having a unique critical point $\xi = \xi(x)$ and fulfilling *(I)*, *(II)* of Lemma 3.3 and let $x_{(i)} \in C^1[t_i, t_{i+1}], 0 \leq i \leq p$, fulfil (1.1).

Lemma 4.1. *Let ω satisfy (H_4) and $i \in \{0, \ldots, p\}$. Then the set*

$$\mathcal{A}_i = \{\omega(x_{(i)}) : x \in \mathcal{B}\}$$

is uniformly absolutely continuous on $[t_i, t_{i+1}]$, i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathcal{M}} \omega(x_{(i)}(t))dt < \varepsilon$$

for each function $x \in \mathcal{B}$ and each set $\mathcal{M} \subset [t_i, t_{i+1}]$ such that $\mu(\mathcal{M}) < \delta$.

Proof. It is sufficient to prove that for any $i \in \{0, \ldots, p\}$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that for each system $\{(\alpha_k, \beta_k)\}_{k=1}^\infty$ of mutually disjoint intervals $(\alpha_k, \beta_k) \subset [t_i, t_{i+1}]$ the implication

$$\sum_{k=1}^\infty (\beta_k - \alpha_k) < \delta \Longrightarrow \sum_{k=1}^\infty \int_{\alpha_k}^{\beta_k} \omega(x_{(i)}(t))dt < \varepsilon \tag{4.1}$$

is valid for each function $x \in \mathcal{B}$.

Let us choose $x \in \mathcal{B}$ and $i \in \{0, \ldots, p\}$. Then x has a critical point $\xi = \xi(x) \in (0, T)$. We are going to estimate the integral

$$I_k = \int_{\alpha_k}^{\beta_k} \omega(x_{(i)}(t))dt.$$

Denote

$$\Omega(z) = \int_0^z \omega(s)ds.$$

We distinguish 3 types of locations of ξ, α_k, β_k.
Location 1: Let $\xi \geq \beta_k$. Then

$$I_k \leq \frac{T}{\gamma} \left(\Omega(\frac{\gamma}{T}(\beta_k - t_i)) - \Omega(\frac{\gamma}{T}(\alpha_k - t_i)) \right). \tag{4.2}$$

Location 2: Let $\xi \leq \alpha_k$. Then

$$I_k \leq \frac{T}{\gamma} \left(\Omega(\frac{\gamma}{T}(t_{i+1} - \alpha_k)) - \Omega(\frac{\gamma}{T}(t_{i+1} - \beta_k)) \right). \tag{4.3}$$

Location 3: Let $\alpha_k < \xi < \beta_k$. Then

$$I_k \leq \frac{T}{\gamma} \left(\Omega(\frac{\gamma}{T}(\xi - t_i)) - \Omega(\frac{\gamma}{T}(\alpha_k - t_i)) \right) +$$
$$+ \frac{T}{\gamma} \left(\Omega(\frac{\gamma}{T}(t_{i+1} - \xi)) - \Omega(\frac{\gamma}{T}(t_{i+1} - \beta_k)) \right). \tag{4.4}$$

Choose an $\varepsilon > 0$ and put $\varepsilon_1 = \varepsilon \frac{\gamma}{2T}$. Since Ω is absolutely continuous on $[0, \gamma]$, we can find $\delta_1 > 0$ such that for any system $\{(a_k, b_k)\}_{k=1}^{\infty}$ of mutually disjoint intervals in $[0, \gamma]$

$$\sum_{k=1}^{\infty} (b_k - a_k) < \delta_1 \implies \sum_{k=1}^{\infty} (\Omega(b_k) - \Omega(a_k)) < \varepsilon_1 \tag{4.5}$$

is valid. Put $\delta = \delta_1 \frac{T}{\gamma}$ and take a system $\{(\alpha_k, \beta_k)\}_{k=1}^{\infty} \subset [t_i, t_{i+1}]$ such that $\sum_{k=1}^{\infty} (\beta_k - \alpha_k) < \delta$. Then, using (4.2)-(4.5) and the inequality $||\beta_k - t_i| - |\alpha_k - t_i|| \leq \beta_k - \alpha_k$, we get

$$\sum_{k=1}^{\infty} I_k < \frac{2T}{\gamma} \varepsilon_1 = \varepsilon,$$

and hence (4.1) is proved. $\qquad\square$

5 Existence principle for regular impulsive BVPs

Lemma 5.1. *Let $h \in L_1[0, T]$ and $c_i, d_i \in \mathbb{R}, 1 \leq i \leq p$. Then there is a unique solution u of problem* (1.5),

$$-u''(t) = h(t), \tag{5.1}$$

$$u(t_i+) = c_i + u(t_i), \ u'(t_i+) = d_i + u'(t_i), \ 1 \leq i \leq p. \tag{5.2}$$

This solution is given by

$$u(t) = \sum_{i=1}^{p} \tilde{G}(t, t_i) c_i + G(t, t_i) d_i - \int_0^T G(t, s) h(s) ds, \tag{5.3}$$

for $t \in [0, T]$, where

$$\tilde{G}(t, s) = \begin{cases} \dfrac{T - t}{T} & if \quad 0 \leq s \leq t \leq T \\[2mm] -\dfrac{t}{T} & if \quad 0 \leq t < s \leq T, \end{cases} \tag{5.4}$$

$$G(t, s) = \begin{cases} \dfrac{s(t - T)}{T} & if \quad 0 \leq s \leq t \leq T \\[2mm] \dfrac{t(s - T)}{T} & if \quad 0 \leq t < s \leq T. \end{cases} \tag{5.5}$$

Proof. We can argue as in the proof of Lemma 2.1 in [18]. $\qquad\square$

Now, suppose that $C_{\mathcal{D}}^1$ is equipped with the norm

$$\|x\|_{\mathcal{D}} = \sup\{|x(t)| + |x'(t)| : t \in [0, T]\}.$$

Then $C_{\mathcal{D}}^1$ becomes a Banach space. (See e.g. [8].) For $R > 0$ define a set

$$K(R) = \{x \in C_{\mathcal{D}}^1 : \|x\|_{\mathcal{D}} < R\}$$

and its closure denote by $cl(K(R))$.

Lemma 5.2. *Let us suppose that $g \in L_1[0,T]$ and (H_1) holds. Then there exists a constant $R^* > 0$ such that each function $u \in AC_{\mathcal{D}}^1$ fulfilling $(1.4),(1.5)$ and*

$$0 < -u''(t) \leq g(t) \quad \textit{for a.e. } t \in [0,T] \tag{5.6}$$

belongs to $K(R^*)$.

Proof. By Lemma 3.1 u has a critical point $\xi \in (0,T)$. First, assume that $\xi \in \mathcal{D}$, i.e. $\xi = t_j$ for some $j \in \{1,\ldots,p\}$. Integrate (5.6) from ξ to $t \in (\xi, t_{j+1}]$. We get

$$0 \leq -u'(t) \leq \|g\|_{L_1} \quad \text{for } t \in [\xi, t_{j+1}].$$

If $j < p$ we integrate (5.6) from t_{j+1} to $t \in (t_{j+1}, t_{j+2}]$ and get

$$0 \leq -u'(t) \leq M_{j+1}(\|g\|_{L_1}) + \|g\|_{L_1} \quad \text{for } t \in [t_{j+1}, t_{j+2}].$$

If $j < p-1$ we integrate again and continue inductively we find a constant $m_1 > 0$ such that

$$0 \leq -u'(t) \leq m_1 \quad \text{for } t \in [\xi, T], \tag{5.7}$$

for any $u \in AC_{\mathcal{D}}^1$ satisfying (1.4), (1.5) and (5.6). Similarly we find $m_2 > 0$ such that

$$0 \leq u'(t) \leq m_2 \quad \text{for } t \in [0,\xi]. \tag{5.8}$$

If $\xi \in (0,T) \setminus \mathcal{D}$ we can use a similar integral procedure and get (5.7), (5.8), as well. Now, integrate (5.8) from 0 to $t \in (0,t_1]$. Then

$$0 \leq u(t) \leq t_1 m_2 \quad \text{for } t \in [0,t_1].$$

Integrate (5.8) from t_1 to $t \in (t_1, t_2]$. Then

$$0 < u(t) \leq (t_2 - t_1)m_2 + J_1(t_1 m_2) \quad \text{for } t \in (t_1, t_2].$$

Continue inductively we deduce that there exists a constant m_3 (independent on u) such that

$$0 < \sup\{u(t) : t \in [0,T]\} \leq m_3.$$

Hence, it suffices to put $R^* = m_1 + m_2 + m_3$ and lemma is proved. □

Theorem 5.3. *Let us suppose that $\tilde{f}$ satisfies the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$, (H_1) holds and that there exists a function $g \in L_1[0,T]$ such that*

$$0 < \tilde{f}(t,x,y) \leq g(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and for all } x,y \in \mathbb{R}. \tag{5.9}$$

Then problem $(1.4),(1.5)$,

$$-u''(t) = \tilde{f}(t, u(t), u'(t)) \tag{5.10}$$

has a positive solution.

Proof. *Step 1.* Choose an arbitrary $y \in C_{\mathcal{D}}^1$ and consider the auxiliary linear problem (1.5)

$$-x''(t) = \tilde{f}(t, y(t), y'(t)), \tag{5.11}$$

$$\begin{cases} x(t_i+) - x(t_i) = J_i(y(t_i)) - y(t_i), \\ x'(t_i+) - x'(t_i) = M_i(y'(t_i)) - y'(t_i), \ 1 \le i \le p. \end{cases} \tag{5.12}$$

Clearly $\tilde{f}(t, y(t), y'(t)) \in L_1[0, T]$, $J_i(y(t_i)) - y(t_i) = c_i \in \mathbb{R}$, $M_i(y'(t_i)) - y'(t_i) = d_i \in \mathbb{R}$, $1 \le i \le p$, and hence, by Lemma 5.1, problem (1.5), (5.11), (5.12) has a unique solution $x \in AC_{\mathcal{D}}^1$. By virtue of (5.3) this solution is of the form

$$x(t) = (Fy)(t) \quad \text{for } t \in [0, T],$$

where $F : C_{\mathcal{D}}^1 \to C_{\mathcal{D}}^1$ is given by

$$(Fy)(t) = \sum_{i=1}^p \tilde{G}(t, t_i)(J_i(y(t_i)) - y(t_i)) +$$

$$+ \sum_{i=1}^p G(t, t_i)(M_i(y'(t_i)) - y'(t_i)) - \int_0^T G(t, s)\tilde{f}(s, y(s), y'(s))ds.$$

Therefore, u is a solution of (1.4), (1.5), (5.10) if and only if u is a fixed point of the operator F. Let $F_1 : C_{\mathcal{D}}^1 \to C_{\mathcal{D}}^1$ be defined by the formula

$$(F_1 y)(t) = \int_0^T G(t, s)\tilde{f}(s, y(s), y'(s))ds.$$

Due to (5.9) we can use the Lebesgue dominated convergence theorem and the Arzelà-Ascoli theorem and get that F_1 is absolutely continuous. Further, since J_i and M_i, $1 \le i \le p$, are continuous, the operator $F_2 = F + F_1$ is continuous, as well. Since F_2 maps $C_{\mathcal{D}}^1$ in a $2p$-dimensional subspace of $C_{\mathcal{D}}^1$, we deduce that F_2 and consequently F are absolutely continuous operators.

Step 2. We are going to prove the existence of a fixed point of F by means of the topological degree arguments. To this aim we consider the operator equation

$$u = Fu \tag{5.13}$$

and the parameter system of equations

$$u = F^*(\lambda, u), \tag{5.14}$$

with $F^* : [0, 1] \times C_{\mathcal{D}}^1 \to C_{\mathcal{D}}^1$,

$$F^*(\lambda, u) = \sum_{i=1}^p \tilde{G}(t, t_i)(J_i(u(t_i)) - u(t_i)) +$$

$$+ \sum_{i=1}^p G(t, t_i)(M_i(u'(t_i)) - u'(t_i)) - \lambda \int_0^T G(t, s)\tilde{f}(s, u(s), u'(s))ds.$$

Clearly $F^*(1, u) = Fu$ and F^* is absolutely continuous. Let us choose $\lambda \in (0, 1]$ and let $u \in C_{\mathcal{D}}^1$ be a corresponding solution of (5.14), i.e. $u = F^*(\lambda, u)$. It means that the function u satisfies

$$-u''(t) = \lambda \tilde{f}(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0, T], \tag{5.15}$$

and $u \in AC_{\mathcal{D}}^1$ fulfils (1.4), (1.5). Hence u is a solution of problem (5.15), (1.4), (1.5). By virtue of (5.9) u satisfies

$$0 < -u''(t) \leq \lambda g(t) \leq g(t) \quad \text{for a.e. } t \in [0, T],$$

and hence, by Lemma 5.2, there exists $R^* > 0$ (independent on u and λ) such that

$$u \in K(R^*). \tag{5.16}$$

Let $u \in C_{\mathcal{D}}^1$ be a solution of (5.14) for $\lambda = 0$, i.e. $u = F^*(0, u)$. Then $u \in AC_{\mathcal{D}}^1$ satisfies (1.4), (1.5) and $-u''(t) = 0$ for a.e. $t \in [0, T]$. Therefore $u(t) = a + bt$, $a, b \in \mathbb{R}$. Since $u(0) = 0$, we get $a = 0$ and the condition $u(T) = 0$ implies $b = 0$. Hence $u(t) = 0$ for $t \in [0, T]$ and consequently u fulfils (5.16). To summarize, we have proved that there exists a constant $R^* > 0$ such that for any $\lambda \in [0, 1]$ each solution of (5.14) belongs to $K(R^*)$. This means that $Iu - F^*(\lambda, u)$ is a homotopy on $[0, 1] \times cl(K(R^*))$ and thus

$$1 = deg(I, K(R^*)) = deg(I - F, K(R^*)). \tag{5.17}$$

Here, deg is the Leray-Schauder topological degree and $I : C_{\mathcal{D}}^1 \to C_{\mathcal{D}}^1$ is the identity operator $Ix = x$. The condition (5.17) implies that F has a fixed point $u \in K(R^*)$. Since fixed points of F are solutions of (1.4), (1.5), (5.10), and due to Lemma 3.1 these solutions are positive on $(0, T)$, Theorem 5.3 is proved. $\qquad \square$

6   Main results

In this section we construct a sequence of auxiliary regular BVPs and, by Theorem 5.3, we get a sequence of their solutions. Then, using the limiting process we prove the existence of a positive solution to our original singular Dirichlet BVP (1.3)-(1.5).

Theorem 6.1.   *Let assumptions (H_1)-(H_5) be satisfied.   Then there exists a positive solution of BVP* (1.3) $-$ (1.5).
Proof. For a.e. $t \in [0, T]$ and all $z \in [0, \infty)$ put

$$\tilde{h}(t, z) = h(t, 1 + z) + q(t)\omega(1). \tag{6.1}$$

Then, due to (H_4), $\tilde{h}$ satisfies the Carathéodory conditions on $[0, T] \times [0, \infty)$, $\tilde{h}$ is non-negative and nondecreasing in its second argument and, by virtue of (2.6),

$$\lim_{z \to \infty} \frac{1}{z} \int_0^T \tilde{h}(t, z) dt = 0.$$

Therefore, we can find positive constants A^*, B^* satisfying Lemma 3.4.

Now, choose an arbitrary $m \in \mathbb{N}$ and put for $x \in [0, \infty), y \in \mathbb{R}$

$$\sigma_1(\frac{1}{m}, x) = \begin{cases} \dfrac{1}{m} & \text{if } 0 \le x \le \frac{1}{m} \\ x & \text{if } \frac{1}{m} \le x \le 1 + A^* \\ 1 + A^* & \text{if } x > 1 + A^*, \end{cases} \qquad \sigma_2(y) = \begin{cases} y & \text{if } |y| \le B^* \\ B^* \text{sgn} y & \text{if } |y| > B^*. \end{cases}$$

Further, for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$ define

$$f_m(t, x, y) = f(t, \sigma_1(\frac{1}{m}, |x|), \sigma_2(y)),$$

and consider the auxiliary equation

$$-u''(t) = f_m(t, u(t), u'(t)). \tag{6.2}$$

Then, by (H_3), f_m satisfies the Carathéodory conditions on $[0, T] \times \mathbb{R}^2$ and for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$

$$0 < f_m(t, x, y) \le g_m(t), \tag{6.3}$$

where

$$g_m(t) = \sup\{f(t, x, y) : x \in [\frac{1}{m}, 1 + A^*], y \in [-B^*, B^*]\} \in L_1[0, T].$$

Therefore, according to Theorem 5.3, for each $m \in \mathbb{N}$ problem (6.2), (1.4), (1.5) has a positive solution u_m. In such a way we get a sequence $\{u_m\}$ of solutions. By Lemma 3.1 and (6.2), (6.3), each u_m has a unique critical point $\xi_m = \xi_m(u_m) \in (0, T)$, $m \in \mathbb{N}$.

Further, by virtue of (H_3)-(H_5), we see that for a.e. $t \in [0, T]$ and all $x \in (0, \infty), y \in \mathbb{R}$, $m \in \mathbb{N}$, the inequalities

$$0 < \psi(t) \le f_m(t, x, y), \tag{6.4}$$

and

$$f_m(t, x, y) \le \tilde{h}(t, x + |y|) + q(t)\omega(x), \tag{6.5}$$

are valid, where ψ is from (H_3) and $\tilde{h}$ (on the place of h), q, ω satisfy (H_4). Note, that (6.5) follows from (2.8), (6.1) and relations

$$\sigma_1(\frac{1}{m}, x) \le 1 + x, \quad |\sigma_2(y)| \le |y|$$

and

$$\omega(\sigma_1(\frac{1}{m}, x)) \le \omega(1 + A^*) + \omega(x) \le \omega(1) + \omega(x).$$

In view of (6.4) we can use Lemma 3.3 and find $\gamma > 0$ (independent on u_m) satisfying (3.10), (3.11), where u is replaced with u_m, $m \in \mathbb{N}$. It means that

$$\{u_m\} \subset \mathcal{B}. \tag{6.6}$$

Moreover, Lemma 3.4 yields

$$\begin{cases} \sup\{u_m(t) : t \in [0, T]\} < A^*, \\ \sup\{|u'_m(t)| : t \in [0, T]\} < B^*. \end{cases} \tag{6.7}$$

Now, choose an arbitrary $i \in \{0, \ldots, p\}$ and denote by $u_{m(i)}$ functions from $C^1[t_i, t_{i+1}]$ which correspond to u_m in the sense of the formula (1.1). Put as before

$$Q = \text{supess}\{q(t) : t \in [0, T]\}.$$

By (6.5), we have for $t, \tau \in [t_i, t_{i+1}]$, $\tau < t$,

$$|u'_{m(i)}(t) - u'_{m(i)}(\tau)| \leq \int_\tau^t \tilde{h}(s, A^* + B^*)ds + Q \int_\tau^t \omega(u_{m(i)}(s))ds. \tag{6.8}$$

Due to (6.6) we can use Lemma 4.1 and obtain that the sequence $\{\omega(u_{m(i)})\}$ is uniformly absolutely continuous on $[t_i, t_{i+1}]$. This, by (6.8), implies that the sequence $\{u'_{m(i)}\}$ is equicontinuous on $[t_i, t_{i+1}]$. Further, by (6.7), we see that the sequence $\{u_{m(i)}\}$ is bounded in $C^1[t_i, t_{i+1}]$. Thus by the Arzelà-Ascoli theorem, we can choose a subsequence $\{u_{k(i)}\}$ which converges in $C^1[t_i, t_{i+1}]$ to a function $u_{(i)} \in C^1[t_i, t_{i+1}]$. Consider the sequence of equalities

$$u'_{k(i)}(t) = u'_{k(i)}(t_i) - \int_{t_i}^t f_k(s, u_{k(i)}(s), u'_{k(i)}(s))ds \quad \text{for } t \in [t_i, t_{i+1}]. \tag{6.9}$$

Denote the set of all $t \in [0, T]$ such that $f(t, \cdot, \cdot) : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous by $\mathcal{U}$. Then $\mu([0, T] \setminus \mathcal{U}) = 0$ and

$$\lim_{k \to \infty} f_k(t, u_{k(i)}(t), u'_{k(i)}(t)) = f(t, u_{(i)}(t), u'_{(i)}(t)) \quad \text{for all } t \in (t_i, t_{i+1}) \cap \mathcal{U},$$

because $u_{(i)}$ is positive on (t_i, t_{i+1}) by (6.6). Using (6.5) and the uniform absolute continuity of $\{\omega(u_{k(i)})\}$ on $[t_i, t_{i+1}]$, we deduce that $\{f_k(t, u_{k(i)}(t), u'_{k(i)}(t))\}$ is also uniformly absolutely continuous on $[t_i, t_{i+1}]$. Therefore we can use the Vitali convergence theorem by which $f(t, u_{(i)}(t), u'_{(i)}(t)) \in L_1[t_i, t_{i+1}]$ and letting $k \to \infty$ in (6.9) we have that

$$u'_{(i)}(t) = u'_{(i)}(t_i) - \int_{t_i}^t f(s, u_{(i)}(s), u'_{(i)}(s))ds \quad \text{for } t \in [t_i, t_{i+1}].$$

It means that $u_{(i)} \in AC^1[t_i, t_{i+1}]$ and $u_{(i)}$ satisfies (1.3) a.e. on $[t_i, t_{i+1}]$. Since $i \in \{0, \ldots, p\}$ has been chosen arbitrarily, we can put

$$u(t) = \begin{cases} u_{(0)}(t) & \text{for} & t \in [0, t_1] \\ u_{(1)}(t) & \text{for} & t \in (t_1, t_2] \\ \ldots & \ldots & \ldots \\ u_{(p)}(t) & \text{for} & t \in (t_p, T] \end{cases}$$

and get $u \in AC^1_\mathcal{D}$, u satisfies (1.3) a.e. on $[0, T]$ and fulfils (1.5). Having in mind that J_i, M_i, $1 \leq i \leq p$ are continuous we deduce that u fulfils (1.4). Really, we have

$$u(t_i+) = u_{(i)}(t_i) = \lim_{k \to \infty} u_{k(i)}(t_i) = \lim_{k \to \infty} J_i(u_{k(i-1)})(t_i)) =$$

$$= J_i(\lim_{k \to \infty} u_{k(i-1)}(t_i)) = J_i(u_{(i-1)}(t_i)) = J_i(u(t_i)).$$

Similarly for $u'(t_i+)$. Finally, due to Lemma 3.1 and (H_3), u is positive on $(0, T)$. Theorem 6.1 is proved.   □

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